Spaces of Analytical Functions and Wavelets
Lecture Notes

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Abstract

This is (raw) lecture notes of the course read on 6th European intensive course on Complex Analysis (Coimbra, Portugal) in 2000.

Our purpose is to describe a general framework for generalizations of the complex analysis. As a consequence a classification scheme for different generalizations is obtained.

The framework is based on wavelets (coherent states) in Banach spaces generated by "admissible" group representations. Reduced wavelet transform allows naturally describe in abstract term main objects of an analytical function theory: the Cauchy integral formula, the Hardy and Bergman spaces, the Cauchy-Riemann equation, and the Taylor expansion.

Among considered examples are classical analytical function theories (one complex variables, several complex variables, Clifford analysis, Segal-Bargmann space) as well as new function theories which were developed within our framework (function theory of hyperbolic type, Clifford version of Segal-Bargmann space).

We also briefly discuss applications to the operator theory (functional calculus) and quantum mechanics.

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Course Outline

Lecture 1

Different Generalizations of Complex Analysis

1.1 Introduction

The classic heritage of complex analysis is contested between several complex variables theory and hypercomplex analysis. The first one was founded long ago by Cauchy and Weierstrass themselves and sometime thought to be the only crown-prince. The hypercomplex analysis is not a single theory but a family of related constructions discovered quite recently [9, 17, 24] (and rediscovered up to now) under hypercomplex framework.

Such a variety of theories puts the question on their classification. One could dream about a Mendeleev-like periodic table for hypercomplex analysis, which clearly explains properties of different theories, relationship between them and indicates how many blank cells are waiting for us. Moreover, because hypercomplex analysis is the recognized background for *classic mechanics* and *quantum physics* theories like the Maxwell and Dirac equations, such a table could play the role of *the Mendeleev table for elementary particles and fields*. We will return to this metaphor and find it is not very superficial.

To make a step in the desired direction we should specify the notion of *function theory* and define the concept of *essential difference*. Probably many people agree that

Definition 1.1.1 The *core of complex analysis* consists of

(i). The Cauchy-Riemann equation and complex derivative $\frac{\partial}{\partial z}$;

(ii). The Cauchy theorem;

(iii). The Cauchy integral formula;

(iv). The Plemeli-Sokhotski formula;

(v). The Taylor and Laurent series.

Any development of several complex variables theory or hypercomplex analysis is beginning from analogies to these notions and results. Thus we adopt the following

Definition 1.1.2 A *function theory* is a collection of notions and results, which includes at least analogies of 1.1.1.(i)–1.1.1.(v).

Of course the definition is more philosophical than mathematical. For example, the understanding of an *analogy* and especially the *right analogy* usually generates many disputes.

Again as a first approximation we propose the following

Definition 1.1.3 Two function theories is said to be *similar* if there is a correspondence between their objects such that analogies of 1.1.1.(i)–1.1.1.(v) in one theory follow from their counterparts in another theory. Two function theories are *essentially different*function theory!(essentially) different if they are not similar.

Unspecified "correspondence" should probably be a linear map and we will look for its meaning soon. It is clear that the *similarity* is an equivalence relation and we are looking for quotient sets with respect to it.

The layout is following. In Subsection 1.2 the classic scheme of hypercomplex analysis is discussed and a possible variety of function theories appears. But we will see in Subsection 1.3 that not all of them are very different. Connection between group representations and (hyper)complex analysis is presented in Section 1.4. It could be a base for classification of essentially different theories.

1.2 Factorizations of the Laplacian

In the next Section we repeat shortly the scheme of development of Clifford analysis as it could be found in [9, 17]. We examine different options arising on this way and demonstrate that some differences are only apparent not essential.

We would like to see how the contents of 1.1.1.(i)–1.1.1.(v) could be realized in a function theory. We are interested in function theories defined in $\mathbb{R}^d$. The Cauchy theorem and integral formula clearly indicates that the behavior of functions inside a domain should be governed by their values on the boundary. Such a property is particularly possessed by solutions to the *second order elliptic differential operator* P

$$P(x, \partial_x) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{d} b_i(x)\partial_i + c(x)$$

with some special properties. Of course, the principal example is the Laplacian

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}. \tag{1.2.1}$$

1.2.1 (i). *Choice of different operators* (for example, the Laplacian or the Helmholtz operator) is the first option which brings the variety in the family of hypercomplex analysis.

The next step is called *linearization.* Namely we are looking for two (possibly coinciding) *first order differential operators* D and D' such that

$$DD' = P(x, \partial_x).$$

The Dirac motivation to do that is to "look for an equation linear in in time derivative $\frac{\partial}{\partial t}$, because the Schrödinger equation is". From the function theory point of view the Cauchy-Riemann operator should be linear also. But the most important gain of the step is an introduction of the Clifford algebra. For example, to factorize the Laplacian (1.2.1) we put

$$D = \sum_{i=1}^{d} e_i\partial_i \tag{1.2.2}$$

where e_i are the *Clifford algebra* generators:

$$e_ie_j + e_je_i = 2\delta_{ij}, \qquad 1 \le i, j \le d. \tag{1.2.3}$$

(ii). *Different linearizations of a second order operator* multiply the spectrum of theories.

Mathematicians and physicists are looking up to now new factorization even for the Laplacian. The essential uniqueness of such factorization was already felt by Dirac himself but it was never put as a theorem. So the idea of the *genuine* factorization becomes the philosophers' stone of our times.

After one made a choice 1.2.1.(i) and 1.2.1.(ii) the following turns to be a routine. The equation

$$D'f(x) = 0,$$

plays the role of the Cauchy-Riemann equation. Having a fundamental solution $F(x)$ to the operator $P(x, \partial_x)$ the Cauchy integral kernel defined by

$$E(x) = D'F(x)$$

with the property $DE(x) = \delta(x)$. Then the Stocks theorem implies the Cauchy theorem and Cauchy integral formula. A decomposition of the Cauchy kernel of the form

$$C(x-y) = \sum_{\alpha} V_\alpha(x)W_\alpha(y),$$

where $V_\alpha(x)$ are some polynomials, yields via integration over the ball the Taylor and Laurent series[1]. In such a way the program-minimum 1.1.1.(i)–1.1.1.(v) could be accomplished.

Thus all possibilities to alter function theory concentrated in 1.2.1.(i) and 1.2.1.(ii). Possible universal algebras arising from such an approach were investigated by F. Sommen [75]. In spite of the apparent wide selection, for operator D and D' with constant coefficients it was found "nothing dramatically new" [75]:

> Of course one can study all these algebras and prove theorems or work out lots of examples and representations of universal algebras. But in the constant coefficient case the most important factorization seems to remain the relation $\Delta = \sum x_j^2$, i.e., the one leading to the definition of the Clifford algebra.

We present an example that there is no dramatical news not only on the level of universal algebras but also for function theory (for the constant coefficient case). We will return to non constant case in Section 1.4.

1.3 Example of Connection

We give a short example of similar theories with explicit connection between them. The full account could be found in [33], another example was considered in [66].

[1] Not all such decompositions give interesting series. The scheme from Section 1.4 gives a selection rule to distinguish them.

Due to physical application we will consider equation

$$\frac{\partial f}{\partial y_0} = (\sum_{j=1}^{n} e_j \frac{\partial}{\partial y_j} + M)f, \tag{1.3.1}$$

where e_j are generators (1.2.3) of the Clifford algebra and $M = M_\lambda$ is an operator of multiplication from the *right*-hand side by the Clifford number λ. Equation (1.3.1) is known in *quantum mechanics* as the Dirac equation *for a particle with a non-zero rest mass* [4, §20], [7, §6.3] and [48]. We will specialize our results for the case $M = M_\lambda$, especially for the simplest (but still important!) case $\lambda \in \mathbb{R}$.

Theorem 1.3.1 *The function $f(y)$ is a solution to the equation*

$$\frac{\partial f}{\partial y_0} = (\sum_{j=1}^{n} e_j \frac{\partial}{\partial y_j} + M_1)f$$

if and only if the function

$$g(y) = e^{y_0 M_2} e^{-y_0 M_1} f(y)$$

is a solution to the equation

$$\frac{\partial g}{\partial y_0} = (\sum_{j=1}^{n} e_j \frac{\partial}{\partial y_j} + M_2)g,$$

where M_1 and M_2 are bounded operators commuting with e_j.

Corollary 1.3.2 *The function $f(y)$ is a solution to the equation (1.3.1) if and only if the function $e^{y_0 M} f(y)$ is a solution to the generalized Cauchy-Riemann equation (1.2.2).*

In the case $M = M_\lambda$ we have $e^{y_0 M_\lambda} f(y) = f(y) e^{y_0 \lambda}$ and if $\lambda \in \mathbb{R}$ then $e^{y_0 M_\lambda} f(y) = f(y) e^{y_0 \lambda} = e^{y_0 \lambda} f(y)$.

In this Subsection we construct a function theory (in the sense of 1.1.1.(i)–1.1.1.(v)) for M-solutions of the generalized Cauchy-Riemann operator based on Clifford analysis and Corollary 1.3.2.

The set of solutions to (1.2.2) and (1.3.1) in a nice domain Ω will be denoted by $\mathfrak{M}(\Omega) = \mathfrak{M}_0(\Omega)$ and $\mathfrak{M}_M(\Omega)$ correspondingly. In the case $M = M_\lambda$ we use the notation $\mathfrak{M}_\lambda(\Omega) = \mathfrak{M}_{M_\lambda}(\Omega)$ also. We suppose that all functions from $\mathfrak{M}_\lambda(\Omega)$ are continuous in the closure of Ω. Let

$$E(y - x) = \frac{\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}} \frac{\overline{y - x}}{|y - x|^{n+1}} \tag{1.3.2}$$

be the Cauchy kernel [17, p. 146] and

$$d\sigma = \sum_{j=0}^{n} (-1)^j e_j dx_0 \wedge \ldots \wedge [dx_j] \wedge \ldots \wedge dx_m.$$

be the differential form of the "oriented surface element" [17, p. 144]. Then for any $f(x) \in \mathfrak{M}(\Omega)$ we have the Cauchy integral formula [17, p. 147]

$$\int_{\partial\Omega} E(y-x)\, d\sigma_y\, f(y) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}.$$

Theorem 1.3.3 (Cauchy's Theorem) *Let $f(y) \in \mathfrak{M}_M(\Omega)$. Then*

$$\int_{\partial\Omega} d\sigma_y\, e^{-y_0 M} f(y) = 0.$$

Particularly, for $f(y) \in \mathfrak{M}_\lambda(\Omega)$ we have

$$\int_{\partial\Omega} d\sigma_y\, f(y) e^{-y_0\lambda} = 0,$$

and

$$\int_{\partial\Omega} d\sigma_y e^{-y_0\lambda}\, f(y) = 0,$$

if $\lambda \in \mathbb{R}$.

Theorem 1.3.4 (Cauchy's Integral Formula) *Let $f(y) \in \mathfrak{M}_M(\Omega)$. Then*

$$e^{x_0 M} \int_{\partial\Omega} E(y-x)\, d\sigma_y\, e^{-y_0 M} f(y) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}. \tag{1.3.3}$$

Particularly, for $f(y) \in \mathfrak{M}_\lambda(\Omega)$ we have

$$\int_{\partial\Omega} E(y-x)\, d\sigma_y\, f(y) e^{(x_0-y_0)\lambda} = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}.$$

and

$$\int_{\partial\Omega} E(y-x) e^{(x_0-y_0)\lambda}\, d\sigma_y\, f(y) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}.$$

if $\lambda \in \mathbb{R}$.

It is hard to expect that formula (1.3.3) may be rewritten as

$$\int_{\partial\Omega} E'(y-x)\, d\sigma_y\, f(y) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}$$

with a simple function $E'(y-x)$.

Because an application of the bounded operator $e^{y_0 M}$ does not destroy uniform convergency of functions we obtain (cf. [17, Chap. II, § 0.2.2, Theorem 2])

Theorem 1.3.5 (Weierstrass' Theorem) *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{M}_M(\Omega)$, which converges uniformly to f on each compact subset $K \in \Omega$. Then*

(i). $f \in \mathfrak{M}_M(\Omega)$.

(ii). For each multi-index $\beta = (\beta_0, \dots, \beta_m) \in \mathbb{N}^{n+1}$, the sequence $\{\partial^\beta f_k\}_{k\in\mathbb{N}}$ converges uniformly on each compact subset $K \in \Omega$ to $\partial^\beta f$.

Theorem 1.3.6 (Mean Value Theorem) *Let $f \in \mathfrak{M}_M(\Omega)$. Then for all $x \in \Omega$ and $R > 0$ such that the ball $\mathbb{B}(x, R) \in \Omega$,*

$$f(x) = e^{x_0 M} \frac{(n+1)\Gamma(\frac{n+1}{2})}{2R^{n+1}\pi^{(n+1)/2}} \int_{\mathbb{B}(x,R)} e^{-y_0 M} f(y)\, dy.$$

Such a reduction of theories could be pushed even future [33] up to the notion of hypercomplex differentiability [55], but we will stop here.

1.4 Hypercomplex Analysis and Group Representations — Towards a Classification

To construct a classification of non-equivalent objects one could use their groups of symmetries. Classical example is Poincaré's proof of bi-holomorphic non-equivalence of the unit ball and polydisk via comparison their groups of bi-holomorphic automorphisms. To employ this approach we need a construction of hypercomplex analysis from its symmetry group. The following scheme will be main theme of this Course.

Let G be a group which acts via transformation of a closed domain $\bar{\Omega}$. Moreover, let $G : \partial\Omega \to \partial\Omega$ and G act on Ω and $\partial\Omega$ transitively. Let us fix a point $x_0 \in \Omega$ and let $H \subset G$ be a stationary subgroup of point x_0. Then domain Ω is naturally identified with the homogeneous space G/H. Till the moment we do not request anything untypical. Now let

- *there exist a H-invariant measure $d\mu$ on $\partial\Omega$.*

We consider the Hilbert space $L_2(\partial\Omega, d\mu)$. Then geometrical transformations of $\partial\Omega$ give us the representation π of G in $L_2(\partial\Omega, d\mu)$. Let $f_0(x) \equiv 1$ and $F_2(\partial\Omega, d\mu)$ be the closed liner subspace of $L_2(\partial\Omega, d\mu)$ with the properties:

(i). $f_0 \in F_2(\partial\Omega, d\mu)$;

(ii). $F_2(\partial\Omega, d\mu)$ is G-invariant;

(iii). $F_2(\partial\Omega, d\mu)$ is G-irreducible, or f_0 is cyclic in $F_2(\partial\Omega, d\mu)$.

The *standard wavelet transform* W is defined by

$$W : F_2(\partial\Omega, d\mu) \to L_2(G) : f(x) \mapsto \widehat{f}(g) = \langle f(x), \pi(g) f_0(x) \rangle_{L_2(\partial\Omega, d\mu)}$$

Due to the property $[\pi(h) f_0](x) = f_0(x)$, $h \in H$ and identification $\Omega \sim G/H$ it could be translated to the embedding:

$$\widetilde{W} : F_2(\partial\Omega, d\mu) \to L_2(\Omega) : f(x) \mapsto \widehat{f}(y) = \langle f(x), \pi(g) f_0(x) \rangle_{L_2(\partial\Omega, d\mu)}, \tag{1.4.1}$$

where $y \in \Omega$ for some $h \in H$. The imbedding (1.4.1) is *an abstract analog of the Cauchy integral formula.* Let functions V_α be the *special functions* generated by the representation of H. Then the decomposition of $\widehat{f_0}(y)$ by V_α gives us the Taylor series.

The scheme is inspired by the following interpretation of complex analysis.

Example 1.4.1 Let the domain Ω be the unit disk $\mathbb{D}$, $\partial\mathbb{D} = \mathbb{S}$. We select the group $SL(2, \mathbb{R}) \sim SU(1, 1)$ acting on $\mathbb{D}$ via the fractional-linear transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

We fix $x_0 = 0$. Then its stationary group is $U(1)$ of rotations of $\mathbb{D}$. Then the Lebesgue measure on $\mathbb{S}$ is $U(1)$-invariant. We obtain $\mathbb{D} \sim SL(2, \mathbb{R})/U(1)$. The subspace of $L_2(\mathbb{S}, dt)$ satisfying to 1.4.0.(i)–1.4.0.(iii) is the Hardy space. The wavelets transform(1.4.1) give exactly the Cauchy formula. The proper functions of $U(1)$ are exactly z^n, which provide the base for the Taylor series. The Riemann mapping theorem allows to apply the scheme to any connected, simply-connected domain.

The conformal group of the Möbius transformations plays the same role in Clifford analysis. One usually says that the conformal group in $\mathbb{R}^n$, $n > 2$ is not so rich as the conformal group in $\mathbb{R}^2$. Nevertheless, the conformal covariance has many applications in Clifford analysis [11, 65]. Notably, groups of conformal mappings of open unit balls $\mathbb{B}^n \subset \mathbb{R}^n$ *on*to itself are similar for all n and as sets can be parametrized by the product of $\mathbb{B}^n$ itself and the group of isometries of its boundary $\mathbb{S}^{n-1}$.

Theorem 1.4.2 *[36] Let $a \in \mathbb{B}^n$, $b \in \Gamma_n$ then the Möbius transformations of the form*

$$\phi_{(a,b)} = \begin{pmatrix} b & 0 \\ 0 & b^{*-1} \end{pmatrix} \begin{pmatrix} 1 & -a \\ a^* & -1 \end{pmatrix} = \begin{pmatrix} b & -ba \\ b^{*-1}a^* & -b^{*-1} \end{pmatrix},$$

constitute the group B_n of conformal mappings of the open unit ball $\mathbb{B}^n$ onto itself. B_n acts on $\mathbb{B}^n$ transitively. Transformations of the form $\phi_{(0,b)}$ constitute a subgroup isomorphic to $\mathrm{O}(n)$. The homogeneous space $B_n/\mathrm{O}(n)$ is isomorphic as a set to $\mathbb{B}^n$. Moreover:

(i). $\phi^2_{(a,1)} = 1$ identically on $\mathbb{B}^n$ ($\phi^{-1}_{(a,1)} = \phi_{(a,1)}$).

(ii). $\phi_{(a,1)}(0) = a$, $\phi_{(a,1)}(a) = 0$.

Obviously, conformal mappings preserve the space of null solutions to the *Laplace operator* (1.2.1) and null solutions the *Dirac operator* (1.2.2). The group B_n is sufficient for construction of the Poisson and the Cauchy integral representation of harmonic functions and Szegö and Bergman projections in Clifford analysis by the formula [35]

$$K(x,y) = c \int_G [\pi_g f](x) \overline{[\pi_g f](y)}\, dg, \tag{1.4.2}$$

where π_g is an irreducible unitary square integrable representation of a group G, $f(x)$ is an arbitrary non-zero function, and c is a constant.

The scheme gives a correspondence between *function theories* and *group representations*. The last are rather well studded and thus such a connection could be a foundation for a classification of function theories. Particularly, the *constant coefficient* function theories in the sense of F. Sommen[75] corresponds to the groups acting only on the function domains in the Euclidean space. Between such groups the Moebius transformations play the leading role. On the contrary, the *variable coefficient* case is described by groups acting on the function space in the non-point sense (for example, combining action on the functions domain and range, see [40]). The set of groups of the second kind should be more profound.

Remark 1.4.3 It is known that many results in real analysis [56] several variables theory [58] could be obtained or even explained via hypercomplex analysis. One could see roots of this phenomenon in relationships between groups of geometric symmetries of two theories: the group of hypercomplex analysis is wider.

Returning to our metaphor on the Mendeleev table we would like recall that it began as linear ordering with respect to atomic masses but have received an explanation only via representation theory of the rotation group.

Lecture 2

Group Representations, Wavelets and Analytic Spaces of Functions

2.1 Introduction

The purpose of this Lecture to introduce the appropriate language of *coherent states* and *wavelet transform*. We suppose some knowledge about groups and their representations. The appropriate material is included in Appendix A and B. We will begin from the standard constructions of coherent states (wavelets) in a Hilbert space (section 2.2) and then will construct an appropriate generalization for Banach spaces (section 2.3).

Wavelet transform considered here is an important example of the interesting object called *token*n [31]. Tokens are kernels of intertwining operators between actions of two cancellative semigroups.

2.2 Wavelets in Hilbert Spaces

2.2.1 Wavelet Transform and Coherent States

We agree with a reader if he/she is not satisfied by the last short proof and would like to see a more detailed account how the core of complex analysis could be reconstructed from representation theory of $SL_2(\mathbb{R})$. We present an abstract scheme, which also could be applied to other analytic function theories, see last two lectures and [13, 39]. We start from a dry construction followed in the next Section by classic examples, which will justify our usage of personal names.

Let X be a topological space and let G be a group that acts $G : X \rightarrow X$ as a transformation $g : x \mapsto g \cdot x$ from the left, i.e., $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$. Moreover, let G act on X transitively. Let there exist a measure dx on X such that a representation $\pi(g) : f(x) \mapsto m(g, x) f(g^{-1} \cdot x)$ (with a function $m(g, h)$) is unitary with respect to the scalar product $\langle f_1(x), f_2(x) \rangle_{L_2(X)} = \int_X f_1(x) \bar{f}_2(x)\, d(x)$, i.e.,

$$\langle [\pi(g) f_1](x), [\pi(g) f_2](x) \rangle_{L_2(X)} = \langle f_1(x), f_2(x) \rangle_{L_2(X)} \quad \forall f_1, f_2 \in L_2(X).$$

We consider the Hilbert space $L_2(X)$ where representation $\pi(g)$ acts by unitary operators.

Remark 2.2.1 It is well known that the most developed part of representation theory consider unitary representations in Hilbert spaces. By this reason we restrict our attention to Hilbert spaces of analytic functions, the area usually done by means of the functional analysis technique. We also assume that our functions are complex valued and this is sufficient for examples explicitly considered in the present paper. However the presented scheme is working also for vector valued functions and this is the natural environment for Clifford analysis [9], for example. One also could start from an abstract Hilbert space H with no explicit realization as $L_2(X)$ given.

Let H be a closed compact[1] subgroup of G and let $f_0(x)$ be such a function that H acts on it as the multiplication

$$[\pi(h) f_0](x) = \chi(h) f_0(x) \qquad , \forall h \in H, \tag{2.2.1}$$

by a function $\chi(h)$, which is a character of H i.e., $f_0(0)$ is a common eigenfunction for all operators $\pi(h)$. Equivalently $f_0(x)$ is a common eigenfunction for operators corresponding under π to a basis of the Lie algebra of H. Note also that $|\chi(h)|^2 = 1$ because π is unitary. $f_0(x)$ is called *vacuum vector* (with respect to subgroup H). We introduce the $F_2(X)$ to be the closed liner subspace of $L_2(X)$ uniquely defined by the conditions:

(i). $f_0 \in F_2(X)$;

(ii). $F_2(X)$ is G-invariant;

(iii). $F_2(X)$ is G-irreducible, or f_0 is cyclic in $F_2(\partial\Omega, d\mu)$.

[1] While the compactness will be explicitly used during our abstract consideration, it is not crucial in fact. One could make a trick for non-compact H [42].

Thus restriction of π on $F_2(X)$ is an irreducible unitary representation.

The *wavelet transform*[2] $\mathcal{W}$ could be defined for square-integral representations π by the formula

$$\begin{aligned} \mathcal{W} &: F_2(X) \rightarrow L_\infty(G) \\ &: f(x) \mapsto \widetilde{f}(g) = \langle f(x), \pi(g) f_0(x) \rangle_{L_2(X)} \end{aligned} \tag{2.2.2}$$

The principal advantage of the wavelet transform $\mathcal{W}$ is that it express the representation π in geometrical terms. Namely it *intertwins* π and left regular representation λ on G:

$$[\lambda_g \mathcal{W} f](g') = [\mathcal{W} f](g^{-1} g') = \langle f, \pi_{g^{-1}g'} f_0 \rangle = \langle \pi_g f, \pi_{g'} f_0 \rangle = [\mathcal{W} \pi_g f](g'), \tag{2.2.3}$$

i.e., $\lambda \mathcal{W} = \mathcal{W} \pi$. Another important feature of W is that it does not lose information, namely function $f(x)$ could *be recovered* as the linear combination of *coherent states* $f_g(x) = [\pi_g f_0](x)$ from its wavelet transform $\widetilde{f}(g)$:

$$f(x) = \int_G \widetilde{f}(g) f_g(x)\, dg = \int_G \widetilde{f}(g) [\pi_g f_0](x)\, dg, \tag{2.2.4}$$

where dg is the Haar measure on G normalized such that

$$\int_G \left| \widetilde{f}_0(g) \right|^2 dg = 1.$$

One also has an orthogonal *projection* $\widetilde{\mathcal{P}}$ from $L_2(G, dg)$ to image $F_2(G, dg)$ of $F_2(X)$ under wavelet transform $\mathcal{W}$, which is just a convolution on g with the image $\widetilde{f}_0(g) = \mathcal{W}(f_0(x))$ of the vacuum vector:

$$[\widetilde{\mathcal{P}} w](g') = \int_G w(g) \widetilde{f}_0(g^{-1} g')\, dg. \tag{2.2.5}$$

2.2.2 Reduced Wavelets Transform

Our main observation will be that one could be much more economical (if subgroup H is non-trivial) with a help of (2.2.1): in this case one need to know $\widetilde{f}(g)$ not on the whole group G but only on the homogeneous space G/H [1, § 3].

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[2] The subject of coherent states or wavelets have been arising many times in many *applied* areas and the author is not able to give a comprehensive history and proper credits. One could mention important books [16, 44, 61]. We give our references by recent paper [34], where applications to *pure* mathematics were considered.

Let $\Omega = G/H$ and $s : \Omega \to G$ be a continuous mapping [29, § 13.1]. Then any $g \in G$ has a unique decomposition of the form $g = s(a)h$, $a \in \Omega$ and we will write $a = s^{-1}(g)$, $h = r(g) = (s^{-1}(g))^{-1}g$. Note that Ω is a left G-homogeneous space[3] with an action defined in terms of s as follow: $g : a \mapsto s^{-1}(g \cdot s(a))$. Due to (2.2.1) one could rewrite (2.2.2) as:

$$\begin{aligned}
\widetilde{f}(g) &= \langle f(x), \pi(g)f_0(x)\rangle_{L_2(X)}\\
&= \langle f(x), \pi(s(a)h)f_0(x)\rangle_{L_2(X)}\\
&= \langle f(x), \pi(s(a))\pi(h)f_0(x)\rangle_{L_2(X)}\\
&= \langle f(x), \pi(s(a))\chi(h)f_0(x)\rangle_{L_2(X)}\\
&= \bar{\chi}(h)\,\langle f(x), \pi(s(a))f_0(x)\rangle_{L_2(X)}
\end{aligned}$$

Thus $\widetilde{f}(g) = \bar{\chi}(h)\widehat{f}(a)$ where

$$\widehat{f}(a) = [\mathcal{C}f](a) = \langle f(x), \pi(s(a))f_0(x)\rangle_{L_2(X)} \tag{2.2.6}$$

and function $\widetilde{f}(g)$ on G is completely defined by function $\widehat{f}(a)$ on Ω. Formula (2.2.6) gives us an embedding $\mathcal{C} : F_2(X) \to L_\infty(\Omega)$, which we will call *reduced wavelet transform*. We denote by $F_2(\Omega)$ the image of $\mathcal{C}$ equipped with Hilbert space inner product induced by $\mathcal{C}$ from $F_2(X)$.

Note a special property of $\widetilde{f}_0(g)$ and $\widehat{f}_0(a)$:

$$\widetilde{f}_0(h^{-1}g) = \langle f_0, \pi_{h^{-1}g}f_0\rangle = \langle \pi_h f_0, \pi_g f_0\rangle = \langle \chi(h)f_0, \pi_g f_0\rangle = \chi(h)\widetilde{f}_0(g).$$

It follows from (2.2.3) that $\mathcal{C}$ intertwines $\rho\mathcal{C} = \mathcal{C}\pi$ representation π with the representation

$$[\rho_g \widehat{f}](a) = \widehat{f}(s^{-1}(g \cdot s(a)))\chi(r(g \cdot s(a))). \tag{2.2.7}$$

While ρ is not completely geometrical as λ in applications it is still more geometrical than original π. In many cases ρ is *representation induced* by the character χ.

If $f_0(x)$ is a vacuum state with respect to H then $f_g(x) = \chi(h)f_{s(a)}(x)$ and we could rewrite (2.2.4) as follows:

$$f(x) = \int_G \widetilde{f}(g)f_g(x)\,dg$$

[3] Ω with binary operation $(a_1, a_2) \mapsto s^{-1}(s(a_1)\cdot s(a_2))$ becomes a loop of the most general form [68]. Thus theory of reduced wavelet transform developed in this subsection could be considered as *wavelet transform associated with loops*. However we prefer to develop our theory based on groups rather on loops.

$$\begin{aligned}
&= \int_\Omega \int_H \widetilde{f}(s(a)h) f_{s(a)h}(x)\, dh\, da \\
&= \int_\Omega \int_H \widehat{f}(a)\bar{\chi}(h)\chi(h) f_{s(a)}(x)\, dh\, da \\
&= \int_\Omega \widehat{f}(a) f_{s(a)}(x)\, da \cdot \int_H |\chi(h)|^2\, dh \\
&= \int_\Omega \widehat{f}(a) f_{s(a)}(x)\, da,
\end{aligned}$$

if the Haar measure dh on H is set in such a way that $\int_H |\chi(h)|^2\, dh = 1$ and $dg = dh\, da$. We define an integral transformation $\mathcal{F}$ according to the last formula:

$$[\mathcal{F}\widehat{f}](x) = \int_\Omega \widehat{f}(a) f_{s(a)}(x)\, da, \tag{2.2.8}$$

which has the property $\mathcal{F}\mathcal{C} = I$ on $F_2(X)$ with $\mathcal{C}$ defined in (2.2.6). One could consider the integral transformation

$$[\mathcal{P}f](x) = [\mathcal{F}\mathcal{C}f](x) = \int_\Omega \left\langle f(y), f_{s(a)}(y)\right\rangle_{L_2(X)} f_{s(a)}(x)\, da \tag{2.2.9}$$

as defined on whole $L_2(X)$ (not only $F_2(X)$). It is known that $\mathcal{P}$ *is an orthogonal projection* $L_2(X) \rightarrow F_2(X)$. If we formally use linearity of the scalar product $\langle\cdot,\cdot\rangle_{L_2(X)}$ (i.e., assume that the Fubini theorem holds) we could obtain from (2.2.9)

$$\begin{aligned}
[\mathcal{P}f](x) &= \int_\Omega \left\langle f(y), f_{s(a)}(y)\right\rangle_{L_2(X)} f_{s(a)}(x)\, da \\
&= \left\langle f(y), \int_\Omega f_{s(a)}(y)\bar{f}_{s(a)}(x)\, da \right\rangle_{L_2(X)} \\
&= \int_X f(y) K(y,x)\, d\mu(y), \qquad (2.2.10)
\end{aligned}$$

where

$$K(y,x) = \int_\Omega \bar{f}_{s(a)}(y) f_{s(a)}(x)\, da$$

With the "probability $\frac{1}{2}$" (see discussion on the Bergman and the Szegö kernels bellow) the integral (2.2.10) exists in the standard sense, otherwise it is a singular integral operator (i.e, $K(y,x)$ is a regular function or a distribution).

Sometimes a reduced form $\widehat{\mathcal{P}} : L_2(\Omega) \rightarrow F_2(\Omega)$ of the projection $\widetilde{\mathcal{P}}$ (2.2.5) is of a separate interest. It is an easy calculation that

$$[\widehat{\mathcal{P}}f](a') = \int_\Omega f(a)\widehat{f}_0(s^{-1}(a^{-1}\cdot a'))\bar{\chi}(r(a^{-1}\cdot a'))\, da, \tag{2.2.11}$$

where $a^{-1} \cdot a'$ is an informal abbreviation for $(s(a))^{-1} \cdot s(a')$. As we will see its explicit form could be easily calculated in practical cases.

And only at the very end of our consideration we introduce the Taylor series and the Cauchy-Riemann equations. One knows that they are *starting points* in the Weierstrass and the Cauchy approaches to complex analysis correspondingly.

For any decomposition $f_a(x) = \sum_\alpha \psi_\alpha(x) V_\alpha(a)$ of the coherent states $f_a(x)$ by means of functions $V_\alpha(a)$ (where the sum could become eventually an integral) we have the *Taylor series* expansion

$$\begin{aligned} \widehat{f}(a) &= \int_X f(x)\bar{f}_a(x)\,dx = \int_X f(x)\sum_\alpha \bar{\psi}_\alpha(x)\bar{V}_\alpha(a)\,dx \\ &= \sum_\alpha \int_X f(x)\bar{\psi}_\alpha(x)\,dx \bar{V}_\alpha(a) \\ &= \sum_\alpha^\infty \bar{V}_\alpha(a) f_\alpha, \end{aligned} \tag{2.2.12}$$

where $f_\alpha = \int_X f(x)\bar{\psi}_\alpha(x)\,dx$. However to be useful within the presented scheme such a decomposition should be connected with structures of G and H. For example, if G is a semisimple Lie group and H its maximal compact subgroup then indices α run through the set of irreducible unitary representations of H, which enter to the representation π of G.

The *Cauchy-Riemann equations* need more discussion. One could observe from (2.2.3) that the image of $\mathcal{W}$ is invariant under action of the left but right regular representations. Thus $F_2(\Omega)$ is invariant under representation (2.2.7), which is a pullback of the left regular representation on G, but its right counterpart. Thus generally there is no way to define an action of left-invariant vector fields on Ω, which are infinitesimal generators of right translations, on $L_2(\Omega)$. But there is an exception. Let $\mathfrak{X}_j$ be a maximal set of left-invariant vector fields on G such that

$$\mathfrak{X}_j \widetilde{f_0}(g) = 0.$$

Because $\mathfrak{X}_j$ are left invariant we have $\mathfrak{X}_j \widetilde{f'_g}(g) = 0$ for all g' and thus image of $\mathcal{W}$, which the linear span of $\widetilde{f'_g}(g)$, belongs to intersection of kernels of $\mathfrak{X}_j$. The same remains true if we consider pullback $\widehat{\mathfrak{X}}_j$ of $\mathfrak{X}_j$ to Ω. Note that the number of linearly independent $\widehat{\mathfrak{X}}_j$ is generally less than for $\mathfrak{X}_j$. We call $\widehat{\mathfrak{X}}_j$ as *Cauchy-Riemann-Dirac operators* in connection with their property

$$\widehat{\mathfrak{X}}_j \widehat{f}(g) = 0 \qquad \forall \widehat{f}(g) \in F_2(\Omega). \tag{2.2.13}$$

Explicit constructions of the Dirac type operator for a discrete series representation could be found in [2, 45].

We do not use Cauchy-Riemann-Dirac operator in our construction, but this does not mean that it is useless. One could found at least such its nice properties:

(i). Being a left-invariant operator it naturally encodes an information about symmetry group G.

(ii). It effectively separates irreducible components of the representation π of G in $L_2(X)$.

(iii). It has a local nature in a neighborhood of a point vs. transformations, which act globally on the domain.

2.3 Wavelets in Banach Spaces

2.3.1 Abstract Nonsence

Let G be a group and H be its closed normal subgroup. Let $X = G/H$ be the corresponding homogeneous space with an invariant measure $d\mu$ and $s : X \to G$ be a Borel section in the principal bundle $G \to G/H$. Let π be a continuous representation of a group G by invertible isometry operators π_g, $g \in G$ in a (complex) Banach space B.

The following definition simulates ones from the Hilbert space case [1, § 3.1].

Definition 2.3.1 Let G, H, $X = G/H$, $s : X \to G$, $\pi : G \to \mathcal{L}(B)$ be as above. We say that $b_0 \in B$ is a *vacuum vector* if for all $h \in H$

$$\pi(h)b_0 = \chi(h)b_0, \qquad \chi(h) \in \mathbb{C}. \tag{2.3.1}$$

We will say that set of vectors $b_x = \pi(x)b_0$, $x \in X$ form a family of *coherent states* if there exists a continuous non-zero linear functional $l_0 \in B^*$ such that

(i). $\|b_0\| = 1$, $\|l_0\| = 1$, $\langle b_0, l_0\rangle \neq 0$;

(ii). $\pi(h)^* l_0 = \bar{\chi}(h) l_0$, where $\pi(h)^*$ is the adjoint operator to $\pi(h)$;

(iii). The following equality holds

$$\int_X \left\langle \pi(x^{-1})b_0, l_0\right\rangle \left\langle \pi(x)b_0, l_0\right\rangle \, d\mu(x) = \langle b_0, l_0\rangle \,. \tag{2.3.2}$$

The functional l_0 is called the *test functional*. According to the strong tradition we call the set (G, H, π, B, b_0, l_0) *admissible* if it satisfies to the above conditions.

We note that mapping $h \to \chi(h)$ from (2.3.1) defines a character of the subgroup H. The following Lemma demonstrates that condition (2.3.2) could be relaxed.

Lemma 2.3.2 *For the existence of a vacuum vector b_0 and a test functional l_0 it is sufficient that there exists a vector b'_0 and continuous linear functional l'_0 satisfying to* (2.3.1) *and* 2.3.1.(ii) *correspondingly such that the constant*

$$c = \int_X \left\langle \pi(x^{-1})b'_0, l'_0 \right\rangle \left\langle \pi(x)b'_0, l'_0 \right\rangle \, d\mu(x) \tag{2.3.3}$$

is non-zero and finite.

Proof. There exist a $x_0 \in X$ such that $\left\langle \pi(x_0^{-1})b'_0, l'_0 \right\rangle \neq 0$, otherwise one has $c = 0$. Let $b_0 = \pi(x^{-1})b'_0 \left\| \pi(x^{-1})b'_0 \right\|^{-1}$ and $l_0 = l'_0 \left\| l'_0 \right\|^{-1}$. For such b_0 and l_0 we have 2.3.1.(i) already fulfilled. To obtain (2.3.2) we change the measure $d\mu(x)$. Let $c_0 = \langle b_0, l_0 \rangle \neq 0$ then $d\mu' = \left\| \pi(x^{-1})b'_0 \right\| \left\| l'_0 \right\| c_0 c^{-1} d\mu$ is the desired measure. □

Remark 2.3.3 Conditions (2.3.2) and (2.3.3) are known for unitary representations in Hilbert spaces as *square integrability* (with respect to a subgroup H). Thus our definition describes an analog of square integrable representations for Banach spaces. Note that in Hilbert space case b_0 and l_0 are often the same function, thus condition 2.3.1.(ii) is exactly (2.3.1). In the particular but still important case of trivial $H = \{e\}$ (and thus $X = G$) all our results take simpler forms.

Convention 2.3.4 In that follow we will usually write $x \in X$ and x^{-1} instead of $s(x) \in G$ and $s(x)^{-1}$ correspondingly. The right meaning of "x" could be easily found from the context (whether an element of X or G is expected there).

The wavelet transform (similarly to the Hilbert space case) could be defined as a mapping from B to a space of bounded continuous functions over G via representational coefficients

$$v \mapsto \widehat{v}(g) = \left\langle \pi(g^{-1})v, l_0 \right\rangle = \left\langle v, \pi(g)^* l_0 \right\rangle .$$

Due to 2.3.1.(ii) such functions have simple transformation properties along orbits gH, i.e. $\widehat{v}(gh) = \bar{\chi}(h)\widehat{v}(g)$, $g \in G$, $h \in H$. Thus they are completely defined by their values indexed by points of $X = G/H$. Therefore we prefer to consider so called reduced wavelet transform.

Definition 2.3.5 The *reduced wavelet transform* $\mathcal{W}$ from a Banach space B to a space of function $F(X)$ on a homogeneous space $X = G/H$ defined by a representation π of G on B, a vacuum vector b_0 and a test functional l_0 is given by the formula

$$\mathcal{W} : B \to F(X) : v \mapsto \widehat{v}(x) = [\mathcal{W}v](x) = \left\langle \pi(x^{-1})v, l_0 \right\rangle = \left\langle v, \pi^*(x) l_0 \right\rangle . \tag{2.3.4}$$

There is a natural representation of G in $F(X)$. For any $g \in G$ there is a unique decomposition of the form $g = s(x)h$, $h \in H$, $x \in X$. We will define $r : G \to H : r(g) = h = (s^{-1}(g))^{-1} g$ from the previous equality and write a formal notation $x = s^{-1}(g)$. Then there is a geometric action of G on $X \to X$ defined as follows

$$g : x \mapsto g^{-1} \cdot x = s^{-1}(g^{-1} s(x)).$$

We define a representation $\lambda(g) : F(X) \to F(X)$ as follow

$$[\lambda(g) f](x) = \chi(r(g^{-1} \cdot x)) f(g^{-1} \cdot x). \tag{2.3.5}$$

We recall that $\chi(h)$ is a character of H defined in (2.3.1) by the vacuum vector b_0. For the case of trivial $H = \{e\}$ (2.3.5) becomes the left regular representation $\rho_l(g)$ of G.

Proposition 2.3.6 *The reduced wavelet transform $\mathcal{W}$ intertwines π and the representation λ (2.3.5) on $F(X)$:*

$$\mathcal{W}\pi(g) = \lambda(g)\mathcal{W}.$$

PROOF. We have:

$$\begin{aligned}
[\mathcal{W}(\pi(g)v)](x) &= \left\langle \pi(x^{-1})\pi(g)v, l_0 \right\rangle \\
&= \left\langle \pi((g^{-1}s(x))^{-1})v, l_0 \right\rangle \\
&= \left\langle \pi(r(g^{-1} \cdot x)^{-1})\pi(s(g^{-1} \cdot x)^{-1})v, l_0 \right\rangle \\
&= \left\langle \pi(s(g^{-1} \cdot x)^{-1})v, \pi^*(r(g^{-1} \cdot x)^{-1}) l_0 \right\rangle \\
&= \chi(r(g^{-1} \cdot x)^{-1})[\mathcal{W}v](g^{-1}x) \\
&= \lambda(g)[\mathcal{W}v](x).
\end{aligned}$$

□

Corollary 2.3.7 *The function space $F(X)$ is invariant under the representation λ of G.*

We will see that $F(X)$ posses many properties of the *Hardy space*. The duality between l_0 and b_0 generates a transform dual to $\mathcal{W}$.

Definition 2.3.8 The *inverse wavelet transform* $\mathcal{M}$ from $F(X)$ to B is given by the formula:

$$\begin{aligned} \mathcal{M} : F(X) \rightarrow B : \widehat{v}(x) \mapsto \mathcal{M}[\widehat{v}(x)] &= \int_X \widehat{v}(x) b_x \, d\mu(x) \\ &= \int_X \widehat{v}(x)\pi(x) \, d\mu(x) b_0. \end{aligned} \quad (2.3.6)$$

Proposition 2.3.9 *The inverse wavelet transform $\mathcal{M}$ intertwines the representation λ on $F(X)$ and π on B:*

$$\mathcal{M}\lambda(g) = \pi(g)\mathcal{M}.$$

PROOF. We have:

$$\begin{aligned} \mathcal{M}[\lambda(g)\widehat{v}(x)] &= \mathcal{M}[\chi(r(g^{-1}\cdot x))\widehat{v}(g^{-1}\cdot x)] \\ &= \int_X \chi(r(g^{-1}\cdot x))\widehat{v}(g^{-1}\cdot x) b_x \, d\mu(x) \\ &= \chi(r(g^{-1}\cdot x)) \int_X \widehat{v}(x') b_{g\cdot x'} \, d\mu(x') \\ &= \pi_g \int_X \widehat{v}(x') b_{x'} \, d\mu(x') \\ &= \pi_g \mathcal{M}[\widehat{v}(x')], \end{aligned}$$

where $x' = g^{-1} \cdot x$. $\square$

Corollary 2.3.10 *The image $\mathcal{M}(F(X)) \subset B$ of subspace $F(X)$ under the inverse wavelet transform $\mathcal{M}$ is invariant under the representation π.*

The following proposition explain the usage of the name for $\mathcal{M}$.

Theorem 2.3.11 *The operator*

$$\mathcal{P} = \mathcal{M}\mathcal{W} : B \rightarrow B \quad (2.3.7)$$

is a projection of B to its linear subspace for which b_0 is cyclic. Particularly if π is an irreducible representation then the inverse wavelet transform $\mathcal{M}$ is a left inverse operator *on B for the wavelet transform $\mathcal{W}$:*

$$\mathcal{M}\mathcal{W} = I.$$

PROOF. It follows from Propositions 2.3.6 and 2.3.9 that operator $\mathcal{MW}: B \to B$ intertwines π with itself. Then Corollaries 2.3.7 and 2.3.10 imply that the image $\mathcal{MW}$ is a π-invariant subspace of B containing b_0. Because $\mathcal{MW}b_0 = b_0$ we conclude that $\mathcal{MW}$ is a projection.

From irreducibility of π by Schur's Lemma [29, § 8.2] one concludes that $\mathcal{MW} = cI$ on B for a constant $c \in \mathbb{C}$. Particularly

$$\mathcal{MW}b_0 = \int_X \left\langle \pi(x^{-1})b_0, l_0 \right\rangle \pi(x)b_0 \, d\mu(x) = cb_0.$$

From the condition (2.3.2) it follows that $\langle cb_0, l_0 \rangle = \langle \mathcal{MW}b_0, l_0 \rangle = \langle b_0, l_0 \rangle$ and therefore $c = 1$. $\square$

We have similar

Theorem 2.3.12 *Operator $\mathcal{WM}$ is a projection of $L_1(X)$ to $F(X)$.*

We denote by $\mathcal{W}^* : F^*(X) \to B^*$ and $\mathcal{M}^* : B^* \to F^*(X)$ the adjoint (in the standard sense) operators to $\mathcal{W}$ and $\mathcal{M}$ respectively.

Corollary 2.3.13 *We have the following identity:*

$$\langle \mathcal{W}v, \mathcal{M}^* l \rangle_{F(X)} = \langle v, l \rangle_B , \qquad \forall v \in B, \quad l \in B^* \tag{2.3.8}$$

or equivalently

$$\int_X \left\langle \pi(x^{-1})v, l_0 \right\rangle \left\langle \pi(x)b_0, l \right\rangle \, d\mu(x) = \langle v, l \rangle . \tag{2.3.9}$$

PROOF. We show the equality in the first form (2.3.9) (but will apply it often in the second one):

$$\langle \mathcal{W}v, \mathcal{M}^* l \rangle_{F(X)} = \langle \mathcal{MW}v, l \rangle_B = \langle v, l \rangle_B .$$

$\square$

Corollary 2.3.14 *The space $F(X)$ has the reproducing formula*

$$\widehat{v}(y) = \int_X \widehat{v}(x) \, \widehat{b}_0(x^{-1} \cdot y) \, d\mu(x), \tag{2.3.10}$$

where $\widehat{b}_0(y) = [\mathcal{W}b_0](y)$ is the wavelet transform of the vacuum vector b_0.

PROOF. Again we have a simple application of the previous formulas:

$$\begin{aligned}
\widehat{v}(y) &= \left\langle \pi(y^{-1})v, l_0 \right\rangle \\
&= \int_X \left\langle \pi(x^{-1})\pi(y^{-1})v, l_0 \right\rangle \left\langle \pi(x)b_0, l_0 \right\rangle \, d\mu(x) \qquad (2.3.11)\\
&= \int_X \left\langle \pi(s(y\cdot x)^{-1})v, l_0 \right\rangle \left\langle \pi(x)b_0, l_0 \right\rangle \, d\mu(x) \\
&= \int_X \widehat{v}(y\cdot x)\, \widehat{b}_0(x^{-1})\, d\mu(x) \\
&= \int_X \widehat{v}(x)\, \widehat{b}_0(x^{-1}y)\, d\mu(x),
\end{aligned}$$

where transformation (2.3.11) is due to (2.3.9). □

REMARK 2.3.15 To possess a reproducing kernel—is a well-known property of spaces of analytic functions. The space $F(X)$ shares also another important property of analytic functions: it belongs to a kernel of a certain first order differential operator with Clifford coefficients (the Dirac operator) and a second order operator with scalar coefficients (the Laplace operator) [2, 41, 39, 45].

Let us now assume that there are two representations π' and π'' of the same group G in two different spaces B' and B'' such that two admissible sets $(G, H, \pi', B', b'_0, l'_0)$ and $(G, H, \pi'', B'', b''_0, l''_0)$ could be constructed for the same normal subgroup $H \subset G$.

Proposition 2.3.16 *In the above situation if $F'(X) \subset F''(X)$ then the composition $\mathcal{T} = \mathcal{M}''\mathcal{W}'$ of the wavelet transform $\mathcal{W}'$ for π' and the inverse wavelet transform $\mathcal{M}''$ for π'' is an intertwining operator between π' and π'':*

$$\mathcal{T}\pi' = \pi''\mathcal{T}.$$

$\mathcal{T}$ is defined as follows

$$\mathcal{T} : b \mapsto \int_X \left\langle \pi'(x^{-1})b, l'_0 \right\rangle \pi''(x) b''_0 \, d\mu(x). \qquad (2.3.12)$$

This transformation defines a B''-valued linear functional (a distribution for function spaces) on B'.

The Proposition has an obvious proof. This simple result is a base for an alternative approach to functional calculus of operators [36, 41, 43]. Note also that formulas (2.3.4) and (2.3.6) are particular cases of (2.3.12) because $\mathcal{W}$ and $\mathcal{M}$ intertwine π and λ.

2.3.2 Singular Vacuum Vectors

In many important cases the above general scheme could not be carried out because the representation π of G is not square-integrable or even not square-integrable modulo a subgroup H. Thereafter the vacuum vector b_0 could not be selected within the original space B which the representation π acts on. The simplest mathematical example is the Fourier transform (see[42]). In physics this is the well-known problem of *absence of vacuum state* in the constructive algebraic *quantum field theory* [71, 72, 73]. The absence of the vacuum within the linear space of system's states is another illustration to the old thesis *Natura abhorret vacuum*[4] or even more specifically *Natura abhorret vectorem vacui*[5].

We will present a modification of our construction which works in such a situation. For a singular vacuum vector the algebraic structure of group representations could not describe the situation alone and requires an essential assistance from analytical structures.

Definition 2.3.17 Let G, H, $X = G/H$, $s : X \to G$, $\pi : G \to \mathcal{L}(B)$ be as in Definition 2.3.1. We assume that there exist a topological linear space $\widehat{B} \supset B$ such that

(i). B is dense in $\widehat{B}$ (in topology of $\widehat{B}$) and representation π could be uniquely extended to the continuous representation $\widehat{\pi}$ on $\widehat{B}$.

(ii). There exists $b_0 \in \widehat{B}$ be such that for all $h \in H$

$$\widehat{\pi}(h)b_0 = \chi(h)b_0, \qquad \chi(h) \in \mathbb{C}. \tag{2.3.13}$$

(iii). There exists a continuous non-zero linear functional $l_0 \in B^*$ such that $\pi(h)^* l_0 = \bar{\chi}(h) l_0$, where $\pi(h)^*$ is the adjoint operator to $\pi(h)$;

(iv). The composition $\mathcal{MW} : B \to \widehat{B}$ of the wavelet transform (2.3.4) and the inverse wavelet transform (2.3.6) maps B to B.

(v). For a vector $p_0 \in B$ the following equality holds

$$\left\langle \int_X \left\langle \pi(x^{-1})p_0, l_0 \right\rangle \pi(x) b_0 \, d\mu(x), l_0 \right\rangle = \left\langle p_0, l_0 \right\rangle, \tag{2.3.14}$$

where the integral converges in the weak topology of $\widehat{B}$.

[4] Nature is horrified by (any) vacuum (Lat.).

[5] Nature is horrified by a carrier of nothingness (Lat.). This illustrates how far a humane beings deviated from Nature.

As before we call the set of vectors $b_x = \pi(x)b_0$, $x \in X$ by *coherent states*; the vector b_0—a *vacuum vector*; the functional l_0 is called the *test functional* and finally p_0 is the *probe vector*.

This Definition is more complicated than Definition 2.3.1. The equation (2.3.14) is a substitution for (2.3.2) if the linear functional l_0 is not continuous in the topology of $\widehat{B}$. The function theory in $\mathbb{R}^{1,1}$ constructed in the next lecture provides a more exotic example of a singular vacuum vector.

We shall show that 2.3.17.(v) could be satisfied by an adjustment of other components.

Lemma 2.3.18 *For the existence of a vacuum vector b_0, a test functional l_0, and a probe vector p_0 it is sufficient that there exists a vector b_0' and continuous linear functional l_0' satisfying to 2.3.17.(i)–2.3.17.(iv) and a vector $p_0' \in B$ such that the constant*

$$c = \left\langle \int_X \left\langle \pi(x^{-1})p_0, l_0 \right\rangle \pi(x)b_0 \, d\mu(x), l_0 \right\rangle$$

is non-zero and finite.

The proof follows the path for Lemma 2.3.2. The following Proposition summarizes results which could be obtained in this case.

Proposition 2.3.19 *Let the wavelet transform $\mathcal{W}$ (2.3.4), its inverse $\mathcal{M}$ (2.3.6), the representation $\lambda(g)$ (2.3.5), and functional space $F(X)$ be adjusted accordingly to Definition 2.3.17. Then*

(i). *$\mathcal{W}$ intertwines $\pi(g)$ and $\lambda(g)$ and the image of $F(X) = \mathcal{W}(B)$ is invariant under $\lambda(g)$.*

(ii). *$\mathcal{M}$ intertwines $\lambda(g)$ and $\widehat{\pi}(g)$ and the image of $\mathcal{M}(F(B)) = \mathcal{M}\mathcal{W}(B) \subset B$ is invariant under $\pi(g)$.*

(iii). *If $\mathcal{M}(F(X)) = B$ (particularly if $\pi(g)$ is irreducible) then $\mathcal{M}\mathcal{W} = I$ otherwise $\mathcal{M}\mathcal{W}$ is a projection $B \to \mathcal{M}(F(X))$. In both cases $\mathcal{M}\mathcal{W}$ is an operator defined by integral*

$$b \mapsto \int_X \left\langle \pi(x^{-1})b, l_0 \right\rangle \pi(x)b_0 \, d\mu(x), \tag{2.3.15}$$

(iv). *Space $F(X)$ has a reproducing formula*

$$\widehat{v}(y) = \left\langle \int_X \widehat{v}(x) \, \pi(x^{-1}y)b_0 \, dx, l_0 \right\rangle \tag{2.3.16}$$

which could be rewritten as a singular convolution

$$\widehat{v}(y) = \int_X \widehat{v}(x)\, \widehat{b}(x^{-1}y)\, dx$$

with a distribution $b(y) = \langle \pi(y^{-1})b_0, l_0 \rangle$ *defined by* (2.3.16).

The proof is algebraic and completely similar to Subsection 2.3.1.

Lecture 3

Analytical Function Theory of Hyperbolic Type

3.1 Introduction

> You should complete your own original research in order to learn when it was done before.

Connections between complex analysis (one variable, several complex variables, Clifford analysis) and its symmetry groups are known from its earliest days. They are an obligatory part of the textbook on the subject [11], [17], [23, § 1.4, § 5.4], [47], [62, Chap. 2] and play an essential role in many research papers [60, 64, 67] just to mention only few. However ideas about fundamental role of symmetries in function theories outlined in [22, 46] were not incorporated in a working toolkit of researchers yet.

It was proposed in the first lecture to distinguish essentially different function theories by corresponding group of symmetries. Such a classification is needed because not all seemingly different function theories are essentially different, see the first lecture and [33]. But it is also important that the group approach gives a constructive way to develop essentially different function theories (see the last two lectures [38, 41, 43]), as well as outlines an alternative ground for functional calculi of operators [36]. In the mentioned papers all given examples consider only well-known function theories. While rearranging of known results is not completely useless there was an appeal to produce a new function theory based on the described scheme.

The theorem proved in [37] underlines the similarity between structure of the group of Möbius transformations in spaces $\mathbb{R}^n$ and $\mathbb{R}^{pq}$. This generates a hope that there exists a non empty function theory in $\mathbb{R}^{pq}$. We construct such a theory in the present lecture for the case of $\mathbb{R}^{1,1}$. Other new function theories based on the same scheme will be described in the next lecture [13].

The format of the lecture is as follows. In Section 3.2 we introduce basic notations and definitions. We construct two function theories—the standard complex analysis and a function theory in $\mathbb{R}^{1,1}$—in Section 3.3. Our consideration is based on two different series of representation of $SL_2(\mathbb{R})$: discrete and principal. We deduce in their terms the Cauchy integral formula, the Hardy spaces, the Cauchy-Riemann equation, the Taylor expansion and their counterparts for $\mathbb{R}^{1,1}$. Finally we collect in Appendices A and B several facts, which we would like (however can not) to assume well known. It may be a good idea to look through the Appendixes C between the reading of Sections 3.2 and 3.3. Finally Appendix 3.4 states few among many open problems. Our examples will be rather lengthy thus their (not always obvious) ends will be indicated by the symbol $\lozenge$.

3.2 Preliminaries

Let $\mathbb{R}^{pq}$ be a real n-dimensional vector space, where $n = p + q$ with a fixed frame e_1, e_2, ..., e_p, e_{p+1}, ..., e_n and with the nondegenerate bilinear form $B(\cdot,\cdot)$ of the signature (p,q), which is diagonal in the frame e_i, i.e.:

$$B(e_i, e_j) = \epsilon_i \delta_{ij}, \text{ where } \epsilon_i = \begin{cases} 1, & i = 1, \dots, p \\ -1, & i = p+1, \dots, n \end{cases}$$

and δ_{ij} is the Kronecker delta. In particular the usual Euclidean space $\mathbb{R}^n$ is $\mathbb{R}^{0n}$. Let $\mathcal{C}\ell(p,q)$ be the *real Clifford algebra* generated by 1, e_j, $1 \leq j \leq n$ and the relations

$$e_i e_j + e_j e_i = -2B(e_i, e_j).$$

We put $e_0 = 1$ also. Then there is the natural embedding $\mathfrak{i} : \mathbb{R}^{pq} \rightarrow \mathcal{C}\ell(p,q)$. We identify $\mathbb{R}^{pq}$ with its image under $\mathfrak{i}$ and call its elements *vectors*. There are two linear anti-automorphisms $*$ (reversion) and $-$ (main anti-automorphisms) and automorphism $'$ of $\mathcal{C}\ell(p,q)$ defined on its basis $A_\nu = e_{j_1} e_{j_2} \cdots e_{j_r}$, $1 \leq j_1 < \cdots < j_r \leq n$ by the rule:

$$(A_\nu)^* = (-1)^{\frac{r(r-1)}{2}} A_\nu, \qquad \bar{A}_\nu = (-1)^{\frac{r(r+1)}{2}} A_\nu, \qquad A'_\nu = (-1)^r A_\nu.$$

In particular, for vectors, $\bar{\mathbf{x}} = \mathbf{x}' = -\mathbf{x}$ and $\mathbf{x}^* = \mathbf{x}$.

It is easy to see that $\mathbf{x}\mathbf{y} = \mathbf{y}\mathbf{x} = 1$ for any $\mathbf{x} \in \mathbb{R}^{pq}$ such that $B(\mathbf{x},\mathbf{x}) \neq 0$ and $\mathbf{y} = \bar{\mathbf{x}} \|\mathbf{x}\|^{-2}$, which is the *Kelvin inverse* of $\mathbf{x}$. Finite products of invertible vectors are invertible in $\mathcal{C}\ell(p,q)$ and form the *Clifford group* $\Gamma(p,q)$. Elements $a \in \Gamma(p,q)$ such that $a\bar{a} = \pm 1$ form the $\mathrm{Pin}(p,q)$ group—the double cover of the group of orthogonal rotations $\mathrm{O}(p,q)$. We also consider [11, § 5.2] $T(p,q)$ to be the set of all products of vectors in $\mathbb{R}^{pq}$.

Let (a,b,c,d) be a quadruple from $T(p,q)$ with the properties:

(i). $(ad^* - bc^*) \in \mathbb{R} \setminus 0$;

(ii). a^*b, c^*d, ac^*, bd^* are vectors.

Then [11, Theorem 5.2.3] 2×2-matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form the group $\Gamma(p+1,q+1)$ under the usual matrix multiplication. It has a representation $\pi_{\mathbb{R}^{pq}}$ by transformations of $\overline{\mathbb{R}^{pq}}$ given by:

$$\pi_{\mathbb{R}^{pq}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{x} \mapsto (a\mathbf{x}+b)(c\mathbf{x}+d)^{-1}, \tag{3.2.1}$$

which form the *Möbius* (or the *conformal*) group of $\overline{\mathbb{R}^{pq}}$. Here $\overline{\mathbb{R}^{pq}}$ the compactification of $\mathbb{R}^{pq}$ by the "necessary number of points" (which form the light cone) at infinity (see [11, § 5.1]). The analogy with fractional-linear transformations of the complex line $\mathbb{C}$ is useful, as well as representations of shifts $\mathbf{x} \mapsto \mathbf{x}+y$, orthogonal rotations $\mathbf{x} \mapsto k(a)\mathbf{x}$, dilations $\mathbf{x} \mapsto \lambda\mathbf{x}$, and the Kelvin inverse $\mathbf{x} \mapsto \mathbf{x}^{-1}$ by the matrixes $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$, $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively. We also use the agreement of [11] that $\frac{a}{b}$ always denotes ab^{-1} for a, $b \in \mathcal{C}\ell(p,q)$.

3.3 Two Function Theories Associated with Representations of $SL_2(\mathbb{R})$

3.3.1 Unit Disks in $\mathbb{R}^{0,2}$ and $\mathbb{R}^{1,1}$

The main example is provided by group $G = SL_2(\mathbb{R})$ (books [26, 53, 76] are our standard references about $SL_2(\mathbb{R})$ and its representations) consisting of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries and determinant $ad - bc = 1$.

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $SL_2(\mathbb{R})$ consists of all 2×2 real matrices of trace zero. One can introduce a basis

$$A = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The commutator relations are

$$[Z,A]=2B,\qquad [Z,B]=-2A,\qquad [A,B]=-\frac{1}{2}Z.$$

We will construct two series of examples. One is connected with discreet series representation and produces the core of standard complex analysis. The second will be its mirror in principal series representations and create parallel function theory. $SL_2(\mathbb{R})$ has also other type representation, which can be of particular interest in other circumstances. However the discreet series and principal ones stay separately from others (in particular by being the support of the Plancherel measure [53, § VIII.4], [76, Chap. 8, (4.16)]) and are in a good resemblance each other.

Example 3.3.1.(a) Via identities

$$\alpha=\frac{1}{2}(a+d-ic+ib),\qquad \beta=\frac{1}{2}(c+b-ia+id)$$

we have isomorphism of $SL_2(\mathbb{R})$ with group $SU(1,1)$ of 2×2 matrices with complex entries of the form $\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}$ such that $|\alpha|^2-|\beta|^2=1$. We will use the last form for $SL_2(\mathbb{R})$ for complex analysis in unit disk $\mathbb{D}$.

$SL_2(\mathbb{R})$ has the only non-trivial compact closed subgroup K, namely the group of matrices of the form $h_\psi=\begin{pmatrix}e^{i\psi} & 0\\ 0 & e^{-i\psi}\end{pmatrix}$. Now any $g\in SL_2(\mathbb{R})$ has a unique decomposition of the form

$$\begin{aligned}\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix} &= |\alpha|\begin{pmatrix}1 & \beta\bar\alpha^{-1}\\ \bar\beta\alpha^{-1} & 1\end{pmatrix}\begin{pmatrix}\frac{\alpha}{|\alpha|} & 0\\ 0 & \frac{\bar\alpha}{|\alpha|}\end{pmatrix}\\ &= \frac{1}{\sqrt{1-|a|^2}}\begin{pmatrix}1 & a\\ \bar a & 1\end{pmatrix}\begin{pmatrix}e^{i\psi} & 0\\ 0 & e^{-i\psi}\end{pmatrix}\end{aligned}\tag{3.3.1}$$

where $\psi=\Im\ln\alpha$, $a=\beta\bar\alpha^{-1}$, and $|a|<1$ because $|\alpha|^2-|\beta|^2=1$. Thus we can identify $SL_2(\mathbb{R})/H$ with the unit disk $\mathbb{D}$ and define mapping $s:\mathbb{D}\to SL_2(\mathbb{R})$ as follows

$$s: a\mapsto \frac{1}{\sqrt{1-|a|^2}}\begin{pmatrix}1 & a\\ \bar a & 1\end{pmatrix}.\tag{3.3.2}$$

Mapping $r:G\to H$ associated to s is

$$r:\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}\mapsto\begin{pmatrix}\frac{\alpha}{|\alpha|} & 0\\ 0 & \frac{\bar\alpha}{|\alpha|}\end{pmatrix}\tag{3.3.3}$$

The invariant measure $d\mu(a)$ on $\mathbb{D}$ coming from decomposition $dg = d\mu(a)\,dk$, where dg and dk are Haar measures on G and K respectively, is equal to

$$d\mu(a) = \frac{da}{(1-|a|^2)^2} \tag{3.3.4}$$

with da—the standard Lebesgue measure on $\mathbb{D}$.

The formula $g : a \mapsto g \cdot a = s^{-1}(g^{-1} * s(a))$ associates with a matrix $g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ the fraction-linear transformation of $\mathbb{D}$ of the form

$$g : z \mapsto g \cdot z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \qquad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{3.3.5}$$

which also can be considered as a transformation of $\dot{\mathbb{C}}$ (the one-point compactification of $\mathbb{C}$). $\Diamond$

Example 3.3.1.(b) We will describe a version of previous formulas corresponding to geometry of unit disk in $\mathbb{R}^{1,1}$. For generators e_1 and e_2 of $\mathbb{R}^{1,1}$ (here $e_1^2 = -e_2^2 = -1$) we see that matrices $\begin{pmatrix} a & be_2 \\ ce_2 & d \end{pmatrix}$ again give a realization of $SL_2(\mathbb{R})$. Making composition with the Caley transform

$$T = \frac{1}{2}\begin{pmatrix} 1 & e_2 \\ e_2 & -1 \end{pmatrix}\begin{pmatrix} 1 & e_1 \\ e_1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+e_2e_1 & e_1+e_2 \\ e_2-e_1 & e_2e_1-1 \end{pmatrix}$$

and its inverse

$$T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -e_1 \\ -e_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & e_2 \\ e_2 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-e_1e_2 & e_2+e_1 \\ e_2-e_1 & -1-e_1e_2 \end{pmatrix}$$

(see analogous calculation in [53, § IX.1]) we obtain another realization of $SL_2(\mathbb{R})$:

$$\frac{1}{4}\begin{pmatrix} 1-e_1e_2 & e_2+e_1 \\ e_2-e_1 & -1-e_1e_2 \end{pmatrix}\begin{pmatrix} a & be_2 \\ ce_2 & d \end{pmatrix}\begin{pmatrix} 1+e_2e_1 & e_1+e_2 \\ e_2-e_1 & e_2e_1-1 \end{pmatrix} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{b}' & \mathsf{a}' \end{pmatrix}, \tag{3.3.6}$$

where

$$\mathsf{a} = \frac{1}{2}(a(1-e_1e_2)+d(1+e_1e_2)), \qquad \mathsf{b} = \frac{1}{2}(b(e_1-e_2)+c(e_1+e_2)). \tag{3.3.7}$$

It is easy to check that the condition $ad-bc=1$ implies the following value of the pseudodeterminant of the matrix $\mathsf{a}(\mathsf{a}')^* - \mathsf{b}(\mathsf{b}')^* = \mathsf{a}\bar{\mathsf{a}} - \mathsf{b}\bar{\mathsf{b}} = 1$. We also observe that a is an even Clifford number and b is a vector thus $\mathsf{a}' = \mathsf{a}$, $\mathsf{b}' = -\mathsf{b}$.

Now we consider the decomposition

$$\begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix} = |\mathsf{a}| \begin{pmatrix} 1 & \mathsf{b}\mathsf{a}^{-1} \\ -\mathsf{b}\mathsf{a}^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{\mathsf{a}}{|\mathsf{a}|} & 0 \\ 0 & \frac{\mathsf{a}}{|\mathsf{a}|} \end{pmatrix}. \qquad (3.3.8)$$

It is seen directly, or alternatively follows from general characterization of $\Gamma(p+1, q+1)$ [11, Theorem 5.2.3(b)], that $\mathsf{b}\mathsf{a}^{-1} \in \mathbb{R}^{1,1}$. Note that now we cannot derive from $\mathsf{a}\bar{\mathsf{a}} - \mathsf{b}\bar{\mathsf{b}} = 1$ that $\mathsf{b}\mathsf{a}^{-1}\overline{\mathsf{b}\mathsf{a}^{-1}} = -(\mathsf{b}\mathsf{a}^{-1})^2 < 1$ because $\mathsf{a}\bar{\mathsf{a}}$ can be positive or negative (but we are sure that $(\mathsf{b}\mathsf{a}^{-1})^2 \neq -1$). For this reason we cannot define the unit disk in $\mathbb{R}^{1,1}$ by the condition $|\mathsf{u}| < 1$ in a way consistent with its Möbius transformations. This topic will be discussed elsewhere with more illustrations [12].

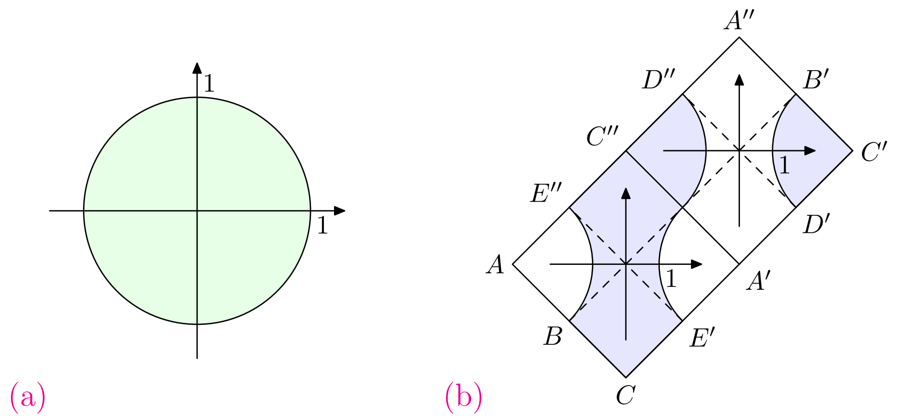

Figure 3.1: Two unit disks in elliptic (a) and hyperbolic (b) metrics. In (b) squares $ACA'C''$ and $A'C'A''C''$ represent two copies of $\mathbb{R}^2$, their boundaries are the image of the light cone at infinity. These cones should be glued in a way to merge points with the same letters (regardless number of dashes). The hyperbolic unit disk $\widetilde{\mathbb{D}}$ (shaded area) is bounded by four branches of hyperbola. Dashed lines are light cones at origins.

We are taking two copies $\mathbb{R}^{1,1}_{+}$ and $\mathbb{R}^{1,1}_{-}$ of $\mathbb{R}^{1,1}$ glued over their light cones at infinity in such a way that the construction is invariant under natural action of the Möbius transformation. This aggregate denoted by $\widetilde{\mathbb{R}}^{1,1}$ is a two-fold cover of $\mathbb{R}^{1,1}$. Topologically $\widetilde{\mathbb{R}}^{1,1}$ is equivalent to the Klein bottle. Similar conformally invariant two-fold cover of the Minkowski space-time was constructed in [70, § III.4] in connection with the red shift problem in extragalactic astronomy.

We define *(conformal) unit disk* in $\widetilde{\mathbb{R}}^{1,1}$ as follows:

$$\widetilde{\mathbb{D}} = \{\mathsf{u} \mid \mathsf{u}^2 < -1,\ \mathsf{u} \in \mathbb{R}^{1,1}_+\} \cup \{\mathsf{u} \mid \mathsf{u}^2 > -1,\ \mathsf{u} \in \mathbb{R}^{1,1}_-\}. \tag{3.3.9}$$

It can be shown that $\widetilde{\mathbb{D}}$ is conformally invariant and has a boundary $\widetilde{\mathbb{T}}$—the two glued copies of unit circles in $\mathbb{R}^{1,1}_+$ and $\mathbb{R}^{1,1}_-$.

We call $\widetilde{\mathbb{T}}$ the *(conformal) unit circle* in $\mathbb{R}^{1,1}$. $\widetilde{\mathbb{T}}$ consists of four parts—branches of hyperbola—with subgroup $A \in SL_2(\mathbb{R})$ acting simply transitively on each of them. Thus we will regard $\widetilde{\mathbb{T}}$ as $\mathbb{R} \cup \mathbb{R} \cup \mathbb{R} \cup \mathbb{R}$ with an exponential mapping $\exp : t \mapsto (+\mathrm{or}-)e_1^{+\mathrm{or}-}$, $e_1^{\pm} \in \mathbb{R}^{1,1}_{\pm}$, where each of four possible sign combinations is realized on a particular copy of $\mathbb{R}$. More generally we define a set of concentric circles for $-1 \le \lambda < 0$:

$$\widetilde{\mathbb{T}}^{\lambda} = \{\mathsf{u} \mid \mathsf{u}^2 = -\lambda^2,\ \mathsf{u} \in \mathbb{R}^{1,1}_+\} \cup \{\mathsf{u} \mid \mathsf{u}^2 = -\lambda^{-2},\ \mathsf{u} \in \mathbb{R}^{1,1}_-\}. \tag{3.3.10}$$

Figure 3.1 illustrates geometry of the conformal unit disk in $\widetilde{\mathbb{R}}^{1,1}$ as well as the "left" half plane conformally equivalent to it.

Matrices of the form

$$\begin{pmatrix} \mathsf{a} & 0 \\ 0 & \mathsf{a}' \end{pmatrix} = \begin{pmatrix} e^{e_1e_2\tau} & 0 \\ 0 & e^{e_1e_2\tau} \end{pmatrix}, \qquad \mathsf{a} = e^{e_1e_2\tau} = \cosh\tau + e_1e_2\sinh\tau, \quad \tau \in \mathbb{R}$$

comprise a subgroup of $SL_2(\mathbb{R})$ which we denote by A. This subgroup is an image of the subgroup A in the Iwasawa decomposition $SL_2(\mathbb{R}) = ANK$ [53, § III.1] under the transformation (3.3.6).

We define an embedding s of $\widetilde{\mathbb{D}}$ for our realization of $SL_2(\mathbb{R})$ by the formula:

$$s : \mathsf{u} \mapsto \frac{1}{\sqrt{1+\mathsf{u}^2}} \begin{pmatrix} 1 & \mathsf{u} \\ -\mathsf{u} & 1 \end{pmatrix}. \tag{3.3.11}$$

The formula $g : \mathsf{u} \mapsto s^{-1}(g \cdot s(\mathsf{u}))$ associated with a matrix $g^{-1} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix}$ gives the fraction-linear transformation $\widetilde{\mathbb{D}} \to \widetilde{\mathbb{D}}$ of the form:

$$g : \mathsf{u} \mapsto g \cdot \mathsf{u} = \frac{\mathsf{a}\mathsf{u} + \mathsf{b}}{-\mathsf{b}\mathsf{u} + \mathsf{a}}, \qquad g^{-1} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix} \tag{3.3.12}$$

The mapping $r : G \to H$ associated to s defined in (3.3.11) is

$$r : \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\mathsf{a}}{|\mathsf{a}|} & 0 \\ 0 & \frac{\mathsf{a}}{|\mathsf{a}|} \end{pmatrix} \tag{3.3.13}$$

And finally the invariant measure on $\widetilde{\mathbb{D}}$

$$d\mu(\mathsf{u}) = \frac{d\mathsf{u}}{(1+\mathsf{u}^2)^2} = \frac{du_1\,du_2}{(1-u_1^2+u_2^2)^2}. \tag{3.3.14}$$

follows from the elegant consideration in [11, § 6.1]. $\diamondsuit$

We hope the reader notes the explicit similarity between these two examples. Following examples will explore it further.

3.3.2 Reduced Wavelet Transform—the Cauchy Integral Formula

Example 3.3.2.(a) We continue to consider the case of $G = SL_2(\mathbb{R})$ and $H = K$. The compact group $K \sim \mathbb{T}$ has a discrete set of characters $\chi_m(h_\phi) = e^{-im\phi}$, $m \in \mathbb{Z}$. We drop the trivial character χ_0 and remark that characters χ_m and χ_{-m} give similar holomorphic and *anti*holomorphic series of representations. Thus we will consider only characters χ_m with $m = 1, 2, 3, \ldots$.

There is a difference in behavior of characters χ_1 and χ_m for $m = 2, 3, \ldots$ and we will consider them separately.

First we describe χ_1. Let us take $X = \mathbb{T}$—the unit circle equipped with the standard Lebesgue measure $d\phi$ normalized in such a way that

$$\int_{\mathbb{T}} |f_0(\phi)|^2 \, d\phi = 1 \text{ with } f_0(\phi) \equiv 1. \tag{3.3.15}$$

From (3.3.2) and (3.3.3) one can find that

$$r(g^{-1} * s(e^{i\phi})) = \frac{\bar{\beta}e^{i\phi} + \bar{\alpha}}{\left|\bar{\beta}e^{i\phi} + \bar{\alpha}\right|}, \qquad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Then the action of G on $\mathbb{T}$ defined by (3.3.5), the equality $d(g \cdot \phi)/d\phi = \left|\bar{\beta}e^{i\phi} + \bar{\alpha}\right|^{-2}$ and the character χ_1 give the following formula:

$$[\pi_1(g)f](e^{i\phi}) = \frac{1}{\bar{\beta}e^{i\phi} + \bar{\alpha}} f\left(\frac{\alpha e^{i\phi} + \beta}{\bar{\beta}e^{i\phi} + \bar{\alpha}}\right). \tag{3.3.16}$$

This is a unitary representation—the *mock discrete series* of $SL_2(\mathbb{R})$ [76, § 8.4]. It is easily seen that K acts in a trivial way by multiplication by $\chi(e^{i\phi})$. The function $f_0(e^{i\phi}) \equiv 1$ mentioned in (3.3.15) transforms as follows

$$[\pi_1(g)f_0](e^{i\phi}) = \frac{1}{\bar{\beta}e^{i\phi} + \bar{\alpha}} \tag{3.3.17}$$

and in particular has an obvious property $[\pi_1(h_\psi)f_0](\phi) = e^{i\psi}f_0(\phi)$, i.e. it is a *vacuum vector* with respect to the subgroup H. The smallest linear subspace $F_2(X) \in L_2(X)$ spanned by (3.3.17) consists of boundary values of analytic functions in the unit disk and is the *Hardy space*. Now the reduced wavelet transform (2.2.6) takes the form

$$\begin{aligned}
\widehat{f}(a) = [\mathcal{W}f](a) &= \langle f(x), \pi_1(s(a))f_0(x)\rangle_{L_2(X)} \\
&= \int_{\mathbb{T}} f(e^{i\phi})\frac{\sqrt{1-|a|^2}}{\bar{a}e^{i\phi}+1}\,d\phi \\
&= \frac{\sqrt{1-|a|^2}}{i}\int_{\mathbb{T}} \frac{f(e^{i\phi})}{a+e^{i\phi}} ie^{i\phi}\,d\phi \\
&= \frac{\sqrt{1-|a|^2}}{i}\int_{\mathbb{T}} \frac{f(z)}{a+z}\,dz, \qquad (3.3.18)
\end{aligned}$$

where $z = e^{i\phi}$. Of course (3.3.18) is the *Cauchy integral formula* up to factor $2\pi\sqrt{1-|a|^2}$. Thus we will write $f(a) = \left(2\pi\sqrt{1-|a|^2}\right)^{-1}\widehat{f}(-a)$ for analytic extension of $f(\phi)$ to the unit disk. The factor 2π is due to our normalization (3.3.15) and $\sqrt{1-|a|^2}$ is connected with the invariant measure on $\mathbb{D}$.

Let us now consider characters χ_m ($m = 2, 3, \ldots$). These characters together with action (3.3.5) of G give following representations:

$$[\pi_m(g)f](w) = f\left(\frac{\alpha w+\beta}{\bar{\beta}w+\bar{\alpha}}\right)(\bar{\beta}w+\bar{\alpha})^{-m}. \qquad (3.3.19)$$

For any integer $m \geq 2$ one can select a measure

$$d\mu_m(w) = 4^{1-m}(1-|w|^2)^{m-2}dw,$$

where dw is the standard Lebesgue measure on $\mathbb{D}$, such that (3.3.19) become unitary representations [53, § IX.3], [76, § 8.4]. These are discrete series.

If we again select $f_0(w) \equiv 1$ then

$$[\pi_m(g)f_0](w) = (\bar{\beta}w+\bar{\alpha})^{-m}.$$

In particular $[\pi_m(h_\phi)f_0](w) = e^{im\phi}f_0(w)$ so this again is a vacuum vector with respect to K. The irreducible subspace $F_2(\mathbb{D})$ generated by $f_0(w)$ consists of analytic functions and is the m-th Bergman space (actually Bergman considered only $m = 2$). Now the transformation (2.2.6) takes the form

$$\begin{aligned}
\widehat{f}(a) &= \langle f(w), [\pi_m(s(a))f_0](w)\rangle \\
&= \left(\sqrt{1-|a|^2}\right)^m \int_{\mathbb{D}} \frac{f(w)}{(a\bar{w}+1)^m}\frac{dw}{(1-|w|^2)^{2-m}},
\end{aligned}$$

which for $m = 2$ is the classical Bergman formula up to factor $\left(\sqrt{1-|a|^2}\right)^m$. Note that calculations in standard approaches are "rather lengthy and must be done in stages" [47, § 1.4]. $\diamondsuit$

Example 3.3.2.(b) Now we consider the same group $G = SL_2(\mathbb{R})$ but pick up another subgroup $H = A$. Let $e_{12} := e_1e_2$. It follows from (C.1.2) that the mapping from the subgroup $A \sim \mathbb{R}$ to even numbers[1] $\chi_\sigma : a \mapsto a^{e_{12}\sigma} = (\exp(e_1e_2\sigma \ln a)) = (a\mathsf{p}_1 + a^{-1}\mathsf{p}_2)^\sigma$ parametrized by $\sigma \in \mathbb{R}$ is a character (in a somewhat generalized sense). It represents an isometric rotation of $\widetilde{\mathbb{T}}$ by the angle a.

Under the present conditions we have from (3.3.11) and (3.3.13):

$$r(g^{-1} * s(\mathsf{u})) = \begin{pmatrix} \frac{-\mathsf{bu}+\mathsf{a}}{|-\mathsf{bu}+\mathsf{a}|} & 0 \\ 0 & \frac{-\mathsf{bu}+\mathsf{a}}{|-\mathsf{bu}+\mathsf{a}|} \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix}.$$

If we again introduce the exponential coordinates t on $\widetilde{\mathbb{T}}$ coming from the subgroup A (i.e., $\mathsf{u} = e_1 e^{e_1e_2t} \cosh te_1 - \sinh te_2 = (x + \frac{1}{x})e_1 - (x - \frac{1}{x})e_2$, $x = e^t$) then the measure dt on $\widetilde{\mathbb{T}}$ will satisfy the transformation condition

$$\frac{d(g \cdot t)}{dt} = \frac{1}{(be^{-t} + a)(ce^t + d)} = \frac{1}{(-\mathsf{bu} + \mathsf{a})(\mathsf{ub} - \mathsf{a})},$$

where

$$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix}.$$

Furthermore we can construct a representation on the functions defined on $\widetilde{\mathbb{T}}$ by the formula:

$$[\pi_\sigma(g)f](\mathsf{v}) = \frac{(-\mathsf{vb} + \bar{\mathsf{a}})^\sigma}{(-\mathsf{bv} + \mathsf{a})^{1+\sigma}} f\left(\frac{\mathsf{av} + \mathsf{b}}{-\mathsf{bv} + \mathsf{a}}\right), \quad g^{-1} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix}. \tag{3.3.20}$$

It is induced by the character χ_σ due to formula $-\mathsf{bv}+\mathsf{a} = (cx+d)\mathsf{p}_1+(bx^{-1}+a)\mathsf{p}_2$, where $x = e^t$ and it is a cousin of the principal series representation (see [53, § VI.6, Theorem 8], [76, § 8.2, Theorem 2.2] and Appendix C.2). The subspaces of vector valued and even number valued functions are invariant under (3.3.20) and the representation is unitary with respect to the following inner product (about Clifford valued inner product see [11, § 3]):

$$\langle f_1, f_2 \rangle_{\widetilde{\mathbb{T}}} = \int_{\widetilde{\mathbb{T}}} \bar{f}_2(t) f_1(t)\, dt.$$

[1] See Appendix C.1 for a definition of functions of even Clifford numbers.

We will denote by $L_2(\widetilde{\mathbb{T}})$ the space of $\mathbb{R}^{1,1}$-even Clifford number valued functions on $\widetilde{\mathbb{T}}$ equipped with the above inner product.

We select function $f_0(\mathsf{u}) \equiv 1$ neglecting the fact that it does not belong to $L_2(\widetilde{\mathbb{T}})$. Its transformations

$$f_g(\mathsf{v}) = [\pi_\sigma(g) f_0](\mathsf{v}) = \left|1+\mathsf{u}^2\right|^{1/2} \frac{(-\mathsf{v}\mathsf{b}+\bar{\mathsf{a}})^\sigma}{(-\mathsf{b}\mathsf{v}+\mathsf{a})^{1+\sigma}} \tag{3.3.21}$$

and in particular the identity $[\pi_\sigma(g)f_0](\mathsf{v}) = \bar{\mathsf{a}}^\sigma \mathsf{a}^{-1-\sigma} f_0(\mathsf{v}) = \mathsf{a}^{-1-2\sigma} f_0(\mathsf{v})$ for $g^{-1} = \begin{pmatrix} \mathsf{a} & 0 \\ 0 & \mathsf{a} \end{pmatrix}$ demonstrates that it is a vacuum vector. Thus we define the reduced wavelet transform accordingly to (3.3.11) and (2.2.6) by the formula[2]:

$$\begin{aligned}
[\mathcal{W}_\sigma f](\mathsf{u}) &= \int_{\widetilde{\mathbb{T}}} \left|1+\mathsf{u}^2\right|^{1/2} \overline{\left(\frac{(-e_1 e^{e_{12}t}\mathsf{u}+1)^\sigma}{(-\mathsf{u}e_1 e^{e_{12}t}+1)^{1+\sigma}}\right)} f(t)\,dt \\
&= \left|1+\mathsf{u}^2\right|^{1/2} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{(-e^{-e_{12}t}e_1\mathsf{u}+1)^{1+\sigma}} f(t)\,dt \qquad (3.3.22)\\
&= \left|1+\mathsf{u}^2\right|^{1/2} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{e^{-e_{12}t(1+\sigma)}(-e_1\mathsf{u}+e^{e_{12}t})^{1+\sigma}} f(t)\,dt \\
&= \left|1+\mathsf{u}^2\right|^{1/2} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{(-e_1\mathsf{u}+e^{e_{12}t})^{1+\sigma}} e^{e_{12}t(1+\sigma)} f(t)\,dt \\
&= \left|1+\mathsf{u}^2\right|^{1/2} e_{12} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{(-e_1\mathsf{u}+e^{e_{12}t})^{1+\sigma}} e^{e_{12}t\sigma} (e_{12}e^{e_{12}t}\,dt)\, f(t) \\
&= \left|1+\mathsf{u}^2\right|^{1/2} e_{12} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1\mathsf{z}+1)^\sigma \mathsf{z}^\sigma}{(-e_1\mathsf{u}+\mathsf{z})^{1+\sigma}}\, d\mathsf{z}\, f(\mathsf{z}) \qquad (3.3.23)
\end{aligned}$$

where $\mathsf{z} = e^{e_{12}t}$ and $d\mathsf{z} = e_{12}e^{e_{12}t}\,dt$ are the new monogenic variable and its differential respectively. The integral (3.3.23) is a singular one, its four singular points are intersections of the light cone with the origin in u with the unit circle $\widetilde{\mathbb{T}}$. See Appendix C.3 about the meaning of this singular integral operator.

The explicit similarity between (3.3.18) and (3.3.23) allows us to consider transformation $\mathcal{W}_\sigma$ (3.3.23) as an analog of the Cauchy integral formula and the linear space $H_\sigma(\widetilde{\mathbb{T}})$ (C.3.1) generated by the coherent states $f_\mathsf{u}(\mathsf{z})$ (3.3.21) as the correspondence of the Hardy space. Due to "indiscrete" (i.e. they are not square integrable) nature of principal series representations there are no counterparts for the Bergman projection and Bergman space. $\diamondsuit$

[2]This formula is not well defined in the Hilbert spaces setting. Fortunately it is possible (see 2.3 and [42]) to define a theory of wavelets in Banach spaces in a way very similar to the Hilbert space case. So we will ignore this difference in this lecture.

3.3.3 The Dirac (Cauchy-Riemann) and Laplace Operators

Consideration of Lie groups is hardly possible without consideration of their Lie algebras, which are naturally represented by left and right invariant vectors fields on groups. On a homogeneous space $\Omega = G/H$ we have also defined a left action of G and can be interested in left invariant vector fields (first order differential operators). Due to the irreducibility of $F_2(\Omega)$ under left action of G every such vector field D restricted to $F_2(\Omega)$ is a scalar multiplier of identity $D|_{F_2(\Omega)} = cI$. We are in particular interested in the case $c = 0$.

Definition 3.3.3 [2, 45] A G-invariant first order differential operator

$$D_\tau : C_\infty(\Omega, \mathcal{S} \otimes V_\tau) \to C_\infty(\Omega, \mathcal{S} \otimes V_\tau)$$

such that $\mathcal{W}(F_2(X)) \subset \ker D_\tau$ is called *(Cauchy-Riemann-)Dirac operator* on $\Omega = G/H$ associated with an irreducible representation τ of H in a space V_τ and a spinor bundle $\mathcal{S}$.

The Dirac operator is explicitly defined by the formula [45, (3.1)]:

$$D_\tau = \sum_{j=1}^{n} \rho(Y_j) \otimes c(Y_j) \otimes 1, \qquad (3.3.24)$$

where Y_j is an orthonormal basis of $\mathfrak{p} = \mathfrak{h}^\perp$—the orthogonal completion of the Lie algebra $\mathfrak{h}$ of the subgroup H in the Lie algebra $\mathfrak{g}$ of G; $\rho(Y_j)$ is the infinitesimal generator of the right action of G on Ω; $c(Y_j)$ is Clifford multiplication by $Y_i \in \mathfrak{p}$ on the Clifford module $\mathcal{S}$. We also define an invariant Laplacian by the formula

$$\Delta_\tau = \sum_{j=1}^{n} \rho(Y_j)^2 \otimes \epsilon_j \otimes 1, \qquad (3.3.25)$$

where $\epsilon_j = c(Y_j)^2$ is $+1$ or -1.

Proposition 3.3.4 *Let all commutators of vectors of $\mathfrak{h}^\perp$ belong to $\mathfrak{h}$, i.e. $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subset \mathfrak{h}$. Let also f_0 be an eigenfunction for all vectors of $\mathfrak{h}$ with eigenvalue 0 and let also $\mathcal{W}f_0$ be a null solution to the Dirac operator D. Then $\Delta f(x) = 0$ for all $f(x) \in F_2(\Omega)$.*

PROOF. Because Δ is a linear operator and $F_2(\Omega)$ is generated by $\pi_0(s(a))\mathcal{W}f_0$ it is enough to check that $\Delta\pi_0(s(a))\mathcal{W}f_0 = 0$. Because Δ and π_0 commute it is enough to check that $\Delta\mathcal{W}f_0 = 0$. Now we observe that

$$\Delta = D^2 - \sum_{i,j} \rho([Y_i, Y_j]) \otimes c(Y_i)c(Y_j) \otimes 1.$$

Thus the desired assertion is follows from two identities: $D\mathcal{W}f_0 = 0$ and $\rho([Y_i, Y_j])\mathcal{W}f_0 = 0$, $[Y_i, Y_j] \in H$. □

Example 3.3.5.(a) Let $G = SL_2(\mathbb{R})$ and H be its one-dimensional compact subgroup generated by an element $Z \in \mathfrak{sl}(2,\mathbb{R})$. Then $\mathfrak{h}^\perp$ is spanned by two vectors $Y_1 = A$ and $Y_2 = B$. In such a situation we can use $\mathbb{C}$ instead of the Clifford algebra $\mathcal{C}\ell(0,2)$. Then formula (3.3.24) takes a simple form $D = r(A + iB)$. Infinitesimal action of this operator in the upper-half plane follows from calculation in [53, VI.5(8), IX.5(3)], it is $[D_{\mathbb{H}} f](z) = -2iy\frac{\partial f(z)}{\partial \bar{z}}$, $z = x + iy$. Making the Caley transform we can find its action in the unit disk $D_{\mathbb{D}}$: again the Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$ is its principal component. We calculate $D_{\mathbb{H}}$ explicitly now to stress the similarity with $\mathbb{R}^{1,1}$ case.

For the upper half plane $\mathbb{H}$ we have following formulas:

$$\begin{aligned}
s &: \mathbb{H} \to SL_2(\mathbb{R}) : z = x + iy \mapsto g = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix};\\
s^{-1} &: SL_2(\mathbb{R}) \to \mathbb{H} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto z = \frac{ai+b}{ci+d};\\
\rho(g) &: \mathbb{H} \to \mathbb{H} : z \mapsto s^{-1}(s(z) * g)\\
&\qquad = s^{-1}\begin{pmatrix} ay^{-1/2} + cxy^{-1/2} & by^{1/2} + dxy^{-1/2} \\ cy^{-1/2} & dy^{-1/2} \end{pmatrix}\\
&\qquad = \frac{(yb + xd) + i(ay + cx)}{ci + d}
\end{aligned}$$

Thus the right action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ is given by the formula

$$\rho(g)z = \frac{(yb + xd) + i(ay + cx)}{ci + d} = x + y\frac{bd + ac}{c^2 + d^2} + iy\frac{1}{c^2 + d^2}.$$

For A and B in $\mathfrak{sl}(2,\mathbb{R})$ we have:

$$\rho(e^{At})z = x + iye^{2t}, \qquad \rho(e^{Bt})z = x + y\frac{e^{2t} - e^{-2t}}{e^{2t} + e^{-2t}} + iy\frac{4}{e^{2t} + e^{-2t}}.$$

Thus

$$\begin{aligned}
[\rho(A)f](z) &= \frac{\partial f(\rho(e^{At})z)}{\partial t}|_{t=0} = 2y\partial_2 f(z),\\
[\rho(B)f](z) &= \frac{\partial f(\rho(e^{Bt})z)}{\partial t}|_{t=0} = 2y\partial_1 f(z),
\end{aligned}$$

where ∂_1 and ∂_2 are derivatives of $f(z)$ with respect to real and imaginary party of z respectively. Thus we get

$$D_{\mathbb{H}} = i\rho(A) + \rho(B) = 2yi\partial_2 + 2y\partial_1 = 2y\frac{\partial}{\partial \bar{z}}$$

as was expected. $\diamondsuit$

Example 3.3.5.(b) In $\mathbb{R}^{1,1}$ the element $B \in \mathfrak{sl}$ generates the subgroup H and its orthogonal completion is spanned by B and Z. Thus the associated Dirac operator has the form $D = e_1\rho(B) + e_2\rho(Z)$. We need infinitesimal generators of the right action ρ on the "left" half plane $\widetilde{\mathbb{H}}$. Again we have a set of formulas similar to the classic case:

$$\begin{aligned}
s &: \widetilde{\mathbb{H}} \rightarrow SL_2(\mathbb{R}) : \mathsf{z} = e_1 y + e_2 x \mapsto g = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix};\\
s^{-1} &: SL_2(\mathbb{R}) \rightarrow \widetilde{\mathbb{H}} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mathsf{z} = \frac{ae_1 + be_2}{ce_2e_1 + d};\\
\rho(g) &: \widetilde{\mathbb{H}} \rightarrow \widetilde{\mathbb{H}} : \mathsf{z} \mapsto s^{-1}(s(\mathsf{z}) * g)\\
&\qquad = s^{-1}\begin{pmatrix} ay^{-1/2} + cxy^{-1/2} & by^{1/2} + dxy^{-1/2} \\ cy^{-1/2} & dy^{-1/2} \end{pmatrix}\\
&\qquad = \frac{(yb + xd)e_2 + (ay + cx)e_1}{ce_2e_1 + d}
\end{aligned}$$

Thus the right action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ is given by the formula

$$\rho(g)\mathsf{z} = \frac{(yb + xd)e_2 + (ay + cx)e_1}{ce_2e_1 + d} = e_1 y\frac{-1}{c^2 - d^2} + e_2 x + e_2 y\frac{ac - bd}{c^2 - d^2}.$$

For A and Z in $\mathfrak{sl}(2, \mathbb{R})$ we have:

$$\begin{aligned}
\rho(e^{At})\mathsf{z} &= e_1 y e^{2t} + e_2 x,\\
\rho(e^{Zt})\mathsf{z} &= e_1 y\frac{-1}{\sin^2 t - \cos^2 t} + e_2 y\frac{-2\sin t\cos t}{\sin^2 t - \cos^2 t} + e_2 x\\
&= e_1 y\frac{1}{\cos 2t} + e_2 y\tan 2t + e_2 x.
\end{aligned}$$

Thus

$$\begin{aligned}
[\rho(A)f](\mathsf{z}) &= \frac{\partial f(\rho(e^{At})\mathsf{z})}{\partial t}|_{t=0} = 2y\partial_2 f(\mathsf{z}),\\
[\rho(Z)f](\mathsf{z}) &= \frac{\partial f(\rho(e^{Zt})\mathsf{z})}{\partial t}|_{t=0} = 2y\partial_1 f(\mathsf{z}),
\end{aligned}$$

where ∂_1 and ∂_2 are derivatives of $f(\mathsf{z})$ with respect of e_1 and e_2 components of z respectively. Thus we get

$$D_{\widetilde{\mathbb{H}}} = e_1\rho(Z) + e_2\rho(A) = 2y(e_1\partial_1 + e_2\partial_2).$$

In this case the Dirac operator is not elliptic and as a consequence we have in particular a singular Cauchy integral formula (3.3.23). Another manifestation of the same property is that primitives in the "Taylor expansion" do not belong to $F_2(\widetilde{\mathbb{T}})$ itself (see Example 3.3.8.(b)). It is known that solutions of a hyperbolic system (unlike the elliptic one) essentially depend on the behavior of the boundary value data. Thus function theory in $\mathbb{R}^{1,1}$ is not defined only by the hyperbolic Dirac equation alone but also by an appropriate boundary condition. $\diamondsuit$

3.3.4 The Taylor expansion

For any decomposition $f_a(x) = \sum_\alpha \psi_\alpha(x) V_\alpha(a)$ of the coherent states $f_a(x)$ by means of functions $V_\alpha(a)$ (where the sum can become eventually an integral) we have the *Taylor expansion*

$$\begin{aligned}
\widehat{f}(a) &= \int_X f(x)\bar{f}_a(x)\,dx = \int_X f(x)\sum_\alpha \bar{\psi}_\alpha(x)\bar{V}_\alpha(a)\,dx \\
&= \sum_\alpha \int_X f(x)\bar{\psi}_\alpha(x)\,dx \bar{V}_\alpha(a) \\
&= \sum_\alpha^\infty \bar{V}_\alpha(a) f_\alpha,
\end{aligned} \tag{3.3.26}$$

where $f_\alpha = \int_X f(x)\bar{\psi}_\alpha(x)\,dx$. However to be useful within the presented scheme such a decomposition should be connected with the structures of G, H, and the representation π_0. We will use a decomposition of $f_a(x)$ by the eigenfunctions of the operators $\pi_0(h)$, $h \in \mathfrak{h}$.

Definition 3.3.6 Let $F_2 = \int_A H_\alpha\,d\alpha$ be a spectral decomposition with respect to the operators $\pi_0(h)$, $h \in \mathfrak{h}$. Then the decomposition

$$f_a(x) = \int_A V_\alpha(a) f_\alpha(x)\,d\alpha, \tag{3.3.27}$$

where $f_\alpha(x) \in H_\alpha$ and $V_\alpha(a) : H_\alpha \to H_\alpha$ is called the Taylor decomposition of the Cauchy kernel $f_a(x)$.

Note that the Dirac operator D is defined in the terms of left invariant shifts and therefor commutes with all $\pi_0(h)$. Thus it also has a spectral decomposition over spectral subspaces of $\pi_0(h)$:

$$D = \int_A D_\delta \, d\delta. \tag{3.3.28}$$

We have obvious property

Proposition 3.3.7 *If spectral measures $d\alpha$ and $d\delta$ from (3.3.27) and (3.3.28) have disjoint supports then the image of the Cauchy integral belongs to the kernel of the Dirac operator.*

For discrete series representation functions $f_\alpha(x)$ can be found in F_2 (as in Example 3.3.7.(a)), for the principal series representation this is not the case. To overcome confusion one can think about the Fourier transform on the real line. It can be regarded as a continuous decomposition of a function $f(x) \in L_2(\mathbb{R})$ over a set of harmonics $e^{i\xi x}$ neither of those belongs to $L_2(\mathbb{R})$. This has a lot of common with the Example 3.3.8.(b).

Example 3.3.8.(a) Let $G = SL_2(\mathbb{R})$ and $H = K$ be its maximal compact subgroup and π_1 be described by (3.3.16). H acts on $\mathbb{T}$ by rotations. It is one dimensional and eigenfunctions of its generator Z are parametrized by integers (due to compactness of K). Moreover, on the irreducible Hardy space these are positive integers $n = 1, 2, 3 \ldots$ and corresponding eigenfunctions are $f_n(\phi) = e^{i(n-1)\phi}$. Negative integers span the space of anti-holomorphic function and the splitting reflects the existence of analytic structure given by the Cauchy-Riemann equation. The decomposition of coherent states $f_a(\phi)$ by means of this functions is well known:

$$f_a(\phi) = \frac{\sqrt{1-|a|^2}}{\bar{a}e^{i\phi} - 1} = \sum_{n=1}^{\infty} \sqrt{1-|a|^2}\,\bar{a}^{n-1} e^{i(n-1)\phi} = \sum_{n=1}^{\infty} V_n(a) f_n(\phi),$$

where $V_n(a) = \sqrt{1-|a|^2}\,\bar{a}^{n-1}$. This is the classical Taylor expansion up to multipliers coming from the invariant measure. $\diamondsuit$

Example 3.3.8.(b) Let $G = SL_2(\mathbb{R})$, $H = A$, and π_σ be described by (3.3.20). Subgroup H acts on $\widetilde{\mathbb{T}}$ by hyperbolic rotations:

$$\tau : \mathsf{z} = e_1 e^{e_{12} t} \rightarrow e^{2e_{12}\tau} \mathsf{z} = e_1 e^{e_{12}(2\tau + t)}, \qquad t, \tau \in \widetilde{\mathbb{T}}.$$

Then for every $p \in \mathbb{R}$ the function $f_p(\mathsf{z}) = (\mathsf{z})^p = e^{e_{12} p t}$ where $\mathsf{z} = e^{e_{12} t}$ is an eigenfunction in the representation (3.3.20) for a generator a of the subgroup

$H = A$ with the eigenvalue $2(p-\sigma)-1$. Again, due to the analytical structure reflected in the Dirac operator, we only need negative values of p to decompose the Cauchy integral kernel.

Proposition 3.3.9 *For $\sigma = 0$ the Cauchy integral kernel (3.3.23) has the following decomposition:*

$$\frac{1}{-e_1\mathsf{u}+\mathsf{z}} = \int_0^\infty \frac{(e_1\mathsf{u})^{[p]}-1}{e_1\mathsf{u}-1}\cdot t\mathsf{z}^{-p}\,dp, \tag{3.3.29}$$

where $\mathsf{u} = u_1e_1+u_2e_2$, $\mathsf{z} = e^{e_{12}t}$, and $[p]$ is the integer part of p (i.e. $k = [p] \in \mathbb{Z}$, $k \le p < k+1$).

Proof. Let

$$f(t) = \int_0^\infty F(p)e^{-tp}\,dp$$

be the Laplace transform. We use the formula [6, Laplace Transform Table, p. 479, (66)]

$$\frac{1}{t(e^{kt}-a)} = \int_0^\infty \frac{a^{[p/k]}-1}{a-1}e^{-tp}\,dp \tag{3.3.30}$$

with the particular value of the parameter $k=1$. Then using $\mathsf{p}_{1,2}$ defined in (C.1.1) we have

$$\begin{aligned}
\int_0^\infty & \frac{(e_1\mathsf{u})^{[p]}-1}{e_1\mathsf{u}-1}\cdot t\mathsf{z}^{-p}\,dp = \\
&= t\int_0^\infty \left(\frac{(-u_1-u_2)^{[p]}-1}{(-u_1-u_2)-1}\mathsf{p}_2+\frac{(-u_1+u_2)^{[p]}-1}{(-u_1+u_2)-1}\mathsf{p}_1\right)(e^{tp}\mathsf{p}_2+e^{-tp}\mathsf{p}_1)\,dp\\
&= t\int_0^\infty \frac{(-u_1-u_2)^{[p]}-1}{(-u_1-u_2)-1}e^{tp}\,dp\,\mathsf{p}_2 + t\int_0^\infty \frac{(-u_1+u_2)^{[p]}-1}{(-u_1+u_2)-1}e^{-tp}\,dp\,\mathsf{p}_1\\
&= \frac{t}{t(e^{-t}+u_1+u_2)}\mathsf{p}_2+\frac{t}{t(e^{t}+u_1-u_2)}\mathsf{p}_1 \qquad (3.3.31)\\
&= \frac{1}{(e^{-t}+u_1+u_2)\mathsf{p}_2+(e^{t}+u_1-u_2)\mathsf{p}_1}\\
&= \frac{1}{-e_1\mathsf{u}+\mathsf{z}},
\end{aligned}$$

where we obtain (3.3.31) by an application of (3.3.30). □

Thereafter for a function $f(\mathsf{z}) \in F_2(\widetilde{\mathbb{T}})$ we have the following Taylor expansion of its wavelet transform:

$$[\mathcal{W}_0 f](u) = \int_0^\infty \frac{(e_1\mathsf{u})^{[p]}-1}{e_1\mathsf{u}-1}f_p\,dp,$$

where

$$f_p = \int_{\widetilde{\mathbb{T}}} t\mathsf{z}^{-p}\, d\mathsf{z} f(\mathsf{z}).$$

The last integral is similar to the Mellin transform [53, § III.3], [76, Chap. 8, (3.12)], which naturally arises in study of the principal series representations of $SL_2(\mathbb{R})$.

I was pointed by Dr. J. Cnops that for the Cauchy kernel $(-e_1\mathsf{u}+\mathsf{z})$ there is still a decomposition of the form $(-e_1\mathsf{u}+\mathsf{z}) = \sum_{j=0}^{\infty}(e_1\mathsf{u})^j\mathsf{z}^{-j-1}$. In this connection one may note that representations π_1 (3.3.16) and π_σ (3.3.20) for $\sigma = 0$ are unitary equivalent. (this is a meeting point between discrete and principal series). Thus a function theory in $\mathbb{R}^{1,1}$ with the value $\sigma = 0$ could carry many properties known from the complex analysis. $\diamondsuit$

3.4 Open problems

This lecture raises more questions than gives answers. Nevertheless it is useful to state some open problems explicitly.

(i). Demonstrate that Cauchy formula (3.3.23) is an isometry between $F_2(\widetilde{\mathbb{T}})$ and $H_\sigma(\widetilde{\mathbb{D}})$ with suitable norms chosen. This almost follows (up to some constant factor) from its property to intertwine two irreducible representations of $SL_2(\mathbb{R})$.

(ii). Formula (3.3.22) contains Szegö type kernel, which is domain dependent. Integral formula (3.3.23) formulated in terms of analytic kernel. Demonstrate using Stocks theorem that (3.3.23) is true for other suitable chosen domains.

(iii). The image of Szegö (or Cauchy) type formulas belong to the kernel of Dirac type operator only if they connected by additional condition (see Proposition 3.3.7). Descriptive condition for the discrete series can be found in [45, Theorem 6.1]. Formulate a similar condition for principal series representations.

Lecture 4

Segal-Barmann Spaces and Nilpotent Groups

4.1 Introduction

This lecture is based on the paper [13].

It is well known, by the celebrated Stone-von Neumann theorem, that all models for the canonical quantisation [54] are isomorphic and provide us with equivalent representations of the Heisenberg group [76, Chap. 1]. Nevertheless it is worthwhile to look for some models which can act as alternatives for the Schrödinger representation. In particular, the Segal-Bargmann representation [3, 69] serves to

- give a geometric representation of the dynamics of the harmonic oscillators;
- present a nice model for the creation and annihilation operators, which is important for quantum field theory;
- allow applying tools of analytic function theory.

The huge abilities of the Segal-Bargmann (or Fock [20]) model are not yet completely employed, see for example new ideas in a recent preprint [59].

We look for similar connections between nilpotent Lie groups and spaces of monogenic [9, 17] Clifford valued functions. Particularly we are interested in a third possible representation of the Heisenberg group, acting on monogenic functions on $\mathbb{R}^n$. There are several reasons why such a model can be of interest. First of all the theory of monogenic functions is (at least) as interesting as several complex variable theory, so the monogenic model should share many pleasant features with the Segal-Bargmann model. Moreover, monogenic functions take their value in a Clifford algebra, which is a natural environment in which to represent internal degrees of freedom of elementary particles such as spin. Thus from the very beginning it has a structure which in the Segal-Bargmann model has to be added, usually by means of the second quantization procedure [18]. So a monogenic representation can be even more relevant to quantum field theory than the Segal-Bargmann one (see Remark 4.2.2).

From the different aspects of the Segal-Bargmann space $F_2(\mathbb{C}^n)$ we select the one giving a unitary representation of the Heisenberg group $\mathbb{H}^n$. The representation is unitary equivalent to the Schrödinger representation on $L_2(\mathbb{R}^n)$ and the Segal-Bargmann transform is precisely the intertwining operator between these two representations (see subsection 4.2.2).

Monogenic functions can be introduced in this scheme in two ways, as either $L_2(\mathbb{R}^n)$ or $F_2(\mathbb{C}^n)$ can be substituted by a space of monogenic functions.

In the first case one defines a new unitary irreducible representation of the Heisenberg group on a space of monogenic functions and constructs an analogue of the Segal-Bargmann transform as the intertwining operator of the new representation and the Segal-Bargmann one. We examine this possibility in section 4.2. In a certain sense the representation of the Heisenberg group constructed here lies between Schrödinger and Segal-Bargmann ones, taking properties both of them.

In the second case we first select a substitute for the Heisenberg group, so we can replace the Segal-Bargmann space by a space of monogenic functions. The space $\mathbb{C}^n$ underlying $F_2(\mathbb{C}^n)$ is intimately connected with the structure of the Heisenberg group $\mathbb{H}^n$ in the sense that $\mathbb{C}^n$ is the quotient of $\mathbb{H}^n$ with respect to its centre. In order to define a space of monogenic functions, say on $\mathbb{R}^{n+1}$, we have to construct a group playing a similar rôle with respect to this space. We describe an option in section 4.3.

Finally we give the basics of coherent states from square integrable group representations and an interpretation of the classic Segal–Bargmann space in terms of these in Appendix B.

This lecture is closely related to [41], where connections between analytic function theories and group representations were described. Representations of another group ($SL_2(\mathbb{R})$) in spaces of monogenic functions can be found in [39]. We hope that the present lecture make only few first steps towards an interesting function theory and other steps will be done elsewhere.

4.2 The Heisenberg Group and Spaces of Analytic Functions

4.2.1 The Schrödinger Representation of the heisenberg Group

We recall here some basic facts on the Heisenberg group $\mathbb{H}^n$ and its Schrödinger representation, see [21, Chap. 1] and [76, Chap. 1] for details.

The Lie algebra of the Heisenberg group is generated by the $2n+1$ elements $p_1, \ldots, p_n, q_1, \ldots, q_n, e$, with the well-known Heisenberg commutator relations:

$$[p_i, q_j] = \delta_{ij} e. \tag{4.2.1}$$

All other commutators vanish. In the standard quantum mechanical interpretation the operators are momentum and coordinate operators [21, § 1.1].

It is common practice to switch between real and complex Lie algebras. Complexify $\mathfrak{h}^n$ to obtain the complex algebra $\mathbb{C}\mathfrak{h}^n$, and take four complex numbers a, b, c and d such that $ad - bc \neq 0$. The *real* $2n+1$-dimensional subspace spanned by

$$A_k = ap_k + bq_k \qquad B_k = cp_k + dq_k$$

and the commutator $[A_k, B_k] = (ad - bc)e$, where $e = [p_k, q_k]$ is of course isomorphic to $\mathfrak{h}^n$, and exponentiating will give a group isomorphic to the Heisenberg group.

An example of this procedure is obtained from the construction of the so-called creation and annihilation operators of Bose particles in the k-th state, a_k^+ and a_k^- (see [21, § 1.1]). These are defined by:

$$a_k^{\pm} = \frac{q_k \mp \imath p_k}{\sqrt{2}}, \tag{4.2.2}$$

giving the commutators $[a_i^+, a_j^-] = (-\imath)\delta_{ij} e$. Putting $-\imath e = \ell$, the real algebra spanned by $a_k^{\pm}$ and ℓ is an alternative realization of $\mathfrak{h}^n$, $\mathfrak{h}_a^n$.

An element g of the Heisenberg group $\mathbb{H}^n$ (for any positive integer n, cf. (A.1.1)) can be represented as $g = (t, \mathbf{z})$ with $t \in \mathbb{R}$, $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. The group law in coordinates $(t, \mathbf{z})$ is given by

$$g * g' = (t, \mathbf{z}) * (t', \mathbf{z}') = (t + t' + \frac{1}{2}\sum_{j=1}^{n} \Im(\bar{z}_j z_j'), \mathbf{z} + \mathbf{z}'), \tag{4.2.3}$$

where $\Im z$ denotes the imaginary part of the complex number z. Of course the Heisenberg group is non-commutative.

The relation between the Heisenberg group and its Lie algebra is given by the exponentiation $\exp : \mathfrak{h}^n_a \to \mathbb{H}^n$. We define the formal vector $\mathbf{a}^+$ as being $(a_1^+, \ldots, a_n^+)$ and $\mathbf{a}^-$ as $(a_1^-, \ldots, a_n^-)$, which allows us to use the formal inner products

$$\begin{aligned} \mathbf{u} \cdot \mathbf{a}^+ &= \sum_{k=1}^{n} u_k a_k^+ \\ \mathbf{v} \cdot \mathbf{a}^- &= \sum_{k=1}^{n} v_k a_k^- . \end{aligned}$$

With these we define, for real vectors $\mathbf{u}$ and $\mathbf{v}$, and real s

$$\exp(\mathbf{u} \cdot (\mathbf{a}^+ + \mathbf{a}^-)) = (0, \sqrt{2}\mathbf{u}) \tag{4.2.4}$$

$$\exp(\mathbf{v} \cdot (\mathbf{a}^- - \mathbf{a}^+) = (0, \imath \mathbf{v}) \tag{4.2.5}$$

$$\exp(s\ell) = (e^{-2s}, 0). \tag{4.2.6}$$

Possible Schrödinger representations are parameterized by the non-zero real number $\hbar$ (the Planck constant). As usual, for considerations where the correspondence principle between classic and quantum mechanics is irrelevant, we consider only the case $\hbar = 1$. The Hilbert space for the Schrödinger representation is $L_2(\mathbb{R}^n)$, where elements of the complex Lie algebra $\mathbb{C}\mathfrak{h}^n$ are represented by the unbounded operators

$$\sigma(a_k^\pm) = \frac{1}{\sqrt{2}} \left(x_k I \mp \frac{\partial}{\partial x_k} \right). \tag{4.2.7}$$

From which it follows, using any j, that

$$\sigma(\ell) = [a_j^+, a_j^-] = -2I.$$

The corresponding representation π of the Heisenberg group is given by exponentiation of the $\sigma(a_k^+)$ and $\sigma(a_k^-)$, but this is most readily expressed by using p_k and q_k, and so is generated by shifts and multiplications $s_\mathbf{c} : f(\mathbf{x}) \mapsto f(\mathbf{x} + \mathbf{c})$ and $m_\mathbf{b} : f(\mathbf{x}) \mapsto e^{\imath \mathbf{x} \cdot \mathbf{b}} f(\mathbf{x})$, with the Weyl commutation relation

$$s_\mathbf{c} m_\mathbf{b} = e^{\imath \mathbf{c} \cdot \mathbf{b}} m_\mathbf{b} s_\mathbf{c}.$$

There is an orthonormal basis of $L_2(\mathbb{R}^n)$ on which the operators $\sigma(a_k^\pm)$ act in an especially simple way. It consists of the functions:

$$\phi_m(\mathbf{y}) = [2^m m! \sqrt{\pi}]^{-1/2} e^{-\mathbf{x} \cdot \mathbf{x}/2} H_m(\mathbf{y}), \tag{4.2.8}$$

where $\mathbf{y} = (y_1, \ldots, y_n)$, $m = (m_1, \ldots, m_n)$, and $H_m(\mathbf{y})$ is the generalized Hermite polynomial

$$H_m(\mathbf{y}) = \prod_{i=1}^{n} H_{m_i}(y_i).$$

For these

$$a_k^+ \phi_m(\mathbf{y}) = \sqrt{m_k + 1}\, \phi_{m'}(\mathbf{y}), \qquad a_k^- \phi_m(\mathbf{y}) = \sqrt{m_k}\, \phi_{m''}(\mathbf{y})$$

where

$$\begin{aligned} m' &= (m_1, m_2, \ldots, m_{k-1}, m_k + 1, m_{k+1}, \ldots, m_n) \\ m'' &= (m_1, m_2, \ldots, m_{k-1}, m_k - 1, m_{k+1}, \ldots, m_n). \end{aligned}$$

This is the most straightforward way to express the creation or annihilation of a particle in the k-th state.

Let us now consider the generating function of the $\phi_m(\mathbf{x})$,

$$A(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{\infty} \frac{x^j}{\sqrt{j!}} \phi_k(\mathbf{y}) = \exp(-\frac{1}{2}(\mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y}) + \sqrt{2}\mathbf{x} \cdot \mathbf{y}). \tag{4.2.9}$$

We state the following elementary fact in Dirac's bra-ket notation.

Lemma 4.2.1 *Let H and H' be two Hilbert spaces with orthonormal bases $\{\phi_k\}$ and $\{\phi'_k\}$ respectively. Then the sum*

$$U = \sum_{j=0}^{\infty} |\phi'_j\rangle \langle \phi_j| \tag{4.2.10}$$

defines a unitary operator $U : H \to H'$ with the following properties:

(i). $U\phi_k = \phi'_k$;

(ii). If an operator $T : H \to H$ is expressed, relative to the basis ϕ_k, by the matrix (a_{ij}) then the operator $UTU^{-1} : H' \to H'$ is expressed relative to the basis ϕ'_k by the same matrix.

Now, if we take the function $A(\mathbf{x}, \mathbf{y})$ from (4.2.9) as a kernel for an *integral transform*,

$$[Af](\mathbf{y}) = \int_{\mathbb{R}^n} A(\mathbf{y}, \mathbf{x}) f(\mathbf{x})\, dx$$

we can consider it subject to the Lemma above. However, for this we need to define the space H' and an orthonormal basis $\{\phi'_k\}$ (we already identified

H with $L_2(\mathbb{R}^n)$ and the $\{\phi_k\}$ are given by (4.2.8)). There is some freedom in doing this.

For example it is possible to take the holomorphic extension $A(\mathbf{z}, \mathbf{y})$ of $A(\mathbf{x}, \mathbf{y})$ with respect to the first variable. Then

(i). H' is the Segal-Bargmann space of analytic functions over $\mathbb{C}^n$ with scalar product defined by the integral with respect to Gaussian measure $e^{-|\mathbf{z}|^2}\, d\mathbf{z}$;

(ii). The Heisenberg group acts on the Segal-Bargmann space as follows:

$$[\beta_{(t,\mathbf{z})} f](\mathbf{u}) = f(\mathbf{u} + \mathbf{z}) e^{\imath t - \bar{\mathbf{z}}\cdot\mathbf{u} - |\mathbf{z}|^2/2}. \tag{4.2.11}$$

This action generates the set of coherent states $f_{(0,\mathbf{v})}(\mathbf{u}) = e^{-\bar{\mathbf{v}}\mathbf{u} - |\mathbf{v}|^2/2}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^n$ from the vacuum vector $f_0(\mathbf{u}) \equiv 1$;

(iii). The operators of creation and annihilation are $a_k^+ = z_k I$, $a_k^- = \frac{\partial}{\partial z_k}$.

(iv). The Segal-Bargmann space is spanned by the orthonormal basis $\phi'_k = \frac{1}{\sqrt{m!}} z^n$ or by the set of coherent states $f_{(0,\mathbf{v})}(\mathbf{u}) = e^{-\bar{\mathbf{v}}\mathbf{u} - |\mathbf{v}|^2/2}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^n$

(v). The intertwining kernel for $\sigma_{(t,\mathbf{z})}$ (4.2.7) and $\beta_{(t,\mathbf{z})}$ (4.2.11) is

$$A(\mathbf{z}, \mathbf{y}) = e^{-(\mathbf{z}\cdot\mathbf{z} + \mathbf{x}\cdot\mathbf{x})/2 - \sqrt{2}\mathbf{z}\cdot\mathbf{x}} = \sum_{k=0}^{\infty} \frac{\mathbf{z}^m}{\sqrt{m!}} \cdot \frac{1}{\sqrt{2^m m! \sqrt[4]{\pi}}} e^{-\mathbf{x}\cdot\mathbf{x}/2} H_m(\mathbf{y})$$

(vi). The Segal-Bargmann space has a reproducing kernel

$$K(\mathbf{u}, \mathbf{v}) = e^{\mathbf{u}\cdot\bar{\mathbf{v}}} = \sum_{k=1}^{\infty} \phi_k(\mathbf{u}) \bar{\phi}_k(\mathbf{v}) = \int e^{\mathbf{u}\cdot\bar{\mathbf{z}}} e^{\mathbf{z}\cdot\bar{\mathbf{v}}} e^{-|\mathbf{z}|^2}\, d\mathbf{z}.$$

4.2.2 The Segal-Bargmann space

We consider a representation of the Heisenberg group $\mathbb{H}^n$ (see Section 4.2) on $L_2(\mathbb{R}^n)$ by shift and multiplication operators [76, § 1.1]:

$$g = (t, \mathbf{z}) : f(\mathbf{x}) \to [\pi_{(t,\mathbf{z})} f](\mathbf{x}) = e^{\imath(2t - \sqrt{2}\mathbf{q}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{p})} f(\mathbf{x} - \sqrt{2}\mathbf{p}), \qquad \mathbf{z} = \mathbf{p} + \imath q, \tag{4.2.12}$$

This is the Schrödinger representation with parameter $\hbar = 1$. As a subgroup H we select the centre of $\mathbb{H}^n$ consisting of elements $(t, 0)$. It is non-compact but using the special form of representation (4.2.12) we can consider the cosets[1] $\widetilde{G}$ and $\widetilde{H}$ of G and H by the subgroup with elements $(\pi m, 0)$, $m \in \mathbb{Z}$. Then (4.2.12) also defines a representation of $\widetilde{G}$ and $\widetilde{H} \sim \Gamma$. We consider the Haar measure on $\widetilde{G}$ such that its restriction on $\widetilde{H}$ has total mass equal to 1.

As "vacuum vector" we will select the original *vacuum vector* of quantum mechanics—the Gauss function $f_0(\mathbf{x}) = e^{-\mathbf{x}\cdot\mathbf{x}/2}$. Its transformations are defined as follows:

$$\begin{aligned} w_g(\mathbf{x}) = \pi_{(t,\mathbf{z})} f_0(\mathbf{x}) &= e^{\imath(2t-\sqrt{2}\mathbf{q}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{p})}\, e^{-(\mathbf{x}-\sqrt{2}\mathbf{p})^2/2} \\ &= e^{2\imath t-(\mathbf{p}\cdot\mathbf{p}+\mathbf{q}\cdot\mathbf{q})/2} e^{-((\mathbf{p}-\imath\mathbf{q})^2+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}(\mathbf{p}-\imath\mathbf{q})\cdot\mathbf{x}} \\ &= e^{2\imath t-\mathbf{z}\cdot\bar{\mathbf{z}}/2} e^{-(\bar{\mathbf{z}}\cdot\bar{\mathbf{z}}+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\bar{\mathbf{z}}\cdot\mathbf{x}}. \end{aligned}$$

In particular $w_{(t,0)}(\mathbf{x}) = e^{-2\imath t} f_0(\mathbf{x})$, i.e. it really is a vacuum vector with respect to $\widetilde{H}$ in the sense of our definition. Of course $\widetilde{G}/\widetilde{H}$ is isomorphic to $\mathbb{C}^n$. Embedding $\mathbb{C}^n$ in G by the identification of $(0,\mathbf{z})$ with $\mathbf{z}$, the mapping $s : \widetilde{G} \to \widetilde{G}$ is defined simply by $s((t,\mathbf{z})) = (0,\mathbf{z}) = \mathbf{z}$; Ω then is identical with $\mathbb{C}^n$.

The Haar measure on $\mathbb{H}^n$ coincides with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ [76, § 1.1] and so the invariant measure on Ω also coincides with Lebesgue measure on $\mathbb{C}^n$. Note also that the composition law sending $\mathbf{z}_1$ $\mathbf{z}_2$ to $s((0,\mathbf{z}_1)(0,\mathbf{z}_2))$ reduces to Euclidean shifts on $\mathbb{C}^n$. We also find $s((0,\mathbf{z}_1)^{-1}\cdot(0,\mathbf{z}_2)) = \mathbf{z}_2 - \mathbf{z}_1$ and $r((0,\mathbf{z}_1)^{-1}\cdot(0,\mathbf{z}_2)) = (\frac{1}{2}\Im\bar{\mathbf{z}}_1\cdot\mathbf{z}_2, 0)$.

The reduced wavelet transform takes the form of a mapping $L_2(\mathbb{R}^n) \to L_2(\mathbb{C}^n)$ and is given by the formula

$$\begin{aligned} \widehat{\mathcal{W}}f(\mathbf{z}) &= \left\langle f, w_{(0,\mathbf{z})}\right\rangle \\ &= \pi^{-n/4}\int_{\mathbb{R}^n} f(\mathbf{x})\, e^{-\mathbf{z}\cdot\bar{\mathbf{z}}/2}\, e^{-(\mathbf{z}\cdot\mathbf{z}+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\mathbf{z}\cdot\mathbf{x}}\, dx \\ &= e^{-|\mathbf{z}|^2/2}\pi^{-n/4}\int_{\mathbb{R}^n} f(\mathbf{x})\, e^{-(\mathbf{z}\cdot\mathbf{z}+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\mathbf{z}\cdot\mathbf{x}}\, dx, \end{aligned} \tag{4.2.13}$$

where $\mathbf{z} = \mathbf{p} + \imath\mathbf{q}$. Then $\widehat{\mathcal{W}}f$ belongs to $L_2(\mathbb{C}^n, dg)$. This can better be expressed by saying that the function $\breve{f}(\mathbf{z}) = e^{|\mathbf{z}|^2/2}\widehat{\mathcal{W}}f(\mathbf{z})$ belongs to $L_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2}dg)$ because $\breve{f}(\mathbf{z})$ is analytic in $\mathbf{z}$. These functions constitute the *Segal-Bargmann space* [3, 69] $F_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2}dg)$ of functions analytic in $\mathbf{z}$ and square-integrable with respect the Gaussian measure $e^{-|\mathbf{z}|^2}d\mathbf{z}$. Analyticity of $\breve{f}(\mathbf{z})$ is equivalent to the condition that $(\frac{\partial}{\partial\bar{\mathbf{z}}_j} + \frac{1}{2}\mathbf{z}_j I)\mathcal{W}f(\mathbf{z})$ equals zero.

[1] $\widetilde{G}$ is sometimes called the *reduced Heisenberg group*Heisenberg group!reduced. It seems that $\widetilde{G}$ is a virtual object, which is important in connection with a selected representation of G.

The integral in (4.2.13) is the well-known Segal-Bargmann transform [3, 69]. Its inverse is given by a realization of (2.2.8):

$$\begin{aligned} f(\mathbf{x}) &= \int_{\mathbb{C}^n} \widehat{\mathcal{W}f(\mathbf{z})} w_{(0,\mathbf{z})}(\mathbf{x})\, d\mathbf{z} \\ &= \int_{\mathbb{C}^n} \breve{f}(\mathbf{z}) e^{-(\bar{\mathbf{z}}^2+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\bar{\mathbf{z}}x}\, e^{-|\mathbf{z}|^2}\, d\mathbf{z}. \end{aligned} \tag{4.2.14}$$

This gives (2.2.8) the name of Segal-Bargmann inverse. The corresponding operator $\mathcal{P}$ (2.3.7) is the identity operator $L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$ and (2.3.7) gives an integral presentation of the Dirac delta.

Meanwhile the orthoprojection $L_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2}dg) \rightarrow F_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2}dg)$ is of interest and is a principal ingredient in Berezin quantisation [5, 14]. We can easy find its kernel. Indeed, $\widehat{\mathcal{W}}f_0(\mathbf{z}) = e^{-|\mathbf{z}|^2}$, and the kernel is

$$\begin{aligned} K(\mathbf{z},\mathbf{w}) &= \widehat{\mathcal{W}}f_0(\mathbf{z}^{-1}\cdot\mathbf{w})\bar{\chi}(r(\mathbf{z}^{-1}\cdot\mathbf{w})) \\ &= \widehat{\mathcal{W}}f_0(\mathbf{w}-\mathbf{z})\exp(\imath\Im(\bar{\mathbf{z}}\cdot\mathbf{w}) \\ &= \exp(\frac{1}{2}(-|\mathbf{w}-\mathbf{z}|^2 + \mathbf{w}\cdot\bar{\mathbf{z}} - \mathbf{z}\cdot\bar{\mathbf{w}})) \\ &= \exp(\frac{1}{2}(-|\mathbf{z}|^2 - |\mathbf{w}|^2) + \mathbf{w}\cdot\bar{\mathbf{z}}). \end{aligned}$$

To obtain the reproducing kernel for functions $\breve{f}(\mathbf{z}) = e^{|\mathbf{z}|^2}\widehat{\mathcal{W}}f(\mathbf{z})$ in the Segal-Bargmann space we multiply $K(\mathbf{z},\mathbf{w})$ by $e^{(-|\mathbf{z}|^2+|\mathbf{w}|^2)/2}$ which gives the standard reproducing kernel, $\exp(-|\mathbf{z}|^2 + \mathbf{w}\cdot\bar{\mathbf{z}})$ [3, (1.10)].

The Segal-Bargmann space is an interesting and important object, but there are also other options. In particular we can consider an alternative representation of the Heisenberg group, this time acting on monogenic functions, an action we introduce in the next subparagraph.

4.2.3 Representation of $\mathbb{H}^n$ in Spaces of Monogenic Functions

We consider the real Clifford algebra $\mathcal{C}\ell(n)$, i.e. the algebra generated by $e_0 = 1$, e_j, $1 \leq j \leq n$, using the identities:

$$e_ie_j + e_je_i = -2\delta_{ij}, \qquad 1 \leq i,j \leq n.$$

For a function f with values in $\mathcal{C}\ell(n)$, the action of the Dirac operator of $\mathbb{R}^{n+1}$ is defined by (here $x = x_0 + \mathbf{x}$ is the $n+1$ dimensional variable)

$$Df(x) = \sum_{i=0}^{n} \partial_i f(x).$$

A function f satisfying $Df = 0$ in a certain domain is called monogenic there; later on we shall use the term 'monogenic' for solutions of more general Dirac operators. Obviously the notion of monogenicity is closely related to the one of holomorphy on the complex plane. As a matter of fact $D^2 = -\Delta$, and monogenic functions are solutions of the Laplacian. The Clifford algebra is not commutative, and so it is necessary to introduce a symmetrized product. For k elements a_i, $1 \leq i \leq k$ of the algebra it is defined by

$$a_1 \times a_2 \times \ldots \times a_k = \frac{1}{k!} \sum_{\sigma} a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)},$$

where the sum is taken over all possible permutations of k elements. If the same element appears several times, we use an exponent notation, e.g. $a^2 \times b^3 = a \times a \times b \times b \times b$.

Let now V_k be the symmetric power monomial defined by the expression

$$V_k(\mathbf{x}) = \frac{1}{\sqrt{k!}} (e_1 x_0 - e_0 x_1)^{k_1} \times (e_2 x_0 - e_0 x_2)^{k_2} \times \cdots \times (e_n x_0 - e_0 x_n)^{k_n}. \quad (4.2.15)$$

It can be proved that these monomials are all monogenic (see e.g. [55]), and even that they constitute a basis for the space of monogenic polynomials (as a module over $\mathcal{C}\ell(n)$). In general the symmetrized product is not associative, and manipulating it can become quite formal. However, if we restrict the monomials defined above to the hyperplane $x_0 = 0$, we obtain

$$V_k(x) = \frac{1}{\sqrt{k!}} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n},$$

and so we have the multiplicative property

$$\sqrt{\frac{k!k'!}{(k+k)!}} V_k V_{k'} = V_{k+k'}, \quad x_0 = 0.$$

Another important function is the monogenic exponential function which is defined by

$$E(\mathbf{u}, x) = \exp(\mathbf{u} \cdot \mathbf{x}) \left(\cos(\|\mathbf{u}\| \, x_0) - \frac{\mathbf{u}}{\|\mathbf{u}\|} \sin(\mathbf{u} x_0) \right).$$

It is not hard to check [9, § 14] that this function is monogenic, and of course its restriction to the hyperplane $x_0 = 0$ is simply the exponential function, $E(\mathbf{u}, \mathbf{x}) = \exp(\mathbf{u} \cdot \mathbf{x})$.

We can therefore extend the symmetric product by the so-called Cauchy-Kovalevskaya product [9, § 14]: If f and g are monogenic in $\mathbb{R}^{n+1}$, then $f \times g$

is the monogenic function equal to fg on $\mathbb{R}^n$. Introducing the monogenic functions $\mathbf{x}_i = e_i x_0 - e_0 x_i$ we can then write

$$V_k(x) = \frac{1}{\sqrt{k!}} x_1^{k_1} \times x_2^{k_2} \times \ldots \times x_n^{k_n}.$$

It is fairly easy to check the V_k form an orthonormal set with respect to the following inner product (see [11, § 3.1] on Clifford valued inner products):

$$\langle V_k, V_{k'} \rangle = \int_{\mathbb{R}^{n+1}} \bar{V}_k(x) V_{k'}(x)\, e^{-|x|^2}\, dx. \tag{4.2.16}$$

Let M_2 be closure of the linear span of $\{V_k\}$, using complex coefficients.

The creation and annihilation operators a_k^+ and a_k^- can be represented by symmetric multiplication (see [55]) with the monogenic variable $\mathbf{x}_j$, which will be written $\mathbf{x}_k I_\times$, and by the (classical) partial derivative $\frac{\partial}{\partial \mathbf{x}_j} = \frac{\partial}{\partial x_j}$ with respect to $\mathbf{x}_j$, which appear in the definition of hypercomplex differentiability. On basis elements they act as follows:

$$\begin{aligned}
\mathbf{x}_j I_\times V_{(k_1,\ldots,k_j,\ldots,k_n)} &= \sqrt{k_j+1} V_{(k_1,\ldots,k_j+1,\ldots,k_n)}, \\
\frac{\partial}{\partial \mathbf{x}_j} V_{(k_1,\ldots,k_j,\ldots,k_n)} &= \sqrt{k_j} V_{(k_1,\ldots,k_j-1,\ldots,k_n)},
\end{aligned}$$

It can be checked that this really is a representation of $a_k^\pm$, and that a_k^+ and a_k^- are each other's adjoint. We use the equalities $a_j^- = \frac{1}{\sqrt{2}}(a_j^+ + a_j^-)$ and $a_j^+ = \frac{\imath}{\sqrt{2}}(a_j^- - a_j^+)$, and the commutation relations $[a_i^+, a_j^-] = e\delta_{ij}$ to obtain a representation of the Heisenberg group. Thus an element $(t, \mathbf{z})$, $\mathbf{z} = \mathbf{u} + i\mathbf{v}$ of the Heisenberg group can be written as

$$\begin{aligned}
(t, \mathbf{z}) &= \left(t + \frac{\mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}}{4}, 0\right)\left(0, \frac{(1+\imath)(\mathbf{u}+\mathbf{v})}{2}\right)\left(0, \frac{(1-\imath)(\mathbf{u}-\mathbf{v})}{2}\right) \\
&= \exp\left(\left(t + \frac{\mathbf{u}^2 - \mathbf{v}^2}{4}\right) e\right) \exp\left(\frac{(\mathbf{u}+\mathbf{v})q}{\sqrt{2}}\right) \exp\left(\frac{(\mathbf{u}-\mathbf{v})\imath p}{\sqrt{2}}\right).
\end{aligned}$$

It is therefore represented by the operator

$$\begin{aligned}
\pi_{(t,\mathbf{z})} &= \exp\left(-\left(t + \frac{\mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}}{4}\right)\right) \\
&\quad \exp\left(\frac{((\mathbf{u}+\mathbf{v})\cdot\mathbf{x}) I_\times}{\sqrt{2}}\right) \exp\left(\frac{(\mathbf{u}-\mathbf{v})\cdot(\partial_{\mathbf{x}})}{\sqrt{2}}\right),
\end{aligned} \tag{4.2.17}$$

where obviously for a monogenic function f we have

$$\begin{aligned}
\exp\left(\frac{(\mathbf{u}-\mathbf{v})\imath p}{\sqrt{2}}\right) f(x) &= f\left(x + \frac{\mathbf{u}-\mathbf{v}}{\sqrt{2}}\right) \\
\exp\left(\frac{((\mathbf{u}+\mathbf{v})\cdot\mathbf{x}) I_\times}{\sqrt{2}}\right) f(x) &= E\left(\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}}, \cdot\right) \times f(x)
\end{aligned}$$

Therefore it is easy to calculate the image of the constant function $f_0(\mathbf{x}) = V_0(\mathbf{x}) \equiv 1$, and we obtain the set of functions

$$\begin{aligned} f_{(t,\mathbf{z})}(\mathbf{x}) &= \pi_{(t,\mathbf{z})} f_0(\mathbf{x}) \\ &= \exp\left(-\left(t + \frac{\mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}}{4}\right)\right) E\left(\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}}, \cdot\right) \times f_0(x) \\ & \exp\left(-\left(t + \frac{\mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}}{4}\right)\right) E\left(\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}}, x\right). \end{aligned} \tag{4.2.18}$$

In the language of quantum physics $f_0(\mathbf{x})$ is the *vacuum vector* and functions $f_{(t,\mathbf{z})}(\mathbf{x})$ are *coherent states* (or *wavelets*) for the representation of $\mathbb{H}^n$ we described. We can summarize the properties of the representation:

(i). All functions in M_2 are complex-vector valued, monogenic in $\mathbb{R}^{n+1}$, and square integrable with respect to the measure $e^{-|x|^2}dx$.

(ii). The representation of the Heisenberg group is given by (4.2.17). This representation generates a set of coherent states $f_{(0,z)}(\mathbf{x})$ (4.2.18) as shifts of the vacuum vector $f_0(\mathbf{x}) \equiv 1$.

(iii). The creation and annihilation operators a_k^+ and a_k^- are represented by symmetric (Cauchy-Kovalevskaya) multiplication by $\mathbf{x}_j$ and by derivation of monogenic functions. They are adjoint with respect to the inner product (4.2.16).

(iv). M_2 is generated as a closed linear space by the orthonormal basis $V_k(\mathbf{x}) = \frac{1}{\sqrt{k!}}(e_1x_0 - e_0x_1)^{k_1} \times (e_2x_0 - e_0x_2)^{k_2} \times \cdots \times (e_nx_0 - e_0x_n)^{k_n}$, and also by the set of coherent states $f_{(t,\mathbf{z})}(\mathbf{x})$ of (4.2.18).

(v). The kernel of the operator intertwining the model constructed here and the Segal-Bargmann one is given by

$$B(\mathbf{z}, x) \sum_{j=0}^{\infty} V_j(x) \frac{\mathbf{z}^j}{\sqrt{j!}} = \exp(\sum_{k=1}^{n} \mathbf{x}_k \bar{z}_k),$$

which is the holomorphic extension in $\mathbf{z} = \mathbf{u} + \imath\mathbf{v}$ of $E(\mathbf{u}, x)$. The transformation pair is given by

$$\begin{aligned} \mathcal{B}f(x) &= \int_{\mathbb{C}^n} B(\mathbf{z}, x) f(\mathbf{z}) \exp\left(\frac{-|\mathbf{z}|^2}{2}\right) d\mathbf{z} \\ \mathcal{B}^{-1}\phi(\mathbf{z}) &= \int_{\mathbb{R}^{n+1}} \overline{B(\mathbf{z}, x)} \phi(x) \exp\left(\frac{-|x|^2}{2}\right) dx \end{aligned}$$

(vi). The space M_2 has a reproducing kernel

$$K(x,y)=\sum_{k=0}^{\infty} V_k(x)\bar{V}_k(y)=\int_{\mathbb{C}^n} B(\mathbf{z},x)\overline{B(\mathbf{z},y)}\, e^{-|z|^2}dz.$$

Notice that $\overline{K(x,y)}$ is monogenic in y; it is the monogenic extension of $\overline{E(\mathbf{y},x)}$.

One can see that some properties of M_2 are closer to those of the Segal-Bargmann space than to those of the space $L_2(\mathbb{R}^n)$ it replaces. It should be noted that the representation of the Heisenberg group we obtained here is new and quite unexpected.

Remark 4.2.2 We construct M_2 as a space of complex-vector valued functions. We can also consider an extended space $\widetilde{M}_2$ being generated by the orthonormal basis $V_k(\mathbf{x})$ or coherent states $f_{(0,z)}(\mathbf{x})$ with Clifford valued coefficients multiplied from the right hand side. Such a space will share many properties of $\mathbb{M}^2$ and have an additional structure: there is a natural representation $s : f(\mathbf{x}) \mapsto s^* f(s\mathbf{x}s^*)s$ of Spin (n) group in $\widetilde{M}_2$. Thus this space provides us with a representation of two main symmetries in quantum field theory: the Heisenberg group of quantized coordinate and momentum (external degrees of freedom) and Spin (n) group of quantified inner degrees of freedom. Another composition of the Heisenberg group and Clifford algebras can be found in [32].

4.3 Another Nilpotent Lie Group and Its Representation

4.3.1 Clifford Algebra and Complex Vectors

Starting from the real Clifford algebra $\mathcal{C}\ell(n)$, we consider complex n-vector valued functions defined on the real line $\mathbb{R}^1$ with values in $\mathbb{C}^n$. Moreover we will look at the j-th component of $\mathbb{C}^n$ as being spanned by the elements 1 and e_j of the Clifford algebra. For two vectors $\mathbf{u}=(u_1,\ldots,u_n)$ and $\mathbf{v}=(v_1,\ldots,v_n)$ we introduce the Clifford vector valued product (see [11, § 3.1] on Clifford valued inner products):

$$\mathbf{u}\cdot\mathbf{v}=\sum_{j=1}^{n}\bar{u}_j v_j=\sum_{j=1}^{n}(u'_j-u''_j e_j)(v'_j+v''_j e_j), \tag{4.3.1}$$

where $u_j = u'_j + u''_j e_j$ and $v_j = v'_j + v''_j e_j$. Of course, $u \cdot u$ coincides with $\|u\|^2 = \sum_1^n (u_j'^2 + u_j''^2)$, the standard norm in $\mathbb{C}^n$. So we can introduce the space $R_2(\mathbb{R}^1)$ of $\mathbb{C}^n$-valued functions on the real line with the product

$$\langle f, f' \rangle = \int_{\mathbb{R}^1} f(x) \cdot f'(x)\, dx. \tag{4.3.2}$$

Again $\langle f, f \rangle^{1/2}$ gives us the standard norm in the Hilbert space of L_2 integrable $\mathbb{C}^n$ valued functions.

4.3.2 A nilpotent Lie group

We introduce a nilpotent Lie group, $\mathbb{G}^n$. As a C^∞-manifold it coincides with $\mathbb{R}^{2n+1}$. Its Lie algebra has generators P, Q_j, T_j, $1 \le j \le n$. The non-trivial commutators between them are

$$[P, Q_j] = T_j; \tag{4.3.3}$$

all others vanish. Particularly $\mathbb{G}^n$ is a step two nilpotent Lie group and the T_j span its centre. It is easy to see that $\mathbb{G}^1$ is just the Heisenberg group $\mathbb{H}^1$.

We denote a point g of $\mathbb{G}^n$ by $2n+1$-tuple of reals $(t_1, \ldots, t_n; p; q_1, \ldots, q_n)$. These are the exponential coordinates corresponding to the basis of the Lie algebra $T_1, \ldots, T_n, P, Q_1, \ldots, Q_n$. The group law is given in exponential coordinates by the formula

$$\begin{aligned} &(t_1, \ldots, t_n; p; q_1, \ldots, q_n) * (t'_1, \ldots, t'_n; p'; q'_1, \ldots, q'_n) = \\ &= \quad (t_1 + t'_1 + \frac{1}{2}(p'q_1 - pq'_1), \ldots, t_n + t'_n + \frac{1}{2}(p'q_n - pq'_n); \\ &\qquad p + p'; q_1 + q'_1, \ldots, q_n + q'_n). \end{aligned} \tag{4.3.4}$$

We consider the homogeneous space $\Omega = \mathbb{G}^n/\mathbb{Z}$. Here $\mathbb{Z}$ is the centre of $\mathbb{G}^n$; its Lie algebra is spanned by T_j, $1 \le j \le n$. It is easy to see that $\Omega \sim \mathbb{R}^{n+1}$. We define the mapping $s : \Omega \to \mathbb{G}^n$ by the rule

$$s : (a_0, a_1, \ldots, a_n) \mapsto (0, \ldots, 0; a_0; a_1, \ldots, a_n). \tag{4.3.5}$$

It is the "inverse" of the natural projection $s^{-1} : \mathbb{G}^n \to \Omega = \mathbb{G}^n/\mathbb{Z}$.

It easy to see that the mapping $\Omega \times \Omega \to \Omega$ defined by the rule $s^{-1}(s(a) * s(a'))$ is just Euclidean (coordinate-wise) addition $a + a'$.

To introduce the Dirac operator we will need the following set of left-invariant differential operators, which generate right shifts on the group:

$$T_j = \frac{\partial}{\partial t_j}, \tag{4.3.6}$$

$$P = \frac{\partial}{\partial p} + \frac{1}{2}\sum_1^n q_j \frac{\partial}{\partial t_j}, \tag{4.3.7}$$

$$Q_j = -\frac{\partial}{\partial q_j} + \frac{1}{2} p \frac{\partial}{\partial t_j}. \tag{4.3.8}$$

The corresponding set of right invariant vector fields generating left shifts is

$$T_j^* = \frac{\partial}{\partial t_j}, \tag{4.3.9}$$

$$P^* = \frac{\partial}{\partial p} - \frac{1}{2}\sum_1^n q_j \frac{\partial}{\partial t_j}, \tag{4.3.10}$$

$$Q_j^* = -\frac{\partial}{\partial q_j} - \frac{1}{2} p \frac{\partial}{\partial t_j}. \tag{4.3.11}$$

A general property is that any left invariant operator commutes with any right invariant one.

4.3.3 A representation of $\mathbb{G}^n$

We introduce a representation ρ of $\mathbb{G}^n$ in the space $R_2(\mathbb{R})$ by the formula:

$$[\rho_g f](x) = (e^{e_1(2t_1+q_1(\sqrt{2}x-p))} f_1(x-\sqrt{2}p), \dots, e^{e_n(2t_n+q_n(\sqrt{2}x-p))} f_n(x-\sqrt{2}p)), \tag{4.3.12}$$

where $f(x) = (f_1(x), \dots, f_n(x))$ and the meaning of $R_2(\mathbb{R})$ was discussed in Subsection 4.3.1. We note that the generators e_j of Clifford algebras do not interact with each other under the representation just defined. One can check directly that (4.3.12) defines a representation of $\mathbb{G}^n$. Indeed:

$$\begin{aligned}
[\rho_g \rho_{g'} f](x) &= \rho_g(e^{e_1(2t_1'+q_1'(\sqrt{2}x-p'))} f_1(x-\sqrt{2}p'), \dots, \\
&\qquad e^{e_n(2t_n'+q_n'(\sqrt{2}x-p'))} f_n(x-\sqrt{2}p')) \\
&= (e^{e_1(2t_1+q_1(\sqrt{2}x-p))} e^{e_1(2t_1'+q_1'(\sqrt{2}(x-\sqrt{2}p)-p'))} f_1(x-\sqrt{2}p-\sqrt{2}p'), \\
&\quad \dots, \\
&\quad e^{e_n(2t_n+q_n(\sqrt{2}x-p))} e^{e_n(2t_n'+q_n'(\sqrt{2}(x-\sqrt{2}p)-p'))} f_n(x-\sqrt{2}p-\sqrt{2}p')) \\
&= (e^{e_1(2(t_1+t_1'+\frac{1}{2}(p'q_1-pq_1'))+(q_1+q_1')(\sqrt{2}x-(p+p')))} f_1(x-\sqrt{2}(p+p')), \\
&\quad \dots, \\
&\quad (e^{e_n(2(t_n+t_n'+\frac{1}{2}(p'q_n-pq_n'))+(q_n+q_n')(\sqrt{2}x-(p+p')))} f_n(x-\sqrt{2}(p+p')) \\
&= [\rho_{gg'} f](x),
\end{aligned} \tag{4.3.13}$$

where gg' is defined by (4.3.4).

ρ_g has the important property that it preserves the product (4.3.2). Indeed:

$$\begin{aligned}
\langle \rho_g f, \rho_g f' \rangle &= \int_{\mathbb{R}} [\rho_g f](x) \cdot [\rho_g f'](x)\, dx \\
&= \int_{\mathbb{R}} \sum_{j=1}^{n} \bar{f}_j(x - \sqrt{2}p) e^{-e_j(2t_j + q_j(\sqrt{2}x - p))} \\
&\qquad\qquad e^{e_j(2t_j + q_j(\sqrt{2}x - p))} f'_j(x - \sqrt{2}p)\, dx \\
&= \int_{\mathbb{R}} \sum_{j=1}^{n} \bar{f}_j(x - \sqrt{2}p) f'_j(x - \sqrt{2}p)\, dx \\
&= \int_{\mathbb{R}} \sum_{j=1}^{n} \bar{f}_j(x) f'_j(x)\, dx \\
&= \langle f, f' \rangle .
\end{aligned}$$

Thus ρ_g is *unitary* with respect to the Clifford valued inner product (4.3.2). Notice this notion is stronger than unitarity for the scalar valued inner product, as the latter is the trace of the Clifford valued one. A proof of unitarity could also consist of proving the action of the Lie algebra is skew-symmetric, i.e. that for an element b of the Lie algebra and f arbitrary

$$\langle d\rho_b f, f \rangle = \langle f, -d\rho_b f \rangle .$$

Here $d\rho_b$ is derived representation of d for an element $b \in \mathfrak{g}_n$ of the Lie algebra of $\mathbb{G}^n$. In the next subsection we will need the explicit form of it. For the selected basis of $\mathfrak{g}_n$ we have:

$$\begin{aligned}
[d\rho(T_j) f](x) &= (0, 0, \ldots, 0, 2e_1 f_j(x), 0, \ldots, 0, 0); \\
[d\rho(P) f](x) &= (-\sqrt{2}\frac{\partial}{\partial x} f_1(x), \ldots, -\sqrt{2}\frac{\partial}{\partial x} f_j(x), \ldots, -\sqrt{2}\frac{\partial}{\partial x} f_n(x)); \\
[d\rho(Q_j) f](x) &= (0, 0, \ldots, 0, \sqrt{2} e_j x f_j(x), 0, \ldots, 0, 0).
\end{aligned}$$

Particularly $d\rho(Q_j) d\rho(Q_k) = 0$ for all $j \neq k$. This does not follow from the structure of $\mathbb{G}$ but is a feature of the described representation.

REMARK 4.3.1 The group $\mathbb{G}^n$ is called as "a generalized Heisenberg group" in [51] where its induced representations are considered.

4.3.4 The wavelet transform for $\mathbb{G}^n$

In $R_2(\mathbb{R})$ we have the $\mathbb{C}^n$-valued function

$$f_0(x) = (e^{-x^2/2}, \ldots, e^{-x^2/2}), \tag{4.3.14}$$

which which is the *vacuum vector* in this case. It is a zero eigenvector for the operator

$$a^- = d\rho(P) - \sum_{j=1}^{n} e_j d\rho(Q_j), \tag{4.3.15}$$

which is *the only annihilating operator* in this model. But we still have n creation operators:

$$a_k^+ = d\rho(P) - \sum_{j=1}^{n}(1-2\delta_{jk})e_j d\rho(Q_j) = a^- + 2e_k d\rho(Q_k). \tag{4.3.16}$$

While a^- and a_k^+ look a little bit exotic for $\mathbb{G}^1 = \mathbb{H}^1$ they are exactly the standard annihilation and creation operators. Another feature of the representation is that the a_k^+ do not commute with each other and have a non-trivial commutator with a^-:

$$[a_j^+, a_k^+] = 2e_k d\rho(T_k) - 2e_j d\rho(T_j), \qquad [a_j^+, a^-] = -2e_j d\rho(T_j)$$

We need the transforms of $f_0(x)$ under the action (4.3.12), i.e. the *coherent states* $f_g(x) = [\rho_g f_0](x)$ in this model:

$$\begin{aligned} f_g(x) &= (\ldots, e^{e_j(2t_j+q_j(\sqrt{2}x-p))}e^{-(x-\sqrt{2}p)^2/2}, \ldots) \\ &= (\ldots, e^{2e_jr_j-(p^2+q_j^2)/2}e^{-((p-e_jq_j)^2+x^2)/2+\sqrt{2}(p-e_jq_j)x}, \ldots) \\ &= (\ldots, e^{2e_jt_j-z_j\bar{z}_j/2}e^{-(\bar{z}_j^2+x^2)/2+\sqrt{2}\bar{z}_jx}, \ldots) \end{aligned} \tag{4.3.17}$$

where $z_j = p + e_j q_j$, $\bar{z}_j = p - e_j q_j$.

Having defined coherent states we can introduce the *wavelet transform* $\mathcal{W} : R_2(\mathbb{R}) \to L_\infty(\mathbb{G}^n)$ by the standard formula:

$$\mathcal{W}f(g) = \langle f, f_g \rangle . \tag{4.3.18}$$

Calculations completely analogous to those of the complex case allow us to find the images $\mathcal{W}f_{(t',\mathbf{a})}(t,\mathbf{z})$ of coherent states $f_{(t',\mathbf{a})}(x)$ under (4.3.18) as follows:

$$\begin{aligned} \mathcal{W}f_{(t',\mathbf{a})}(t,\mathbf{z}) &= \left\langle f_{(t',\mathbf{a})}, f_{(t,\mathbf{z})} \right\rangle \\ &= \int_{\mathbb{R}} \sum_{j=1}^{n} \exp\left(-2e_jt_j - \frac{z_j\bar{z}_j}{2} - \frac{z_j^2+x^2}{2} + \sqrt{2}z_jx\right) \\ &\qquad \exp\left(+2e_jt'_j - \frac{a_j\bar{a}_j}{2} - \frac{\bar{a}_j^2+x^2}{2} + \sqrt{2}\bar{a}_jx\right) dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \exp\left(-2e_j t_j - \frac{z_j \bar{z}_j}{2} + 2e_j t'_j - \frac{a_j \bar{a}_j}{2} + \bar{a}_j z_j\right) \\
&\qquad \times \int_{\mathbb{R}} \exp(-x^2 + 2x\frac{z_j + \bar{a}_j}{\sqrt{2}} - \frac{(z_j + \bar{a}_j)^2}{2})\, dx \\
&= \sum_{j=1}^{n} \exp\left(-2e_j (t_j - t'_j) - \frac{z_j \bar{z}_j + a_j \bar{a}_j}{2} + \bar{a}_j z_j\right) \\
&\qquad \times \int_{\mathbb{R}} \exp(-(x - \frac{z_j + \bar{a}_j}{\sqrt{2}})^2)\, dx \\
&= \sum_{j=1}^{n} \exp\left(-2e_j (t_j - t'_j) - \frac{z_j \bar{z}_j + a_j \bar{a}_j}{2} + \bar{a}_j z_j\right) \qquad (4.3.19)
\end{aligned}
$$

Here $a_j = a_0 + e_j a_j$, $\bar{a}_j = a_0 - e_j a_j$; z_j, $\bar{z}_j$ were defined above.

In this case all $\mathcal{W} f_{(t',\mathbf{a})}(t, \mathbf{z})$ are *monogenic* functions with respect to the following Dirac operator:

$$
\frac{\partial}{\partial p} - \sum_{j=1}^{n} e_j \frac{\partial}{\partial q_j} + \frac{1}{2} \sum_{j=1}^{n} (e_j p + q_j) \frac{\partial}{\partial t_j}, \qquad (4.3.20)
$$

with z_j related to p and q_j as above. This can be checked by the direct calculation or follows from the observation: the Dirac operator (4.3.20) is the image of the annihilation operator a^- (4.3.15) under the wavelet transform (4.3.18). The situation is completely analogous to the Segal-Bargmann case, where holomorphy is defined by the operators $\frac{\partial}{\partial \bar{z}_k}$, which are the images of the annihilation operators a_k^-. Actually, it is the Dirac operator associated with the unique left invariant metric on $\mathbb{G}^n/\mathbb{Z}$ for which P together with the Q_k forms an orthonormal basis in the origin, and therefore everywhere.

The operator (4.3.20) is a realization of a generic Dirac operator constructed for a nilpotent Lie group, see [15]. Indeed the operator (4.3.20) is defined by the formula $D = P + \sum_1^n e_j Q_j$, where P and Q_j are the left invariant vector fields in (4.3.6)–(4.3.8). So the operator (4.3.20) is left invariant and one has only to check the monogenicity of $\mathcal{W} f_{(0,0)}(t, \mathbf{z})$—all other functions $\mathcal{W} f_{(t',\mathbf{a})}(t, \mathbf{z})$ are its left shifts.

Of course all linear combinations of the $\mathcal{W} f_{(t',\mathbf{a})}(t, \mathbf{z})$ are also monogenic. So if we define two function spaces, R_2 and M_2, as being the closure of the linear span of all $f_g(x)$ and $\mathcal{W} f_{(t',\mathbf{a})}(t, \mathbf{z})$ respectively, then

(i). M_2 is a space of monogenic function on $\mathbb{G}^n$ in the sense above.

(ii). $\mathbb{G}^n$ has representations both in R_2 and in M_2. On the second space the group acts via left regular representation.

(iii). These representation are intertwining by the integral transformation with the kernel $T(t', \mathbf{a}, x) = f_{(t',\mathbf{a})}(x)$.

(iv). The space M_2 has a reproducing kernel $K(t', \mathbf{a}, t, z) = \mathcal{W} f_{(t',\mathbf{a})}(t, \mathbf{z})$.

The standard wavelet transform can be processed as expected.

For the reduced wavelet transform associated with the mapping $s : \Omega \to \mathbb{G}^n$ in particular we have

$$\widehat{\mathcal{W}} f_{\mathbf{a}}(\mathbf{z}) = \mathcal{W} f_{(0,\mathbf{a})}(0, \mathbf{z}) = \sum_{j=1}^{n} \exp \bar{a}_j z_j.$$

However the reduced wavelet transform cannot be constructed from a single vacuum vector. We need exactly n linearly independent vacuum vectors and the corresponding multiresolution wavelet analysis (wavelet transform with several independent vacuum vectors) which is outlined in [10] (see also M.G. Krein's works [49] on "directing functionals"). Indeed we have n different vacuum vectors $(\ldots, 0, e^{-x^2/2}, 0, \ldots)$ each of which is an eigenfunction for the action of the centre of $\mathbb{G}^n$. All functions $\widehat{\mathcal{W}} f_{\mathbf{a}}(\mathbf{z})$ are *monogenic* with respect to the Dirac operator

$$D = \frac{\partial}{\partial p} + \sum_{j=1}^{n} e_j \frac{\partial}{\partial q_j}. \tag{4.3.21}$$

REMARK 4.3.2 It may seem on the first glance that the theory constructed in this section is only (if any) of pure mathematical interest and could not be related to physical reality. However the situation with an matching number of coordinates and momenta appears in very promising approach to *quantum field theory*, see [27] and references therein. This series of papers also come to the conclusion that a proper quantum picture in that setting requires Clifford algebras.

Appendix A

Groups and Homogeneous Spaces

The group theory and the representation theory are two enormous and interesting subjects themselves. However they are auxiliary in our consideration and we are forced to restrict ourselves only to brief and very dry overview.

Besides introduction to that areas presented in [57, 77] we recommend additionally the books [29, 76]. The representation theory intensively uses tools of functional analysis and on the other hand inspires its future development. We use the book [30] for references on functional analysis here and recommend it as a nice reading too.

A.1 Basics of Group Theory

We start from the definition of central object which formalizes the universal notion of symmetries.

Definition A.1.1 A *transformation group* G is a nonvoid set of mappings of a certain set X into itself with the following properties:

(i). if $g_1 \in G$ and $g_2 \in G$ then $g_1 g_2 \in G$;

(ii). if $g \in G$ then g^{-1} exists and belongs to G.

Exercise A.1.2 List all transformation groups on a set of three elements.

Exercise A.1.3 Verify that the following are groups in fact:

(i). Group of permutations of n elements;

(ii). Group of $n \times n$ matrixes with non zero determinant over a field $\mathbb{F}$ under matrix multiplications;

(iii). Group of rotations of the unit circle $\mathbb{T}$;

(iv). Groups of shifts of the real line $\mathbb{R}$ and plane $\mathbb{R}^2$;

(v). Group of linear fractional transformations of the extended complex plane.

Definition A.1.4 An *abstract group* (or simply *group*) is a nonvoid set G on which there is a law of *group multiplication* (i.e. mapping $G \times G \to G$) with the properties

(i). *associativity*: $g_1(g_2g_3) = (g_1g_2)g_3$;

(ii). the existence of *identity*: $e \in G$ such that $eg = ge = g$ for all $g \in G$;

(iii). the existence of *inverse*: for every $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

Exercise A.1.5 Check that any transformation group is an abstract group.

Exercise A.1.6 Check that the following transformation groups (cf. Example A.1.3) have the same law of multiplication, i.e. are equivalent as abstract groups:

(i). The group of isometric mapping of an equilateral triangle onto itself;

(ii). The group of all permutations of a set of free elements;

(iii). The group of invertible matrix of order 2 with coefficients in the field of integers modulo 2;

(iv). The group of linear fractional transformations of the extended complex plane generated by the mappings $z \mapsto z^{-1}$ and $z \mapsto 1 - z$.

Exercise* A.1.7 Expand the list in the above exercise.

It is simpler to study groups with the following additional property.

Definition A.1.8 A group G is *commutative* if for all g_1, $g_2 \in G$, we have $g_1g_2 = g_2g_1$.

Most of interesting and important groups are *noncommutative*, however.

Exercise A.1.9 (i). Which groups among found in Exercise A.1.2 are commutative?

(ii). Which groups among listed in Exercise A.1.3 are noncommutative?

Groups could have some additional analytical structures, e.g. they could be a topological sets with a corresponding notion of limit. We always assume that our groups are *locally compact* [29, § 2.4].

Definition A.1.10 If for a group G the group multiplication and the taking of inverse are continuous mappings then G is *continuous group*.

Even a better structure could be found among *Lie groups* [29, § 6], e.g. groups with a differentiable law of multiplication. Investigating such groups we could employ the whole arsenal of analytical tools, thereafter most of groups studied in this notes will be Lie groups.

Exercise A.1.11 Check that the following are noncommutative Lie (and thus continuous) groups:

(i). [76, Chap. 7] The *$ax+b$ group*: set of elements (a,b), $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ with the group law:

$$(a,b) * (a',b') = (aa', ab'+b).$$

The identity is $(1,0)$, and $(a,b)^{-1} = (a^{-1}, -b/a)$.

(ii). The *Heisenberg group* [25], [76, Chap. 1]: a set of triples of real numbers (s,x,y) with the group multiplication:

$$(s,x,y) * (s',x',y') = (s+s'+\frac{1}{2}(x'y-xy'), x+x', y+y'). \tag{A.1.1}$$

The identity is $(0,0,0)$, and $(s,x,y)^{-1} = (-s,-x,-y)$.

(iii). The $SL_2(\mathbb{R})$ group [26, 53]: a set of 2×2 matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries a, b, c, $d \in \mathbb{R}$, the determinant $\det = ad - bc$ equal to 1 and the group law coinciding with matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}.$$

The identity is the unit matrix and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The above three groups are behind many important results of real and complex analysis [25, 26, 53] and we meet them many times in these notes.

A.2 Homogeneous Spaces and Invariant Measures

While abstract group are a suitable language for investigation of their general properties we meet groups in applications as transformation groups acting on a set X.

Let X be a set and let be defined an operation $G : X \to X$ of G on X. There is an equivalence relation on X, say, $x_1 \sim x_2 \Leftrightarrow \exists g \in G : gx_1 = x_2$, with respect to which X is a disjoint union of distinct *orbits* [52, § I.5].

Exercise A.2.1 Let action of $SL_2(\mathbb{R})$ group on $\mathbb{C}$ by means of *linear-fractional transformations*:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

Show that there three orbits: the real axis $\mathbb{R}$, *upper (lower) half plane* $\mathbb{R}^n_\pm$:

$$\mathbb{R}^n_\pm = \{x \pm iy \mid x, y \in \mathbb{R},\ y > 0\}.$$

Thus from now on, without lost of a generality, we assume that the operation of G on X is *transitive*, i. e. for every $x \in X$ we have

$$Gx := \bigcup_{g \in G} g(x) = X.$$

In this case X is *G-homogeneous space*.

Exercise A.2.2 Show that for any group G we could define its action on $X = G$ as follows:

(i). The *conjugation* $g : x \mapsto gxg^{-1}$ (which is even a group homomorphism, but is trivial for all commutative groups).

(ii). The *left shift* $\lambda(g) : x \mapsto gx$ and the *right shift* $\rho(g) : x \mapsto xg^{-1}$.

If we fix a point $x \in X$ then the set of elements $G_x = \{g \in G \mid g(x) = x\}$ obviously forms the *isotropy (sub)group* of x in G [52, § I.5]. The set X is in the bijection with the factor set G/G_x for any $x \in X$.

Exercise A.2.3 Find a subgroup which correspond to the given action of G on X:

(i). Action of $ax + b$ group on $\mathbb{R}$ by the formula: $(a, b) : x \mapsto ax + b$.

(ii). Action of $SL_2(\mathbb{R})$ group on one of three orbit from Exercise A.2.1.

To do some analysis on groups we need suitably defined basic operation: differentiation and integration. The first operation is naturally defined for Lie group. If G is a Lie group then the homogeneous space G/G_x is a smooth manifold (and a *loop* as an algebraic object) for every $x \in X$. Therefore the one-to-one mapping $G/G_x \rightarrow X : g \mapsto g(x)$ induces a structure of C^∞-manifold on S. Thus the class $C_0^\infty(X)$ of smooth functions with compact supports on x has the evident definition.

In order to perform an integration we need a suitable *measure*. A smooth measure $d\mu$ on X is called (left) *invariant measure* with respect to an operation of G on X if

$$\int_X f(x)\, d\mu(x) = \int_X f(g(x))\, d\mu(x), \quad \text{for all } g \in G,\ f(x) \in C_0^\infty(X). \tag{A.2.1}$$

Exercise A.2.4 Show that measure $y^{-2}dy\, dx$ on the upper half plane $\mathbb{R}^2_+$ is invariant under action from Exercise A.2.1.

Left invariant measures on $X = G$ is called the *Haar measure*. It always exists and is uniquely defined up to a scalar multiplier [76, § 0.2]. An equivalent formulation of (A.2.1) is: *G operates on $L_2(X, d\mu)$ by unitary operators.* We will transfer the Haar measure $d\mu$ from G to $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \rightarrow G$ and will call it as the *invariant measure on a Lie algebra* $\mathfrak{g}$.

Exercise A.2.5 Check that the following are Haar measures for corresponding groups:

(i). The *Lebesgue measure* dx on the real line $\mathbb{R}$.

(ii). The Lebesgue measure $d\phi$ on the unit circle $\mathbb{T}$.

(iii). dx/x is a Haar measure on the multiplicative group $\mathbb{R}_+$;

(iv). $dx\, dy/(x^2 + y^2)$ is a Haar measure on the multiplicative group $\mathbb{C} \setminus \{0\}$, with coordinates $z = x + iy$.

(v). $a^{-2}\, da\, db$ and $a^{-1}\, da\, db$ are the left and right invariant measure on $ax+b$ group.

(vi). The Lebesgue measure $ds\, dx\, dy$ of $\mathbb{R}^3$ for the Heisenberg group $\mathbb{H}^1$.

In this notes we assume *all integrations on groups performed over the Haar measures.*

Exercise A.2.6 Show that invariant measure on a compact group G is finite and thus may be normalized to total measure 1.

The above simple result has surprisingly important consequences.

Definition A.2.7 The left *convolution* $f_1 * f_2$ of two functions $f_1(g)$ and $f_2(g)$ defined on a group G is

$$f_1 * f_2(g) = \int_G f_1(h)\, f_2(h^{-1}g)\, dh$$

Exercise A.2.8 Let $k(g) \in L_1(G, d\mu)$ and operator K on $L_1(G, d\mu)$ is the left *convolution operator* with k, .i.e. $K : f \mapsto k * f$. Show that K commutes with all right shifts on G.

The following Lemma characterizes *linear subspaces* of $L_2(G, d\mu)$ invariant under shifts in the term of *ideals of convolution algebra* $L_2(G, d\mu)$ and is of the separate interest.

Lemma A.2.9 *A closed linear subspace H of $L_2(G, d\mu)$ is invariant under left (right) shifts if and only if H is a left (right) ideal of the right group convolution algebra $L_2(G, d\mu)$.*

A closed linear subspace H of $L_2(G, d\mu)$ is invariant under left (right) shifts if and only if H is a right (left) ideal of the left group convolution algebra $L_2(G, d\mu)$.

Proof. Of course we consider only the “right-invariance and right-convolution” case. Then the other three cases are analogous. Let H be a closed linear subspace of $L_2(G, d\mu)$ invariant under right shifts and $k(g) \in H$. We will show the inclusion

$$[f * k]_r(h) = \int_G f(g)k(hg)\, d\mu(g) \in H, \tag{A.2.2}$$

for any $f \in L_2(G, d\mu)$. Indeed, we can treat integral (A.2.2) as a limit of sums

$$\sum_{j=1}^{N} f(g_j)k(hg_j)\Delta_j. \tag{A.2.3}$$

But the last sum is simply a linear combination of vectors $k(hg_j) \in H$ (by the invariance of H) with coefficients $f(g_j)$. Therefore sum (A.2.3) belongs to H and this is true for integral (A.2.2) by the closeness of H.

Otherwise, let H be a right ideal in the group convolution algebra $L_2(G, d\mu)$ and let $\phi_j(g) \in L_2(G, d\mu)$ be an approximate unit of the algebra [19, § 13.2], i. e. for any $f \in L_2(G, d\mu)$ we have

$$[\phi_j * f]_r(h) = \int_G \phi_j(g) f(hg)\, d\mu(g) \rightarrow f(h), \text{ when } j \rightarrow \infty.$$

Then for $k(g) \in H$ and for any $h' \in G$ the right convolution

$$[\phi_j * k]_r(hh') = \int_G \phi_j(g) k(hh'g)\, d\mu(g) = \int_G \phi_j(h'^{-1}g') k(hg')\, d\mu(g'),\ g' = h'g,$$

from the first expression is tensing to $k(hh')$ and from the second one belongs to H (as a right ideal). Again the closeness of H implies $k(hh') \in H$ that proves the assertion. $\square$

Appendix B

Elements of the Representation Theory

B.1 Representations of Groups

Objects unveil their nature in actions. Groups act on other sets by means of *representations*. A representation of a group G is a group homomorphism of G in a transformation group of a set. It is a fundamental observation that *linear* objects are easer to study. Therefore we begin from linear representations of groups.

Definition B.1.1 A linear continuous *representation of a group* G is a continuous function $T(g)$ on G with values in the group of non-degenerate linear continuous transformation in a linear space H (either finite or infinite dimensional) such that $T(g)$ satisfies to the functional identity:

$$T(g_1 g_2) = T(g_1)\,T(g_2). \tag{B.1.1}$$

Exercise B.1.2 Show that $T(g^{-1}) = T^{-1}(g)$ and $T(e) = I$, where I is the identity operator on B.

Exercise B.1.3 Show that these are linear continuous representations of corresponding groups:

(i). Operators $T(x)$ such that $[T(x)\,f](t) = f(t + x)$ form a representation of $\mathbb{R}$ in $L_2(\mathbb{R})$.

(ii). Operators $T(n)$ such that $T(n)a_k = a_{k+n}$ form a representation of $\mathbb{Z}$ in ℓ_2.

(iii). Operators $T(a,b)$ defined by

$$[T(a,b)\,f](x) = \sqrt{a}f(ax+b), \qquad a \in \mathbb{R}_+,\ b \in \mathbb{R} \tag{B.1.2}$$

form a representation of $ax+b$ group in $L_2(\mathbb{R})$.

(iv). Operators $T(s,x,y)$ defined by

$$[T(s,x,y)\,f](t) = e^{i(2s-\sqrt{2}yt+xy)}f(t-\sqrt{2}x) \tag{B.1.3}$$

form *Schrödinger representation* of the Heisenberg group $\mathbb{H}^1$ in $L_2(\mathbb{R})$.

(v). Operators $T(g)$ defined by

$$[T(g)f](t) = \frac{1}{ct+d}f\left(\frac{at+b}{ct+d}\right), \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{B.1.4}$$

form a representation of $SL_2(\mathbb{R})$ in $L_2(\mathbb{R})$.

In the sequel a representation *always means* linear continuous representation. $T(g)$ is an *exact representation* (or *faithful representation* if $T(g) = I$ only for $g = e$. The opposite case when $T(g) = I$ for all $g \in G$ is a *trivial representation*. The space H is *representation space* and in most cases will be a *Hilber space* [30, § III.5]. If dimensionality of H is finite then T is a *finite dimensional representation*, in the opposite case it is *infinite dimensional representation*.

We denote the *scalar product* on H by $\langle\cdot,\cdot\rangle$. Let $\{\mathbf{e}_j\}$ be an (finite or infinite) *orthonormal basis* in H, i.e.

$$\langle \mathbf{e}_j, \mathbf{e}_j \rangle = \delta_{jk},$$

where δ_{jk} is the *Kroneker delta*, and linear span of $\{\mathbf{e}_j\}$ is dense in H.

Definition B.1.4 The *matrix elements* $t_{jk}(g)$ of a representation T of a group G (with respect to a basis $\{\mathbf{e}_j\}$ in H) are complex valued functions on G defined by

$$t_{jk}(g) = \langle T(g)\mathbf{e}_j, \mathbf{e}_k \rangle\,. \tag{B.1.5}$$

Exercise B.1.5 Show that [77, § 1.1.3]

(i). $T(g)\,\mathbf{e}_k = \sum_j t_{jk}(g)\,\mathbf{e}_j$.

(ii). $t_{jk}(g_1g_2) = \sum_n t_{jn}(g_1)\,t_{nk}(g_2)$.

It is typical mathematical questions to determine identical objects which may have a different appearance. For representations it is solved in the following definition.

Definition B.1.6 Two representations T_1 and T_2 of the same group G in spaces H_1 and H_2 correspondingly are *equivalent representations* if there exist a linear operator $A : H_1 \rightarrow H_2$ with the continuous inverse operator A^{-1} such that:

$$T_2(g) = A\, T_1(g)\, A^{-1}, \qquad \forall g \in G.$$

Exercise B.1.7 Show that representation $T(a, b)$ of $ax + b$ group in $L_2(\mathbb{R})$ from Exercise B.1.3.(iii) is equivalent to the representation

$$[T_1(a, b)\, f](x) = \frac{e^{i\frac{b}{a}}}{\sqrt{a}}\, f\left(\frac{x}{a}\right). \qquad \text{(B.1.6)}$$

HINT. Use the Fourier transform. □

The *relation of equivalence* is reflexive, symmetric, and transitive. Thus it splits the set of all representations of a group G into *classes of equivalent representations. In the sequel we study group representations up to their equivalence classes only.*

Exercise B.1.8 Show that equivalent representations have the same matrix elements in appropriate basis.

Definition B.1.9 Let T is a representation of a group G in H The *adjoint representation* $T'(g)$ of G in H is defined by

$$T'(g) = \left(T(g^{-1})\right)^*,$$

where * denotes the adjoint operator in H.

Exercise B.1.10 Show that

(i). T' is indeed a representation.

(ii). $t'_{jk}(g) = \bar{t}_{kj}(g^{-1})$.

Recall [30, § III.5.2] that a bijection $U : H \rightarrow H$ is a *unitary operator* if

$$\langle Ux, Uy\rangle = \langle x, y\rangle, \qquad \forall x, y \in H.$$

Exercise B.1.11 Show that $UU^* = I$.

Definition B.1.12 T is a *unitary representation* of a group G in a space H if $T(g)$ is a unitary operator for all $g \in G$. T_1 and T_2 are *unitary equivalent representations* if $T_2 = UT_2U^{-1}$ for a unitary operator U.

Exercise B.1.13 (i). Show that all representations from Exercises B.1.3 are unitary.

(ii). Show that representations from Exercises B.1.3.(iii) and B.1.7 are unitary equivalent.

HINT. Take that the Fourier transform is unitary for granted. □

Exercise B.1.14 Show that if a Lie group G is represented by unitary operators in H then its Lie algebra $\mathfrak{g}$ is represented by self-adjoint (possibly unbounded) operators in H.

The following definition have a sense for *finite* dimensional representations.

Definition B.1.15 A *character of representation* T is equal $\chi(g) = \operatorname{tr}(T(g))$, where tr is the *trace* [30, § III.5.2 (Probl.)] of operator.

Exercise B.1.16 Show that

(i). Characters of a representation T are constant on the adjoint elements $g^{-1}hg$, for all $g \in G$.

(ii). Character is an algebra homomorphism from an algebra of representations with Kronecker's (tensor) multiplication [77, § 1.9] to complex numbers.

HINT. Use that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$, and $\operatorname{tr}(A \otimes B) = \operatorname{tr} A \operatorname{tr} B$. □

For *infinite* dimensional representation characters could be defined either as distributions [29, § 11.2] or in infinitesimal terms of Lie algebras [29, § 11.3].

The characters of a representation should not be confused with the following notion.

Definition B.1.17 A *character of a group* G is a one-dimensional representation of G.

Exercise B.1.18 (i). Let χ be a character of a group G. Show that a character of representation χ coincides with it and thus is a character of G.

(ii). A matrix element of a group character χ coincides with χ.

(iii). Let χ_1 and χ_1 be characters of a group G. Show that $\chi_1 \otimes \chi_2 = \chi_1\chi_2$ and $\chi'(g) = \chi_1(g^{-1})$ are again characters of G. In other words *characters of a group form a group themselves.*

B.2 Decomposition of Representations

The important part of any mathematical theory is classification theorems on structural properties of objects. Very well known examples are:

(i). The main theorem of arithmetics on unique representation an integer as a product of powers of prime numbers.

(ii). Jordan's normal form of a matrix.

The similar structural results in the representation theory are very difficult. The easiest (but still rather difficult) questions are on classification of unitary representations up to unitary equivalence.

Definition B.2.1 Let T be a representation of G in H. A linear subspace $L \subset H$ is *invariant subspace* for T if for any $\mathbf{x} \in L$ and any $g \in G$ the vector $T(g)\mathbf{x}$ again belong to L.

There are always two trivial invariant subspaces: the null and entire H. All other are *nontrivial invariant subspaces.*

Definition B.2.2 If there are only two trivial invariant subspaces then T is *irreducible representation*. In the opposite case we have *reducible representation*.

For any nontrivial invariant subspace we could define the *restriction of representation* of T on it. In this way we obtain a *subrepresentation* of T.

Example B.2.3 Let $T(a)$, $a \in \mathbb{R}_+$ be defined as follows: $[T(a)]f(x) = f(ax)$. Then spaces of even and odd functions are invariant.

Definition B.2.4 If the closure of liner span of all vectors $T(g)v$ is dense in H then v is called *cyclic vector* for T.

Exercise B.2.5 Show that for an irreducible representation any non zero vector is cyclic.

The important property of unitary representation is complete reducibility.

Exercise B.2.6 Let a unitary representation T has an invariant subspace $L \subset H$, then its orthogonal completion $L^\perp$ is also invariant.

Theorem B.2.7 *[29, § 8.4] Any unitary representation T of a locally compact group G could be decomposed in a (continuous) direct sum irreducible representations: $T = \int_X T_x \, d\mu(x)$.*

The necessity of continuous sums appeared in very simple examples:

Exercise B.2.8 Let T be a representation of $\mathbb{R}$ in $L_2(\mathbb{R})$ as follows: $[T(a)f](x) = e^{iax}f(x)$. Show that

(i). Any measurable set $E \subset \mathbb{R}$ define an invariant subspace of functions vanishing outside E.

(ii). T does not have invariant irreducible subrepresentations.

Definition B.2.9 The set of equivalence classes of unitary irreducible representations of a group G is denoted be $\hat{G}$ and called *dual object* (or *dual space*) of the group G.

Definition B.2.10 A left *regular representation* $\Lambda(g)$ of a group G is the representation by left shifts in the space $L_2(G)$ of square-integrable function on G with the left Haar measure

$$\Lambda g : f(h) \mapsto f(g^{-1}h). \tag{B.2.1}$$

The *main problem of representation theory* is to decompose a left regular representation $\Lambda(g)$ into irreducible components.

B.3 Invariant Operators and Schur's Lemma

It is a pleasant feature of an abstract theory that we obtain important general statements from simple observations. Finiteness of invariant measure on a compact group is one such example. Another example is Schur's Lemma presented here.

To find different classes of representations we need to compare them each other. This is done by *intertwining operators*.

Definition B.3.1 Let T_1 and T_2 are representations of a group G in a spaces H_1 and H_2 correspondingly. An operator $A : H_1 \rightarrow H_2$ is called an *intertwining operator* if

$$A\,T_1(g) = T_2(g)\,A, \qquad \forall g \in G.$$

If $T_1 = T_2 = T$ then A is *interntwinig operator* or *commuting operator* for T.

Exercise B.3.2 Let G, H, $T(g)$, and A be as above. Show that [77, § 1.3.1]

(i). Let $\mathbf{x} \in H$ be an eigenvector for A with eigenvalue λ. Then $T(g)\mathbf{x}$ for all $g \in G$ are eigenvectors of A with the same eigenvalue λ.

(ii). All eigenvectors of A with a fixed eigenvalue λ for a linear subspace invariant under all $T(g)$, $g \in G$.

(iii). If an operator A is commuting with irreducible representation T then $A = \lambda I$.

HINT. Use the spectral decomposition of selfadjoint operators [30, § V.2.2]. □

The next result have very important applications.

Lemma B.3.3 (Schur) *[29, § 8.2] If two representations T_1 and T_2 of a group G are irreducible, then every intertwining operator between them either zero or is invertible.*

HINT. Consider subspaces $\ker A \subset H_1$ and $\operatorname{im} A \subset H_2$. □

Exercise B.3.4 Show that

(i). Two irreducible representations either equivalent or disjunctive.

(ii). All operators commuting with an irreducible representation form a field.

(iii). Irreducible representation of commutative group are one-dimensional.

(iv). If T is unitary irreducible representation in H and $B(\cdot,\cdot)$ is a bounded semi linear form in H invariant under T: $B(T(g)\mathbf{x}, T(g)\mathbf{y}) = B(\mathbf{x}, \mathbf{y})$ then $B(\cdot,\cdot) = \lambda\,\langle\cdot,\cdot\rangle$.

HINT. Use that $B(\cdot,\cdot) = \langle A\cdot,\cdot\rangle$ for some A [30, § III.5.1]. □

Appendix C

Miscellanea

C.1 Functions of even Clifford numbers

Let

$$\mathsf{a} = a_1\mathsf{p}_1 + a_2\mathsf{p}_2, \qquad \mathsf{p}_1 = \frac{1+e_1e_2}{2}, \quad \mathsf{p}_2 = \frac{1-e_1e_2}{2}, \quad a_1, a_2 \in \mathbb{R} \tag{C.1.1}$$

be an even Clifford number in $\mathcal{C}\ell(1,1)$. It follows from the identities

$$\mathsf{p}_1\mathsf{p}_2 = \mathsf{p}_2\mathsf{p}_1 = 0, \qquad \mathsf{p}_1^2 = \mathsf{p}_1, \qquad \mathsf{p}_2^2 = \mathsf{p}_2, \qquad \mathsf{p}_1 + \mathsf{p}_2 = 1 \tag{C.1.2}$$

that $p(\mathsf{a}) = p(a_1)\mathsf{p}_1 + p(a_2)\mathsf{p}_2$ for any polynomial $p(x)$. Let P be a topological space of functions $\mathbb{R} \to \mathbb{R}$ such that polynomials are dense in it. Then for any $f \in P$ we can define $f(\mathsf{a})$ by the formula

$$f(\mathsf{a}) = f(a_1)\mathsf{p}_1 + f(a_2)\mathsf{p}_2. \tag{C.1.3}$$

This definition gives continuous algebraic homomorphism.

C.2 Principal series representations of $SL_2(\mathbb{R})$

We describe a realization of the principal series representations of $SL_2(\mathbb{R})$. The realization is deduced from the realization by left regular representation on the a space of homogeneous function of power $-is-1$ on $\mathbb{R}^2$ described in [76, § 8.3]. We consider now the restriction of homogeneous function not to the unit circle as in [76, Chap. 8, (3.23)] but to the line $x_2 = 1$ in $\mathbb{R}^2$. Then an equivalent unitary representation of $SL_2(\mathbb{R})$ acts on the Hilbert space $L_2(\mathbb{R})$ with the standard Lebesgue measure by the transformations:

$$[\pi_{is}(g)f](x) = \frac{1}{|cx+d|^{1+is}} f\left(\frac{ax+b}{cx+d}\right), \qquad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{C.2.1}$$

C.3 Boundedness of the Singular Integral Operator $\mathcal{W}_\sigma$

The kernel of integral operator $\mathcal{W}_\sigma$ (3.3.22) is singular in four points, which are the intersection of $\widetilde{\mathbb{T}}$ and the light cone with the origin in u. One can easily see

$$\left|\frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{(-e^{-e_{12}t}e_1\mathsf{u}+1)^{1+\sigma}}\right| = \left|1+\mathsf{u}^2\right|^{1/2}\frac{1}{|t-t_0|}+O(\frac{1}{|t-t_0|^2}).$$

where t_0 is one of four singular points mentioned before for a fixed u and t is a point in its neighborhood. More over the kernel of integral operator $\mathcal{W}_\sigma$ is changing the sign while t crossing the t_0. Thus we can define $\mathcal{W}_\sigma$ in the sense of the principal value as the standard singular integral operator.

Such defined integral operator $\mathcal{W}_\sigma$ becomes a bounded linear operator $L_2(\widetilde{\mathbb{T}}) \rightarrow L_2(\widetilde{\mathbb{T}}^\lambda)$, where $\widetilde{\mathbb{T}}^\lambda$ is the circle (3.3.10) in $\widetilde{\mathbb{R}}^{1,1}$ with center in the origin and the "radius" λ. Moreover the norm of the operator $\lambda^{-2}\mathcal{W}_\sigma$ is uniformly bounded for all λ and thus we can consider it as bounded operator

$$L_2(\widetilde{\mathbb{T}}) \rightarrow H_\sigma(\widetilde{\mathbb{D}}),$$

where

$$H_\sigma(\widetilde{\mathbb{D}}) = \{f(\mathsf{u}) \mid D_{\widetilde{\mathbb{D}}}f(\mathsf{u})=0,\ \mathsf{u}\in\widetilde{\mathbb{D}},\ |\lambda|^{-2}\int_{\widetilde{\mathbb{T}}^\lambda}|f(\mathsf{u})|^2\,du<\infty,\ \forall\lambda<0\}. \tag{C.3.1}$$

is an analog of the classic Hardy space. Note that $|\lambda|^{-2}\,d\mathsf{u}$ is exactly the invariant measure (3.3.14) on $\widetilde{\mathbb{D}}$.

One can note the similarity of arising divergency and singularities with the ones arising in *quantum field theory*. The similarity generated by the same mathematical object in basement: a pseudoeuclidean space with an indefinite metric.

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