Numbers and Vectors
Lecture Notes

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Module summary. This module introduces students to three outstandingly influential developments from 19th century mathematics:

- complex numbers;
- the rigorous notion of limit; and
- vectors in a three-dimensional space.

Complex numbers are the natural setting for much pure and applied mathematics, and vectors provide the natural language to describe mechanics, gravitation and electromagnetism, while the rigorous notion of limit is fundamental to calculus. Along the way, students will go beyond the straightforward calculation and problem solving skills emphasized in A-level Mathematics, and learn to formulate rigorous mathematical proofs.

Objectives. On completion of this module, students should be able to:

(1) perform algebraic calculations with complex numbers and solve simple equations for a complex variable;
(2) determine whether simple sequences and series converge;
(3) perform calculations with vectors, write down the equations of lines, planes and spheres in vector language, and, conversely, describe the geometry of the solution sets of simple vector equations;
(4) construct rigorous mathematical proofs of simple propositions, including proofs by mathematical induction.

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CHAPTER 1

Numbers

> God has created the Natural Numbers, but everything else is a man's work. (Leopold Kronecker)

In the first part of the course we will deal with numbers. You should be familiar with natural numbers and integers, rational numbers and real numbers. We will recall their properties and uses. The main topic in this part will be *complex numbers*. We will discuss how to calculate with complex numbers and how to represent them graphically. We will also discuss one of the most famous formulas in mathematics: *Euler's Formula*.

1. Natural numbers and integers

In mathematics, we use *sets* to collect objects, which often are united by common properties. The simplest examples are sets of numbers, which we consider in this chapter.

Firstly, we denote the set of *natural numbers* by

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Note, that we do not include 0 in natural numbers, however this agreement is not universal. We also use the notation

$$\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

(The symbol '$\cup$' denotes the union of the two sets; please have a look at basic notations from set theory if you are not familiar with them.)

There are two arithmetic operations—*addition* and *multiplication*—which, for a given pair of natural numbers, produce the result produce. The result is a natural number again and we say that *natural numbers form a set closed under the binary operation of addition and multiplication*. Furthermore, these operation have the following properties:

(1) $n + m = m + n,$ (commutativity of addition),

(2) $(n + m) + k = n + (m + k),$ (associativity of addition),

(3) $n \cdot m = m \cdot n,$ (commutativity of multiplication),

(4) $(n \cdot m) \cdot k = n \cdot (m \cdot k),$ (associativity of multiplication),

(5) $(n + m) \cdot k = n \cdot k + n \cdot k,$ (distributive law),,

for all $n, m, k \in \mathbb{N}$.

2. Proof by induction

In mathematics we adopt rigour standards how we can decide whether things are right or wrong. A small amount of simple statements (called *axioms*) are taken to be true without any justification. For example, to define natural numbers rigorously we need only five Peano's axioms. One of them is:

The set of natural numbers is infinite.

For any statement, which is not an axiom, we need to use the mathematical process to establish whether a statement is true, the process is called *proof*. We do not go into the details and theory behind proofs (this happens in the field known as *mathematical logic*); for us a proof of a statement means to turn this statement into something that is obviously true, in a series of mathematically correct steps.

The method of proof by induction is a powerful and important way to prove statements of the type "For all natural numbers, it is true that. . . ".

2.1. First example. To prove a mathematical statement as above, *mathematical induction* works as follows: We first check that the statement is true for a specific number (this is called the basis of induction). Then, in the proof's main part, we assume that the statement is true for some number n (induction assumption) and use this to show that it must be also true for the successor $n + 1$ (induction step).

Therefore, this method works in the same way as a domino effect. If you are presented with a long row of dominoes, you can be sure that whenever a domino falls, its next neighbour will also fall (this corresponds to the induction proof). To get the whole process going, however, you must make sure that one specific (usually the first) domino falls (which corresponds to the basis of the induction).

Let us take a look at some examples to make things more specific.

Claim. For all numbers $n \in \mathbb{N}$, we have

$$1 + 2 + \ldots + (n - 1) + n = \sum_{k=1}^{n} k = \frac{n(n + 1)}{2}$$

(Note the use of the 'sum symbol' $\sum$ to abbreviate the above sum.)

Proof. We use the proof by induction.

Step 1. Show that the statement is true for $n = 1$:

We have $1 = \frac{1\cdot 2}{2}$, such that for $n = 1$ the assertion is true.

Step 2. Assume that the statement is true for some (unspecified) number n: For an $n \in \mathbb{N}$, we assume that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Step 3. Show that the statement is true for the successor $n+1$: We want to show that

$$\sum_{k=1}^{n+1} k = \frac{(n+2)(n+1)}{2}.$$

Proof of Step 3:

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \left(\sum_{j=1}^{n} k\right) + (n+1) \\
\text{(use assumption in Step 2 for the 1st term)} &= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{(n+2)(n+1)}{2}
\end{aligned}$$

and so the statement is also true for $n+1$.

We have thus proved by induction that the claim is true for all $n \in \mathbb{N}$. $\square$

Remark 2.1. The above formula can also be proved in a different way. For this, consider the following.

$$\begin{array}{rrcccccccccc}
 & \sum_{k=1}^{n} k & = & 1 & + & 2 & +\ldots+ & (n-1) & + & n \\
+ & \sum_{k=1}^{n} k & = & n & + & (n-1) & +\ldots+ & 2 & + & 1 \\
\hline
 & 2\sum_{k=1}^{n} k & = & (n+1) & + & (n+1) & +\ldots+ & (n+1) & + & (n+1).
\end{array}$$

So, the columns on the right-hand side each add up to $n+1$, and there are n of these terms, which implies

$$2 \cdot \sum_{k=1}^{n} k = n(n+1).$$

This little trick proves the formula, too.

The story goes that the famous German mathematician Carl Friedrich Gauss used this clever idea as a school boy to compute the sum of the first 100 natural numbers; much to the annoyance of his school-teacher, who wanted to use this task to keep his children busy for a while.

In mathematics, there is usually more than one way to prove a statement. For the example above, we see that adding up the numbers in 2 different ways provides a very brief (and what mathematicians call 'elegant' proof). But of course, first of all you need to have such a clever idea. The advantage of mathematical induction is that it gives you a clear framework, in which to prove a statement.

Also, we have so far presented the simplest version of mathematical induction. There are several extensions:

- In step 1, the induction basis can start at a different number (see the second example below).
- In the induction step we can go from n to $n+2$ to prove statements about even numbers, or go from n to $n-1$ to prove statements about negative integers.

2.2. Second example. Mathematical induction can be used in different situations. The second example concerns an inequality.

LEMMA 2.2. *For all natural numbers $n > 4$ we have $n^2 < 2^n$.*

Before we prove the lemma, you might want to check why the statement is not true for all natural numbers.

PROOF. We use again proof by induction.

Step 1. As required by the statement, we start the induction proof by checking it for $n = 5$:

We have $5^2 = 25 < 32 = 2^5$.

Step 2. Assume that the Lemma is true for some n, i.e.

$$n^2 < 2^n.$$

Step 3. Prove the statement for $n + 1$.

We want to prove that $(n+1)^2 < 2^{n+1}$.

We first transform the left-hand side

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &< 2^n + 2n + 1,\end{aligned}$$

where we use the induction assumption for the n^2-term.

Now, let us assume for the moment, that the following is true:

$$2n + 1 < 2^n \quad \text{for all } n > 4. \tag{6}$$

We can use the Inequality (6) to complete the above proof, as follows

$$\begin{aligned} 2^n + 2n + 1 &< 2^n + 2^n \\ &= 2 \cdot 2^n \\ &= 2^{n+1}. \end{aligned}$$

Going through the list of inequalities, we find that $(n+1)^2 < 2^{n+1}$.

Hence, assuming that (6) is true, we can prove the Lemma using mathematical induction.

It remains, of course, to prove (6); i.e., we want to show that $2n + 1 < 2^n$ for all numbers $n > 4$. This can be done by induction again:

Step 1. For $n = 5$ we have $11 < 32$, as required.

Step 2. Assume that $2n + 1 < 2^n$ for some number n.

Step 3. Show that $2(n + 1) + 1 < 2^{n+1}$.

We have

$$\begin{aligned} 2(n+1) + 1 &= 2n + 1 + 2 \\ &< 2^n + 2 \\ &< 2^n + 2^n \\ &= 2^{n+1}, \end{aligned}$$

where for the first inequality we have used the induction assumption.

This proves that the Inequality (6) is correct. Putting it all together we have proved Lemma 1 using mathematical induction. □

2.3. Example 3: Geometric sums. We will mainly use proofs by induction to verify formulas for sums. A particular important example is given by the sum of geometric progression, and so we want to prove this sum formula here. (We already use the concept of a real number here, but will only discuss these numbers briefly afterwards.)

LEMMA 2.3. *Let $q \in \mathbb{R}$ be a real number. Then*

$$\sum_{k=0}^{n} q^k = \begin{cases} \frac{q^{n+1}-1}{q-1} & q \neq 1 \\ n+1 & q = 1 \end{cases}.$$

PROOF. In the case $q = 1$ the proof is simple, since we simply add up '1's for $n + 1$ times. So, we will concentrate on $q \neq 1$, and present a proof by induction.

Step 1. If $n = 0$, then we find $1 = \frac{q-1}{q-1}$, and so the statement is correct.

Step 2. We assume that for some n

$$\sum_{k=0}^{n} q^k = \frac{q^{n+1}-1}{q-1}.$$

Step 3. We now want to show that

$$\sum_{k=0}^{n+1} q^k = \frac{q^{n+2}-1}{q-1}.$$

Proof of Step 3. We have

$$\begin{aligned}
\sum_{k=0}^{n+1} q^k &= \left(\sum_{k=0}^{n} q^k\right) + q^{n+1} \\
\text{(using Step 2.)} \quad &= \frac{q^{n+1}-1}{q-1} + q^{n+1} \\
&= \frac{1}{q-1}\left(q^{n+1} - 1 + q^{n+2} - q^{n+1}\right) \\
&= \frac{q^{n+2}-1}{q-1}.
\end{aligned}$$

Therefore, we have proved the lemma, using mathematical induction. □

2.4. Example 4: Be careful! Sometimes we may be mislead by reasoning which looks like a proof but is not. The following statement is obviously false:

> For any n different points of the plane there is a straight line passing them.

However we may construct the following "proof" of this statement by mathematical induction:

Step 1. If $n = 1$ the statement is obviously true: for every point there is a line passing it, in fact an infinite number of such lines. The statement is also true for $n = 2$: every two distinct points define the unique straight line.

Step 2. Now we assume that any n different points admit a line passing them.

Step 3. Take any $n + 1$ different points A_1, A_2,... A_n, A_{n+1}. By the previous step there is a line, call it l_1 passing points A_1, A_2,... A_n. For the same reason, there is a line, call it l_2 passing points A_2,... A_n, A_{n+1}. Lines l_1 and l_2 both pass points A_2,... A_n, thus these two lines coincide and pass all points A_1, A_2,... A_n, A_{n+1}. Our "proof" is complete.

Can you spot an error in the above arguments?

3. Extending natural numbers: integers, rationals, reals

3.1. Integers. It is a common situation, that we need to find a quantity through its relation to others. Typically, such a relation may be expressed as an *equation*, e.g. $x + 3 = 5$ which has the only solution (*root*) $x = 2$. We quickly discover many equations with natural numbers, which do not have any solution among natural numbers, e.g. $x + 5 = 3$.

A way out of this situation is to extend our notion of number. This will be a common theme in this part of the course: If there are equations for which no solutions exist in the set of numbers we have, we 'simply' extend the set of numbers, so that we are able to write down a solution.

As the first step we introduce the set of integers as the set of all solutions to equations $x + n = m$ with natural n and m. Integers will be denoted by

$$\begin{aligned} \mathbb{Z} &= \mathbb{N}_0 \cup (-\mathbb{N}) \\ &= \{0, 1, -1, 2, -2, 3, -3, \ldots\} \\ &= \{\ldots, -2, -1, 0, 1, 2, \ldots\}. \end{aligned}$$

Now the equation $x + 5 = 3$ has the only integer solution $x = -2$. Also, addition and multiplication are extended to integers: the set $\mathbb{Z}$ is closed under these two operation and all properties (1)–(5) are preserved.

All of those sets are infinite. Since all natural numbers are integers, we can write

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z},$$

Usually, we will use the letters j, k, l, m and n to denote integer variables.

3.2. Rational numbers. Consider further equation with integer coefficients: $x \cdot 2 = -6$, it has the unique integer root $x = -3$. However a similar equation $x \cdot 6 = -2$ does not have an integer solution. Thus we again looking for an appropriate extension of numbers.

Rational numbers are defined as fractions $r = \frac{p}{q}$, that is solutions of the equation $r \cdot q = p$. We denote the set of all rational numbers by

$$\mathbb{Q} = \left\{ r = \frac{p}{q}, \text{ with } p \in \mathbb{Z},\ q \in \mathbb{N} \right\}.$$

Note, that our agreement $0 \notin \mathbb{N}$ allows to avoid meaningless expressions like $\frac{1}{0}$.

Of course, if we write a rational number as a fraction, then this representation is not unique, since for example

$$2 = \frac{2}{1} = \frac{8}{4}.$$

We usually assume that p and q have no common divisors.

As before addition and multiplication are extended to rationals (by arithmetic of fractions) with preservation of properties (1)–(5). Also rationals are sufficient to solve any linear equation $ax + b = c$ with rational coefficients a, b and c, where $a \neq 0$.

3.3. Irrational numbers. We next move on to real numbers. They are 'needed' if we want to solve certain quadratic equations. The standard example is the equation $x^2 = 2$. We call the (positive) solution $x = \sqrt{2}$, and we claim that it is not a rational number. Such numbers are called *irrationals*.

How can we actually prove that $\sqrt{2}$ is irrational? For this, we will use another, very important mathematical technique - the *proof by contradiction* or *indirect proof*. We will demonstrate this method using the before-mentioned example.

LEMMA 3.1. *The equation* $x^2 = 2$ *does not have a rational solution.*

REMARK 3.2 (Proof by Contradiction). Suppose that S is a mathematical statement that we want to prove. (For example S might be the statement "For all integers n and m, if $n \times m$ is odd then n and m are both odd.") In order to carry out a *proof by contradiction* or *indirect proof* of S we start by assuming that S is false. In other words we assume a statement $\bar{S}$ which is equivalent to saying "S is false". The idea is then to deduce from the statement $\bar{S}$ something that is obviously false or a contradiction. If all the steps in this deduction are mathematically correct, then our starting point $\bar{S}$ must be false. This means that the statement "S is false" is itself false. Therefore the statement S is true, which we wanted to prove.

PROOF. So let us start with the proof of the lemma. We want to prove that there is no rational solution for $x^2 = 2$, and so we assume that there *does* exist such a solution, say $x = p/q$. We also assume that p and q have no common divisors.

We then have, that

$$x^2 = \frac{p^2}{q^2} = 2,$$

that is

$$p^2 = 2q^2, \tag{7}$$

and so p^2 is an even number. Therefore, p is even, too, because only the square of an even number is an even number. Thus $p = 2m$ for some integer m. So

$$(2m)^2 = 2^2 m^2 = 2q^2 \tag{8}$$

so that (dividing through by 2) we have

$$2m^2 = q^2\,. \tag{9}$$

Thus q^2 is even and so q is also even (by the same argument applied to p). In other words both p and q are even, i.e. 2 is a common divisor of both p and q. However we assumed that p and q had no common divisors.

We have thus arrived at a contradiction and can conclude that our starting assumption must be wrong. And this means that the lemma is correct, i.e. it means that $\sqrt{2}$ is not a rational number. $\square$

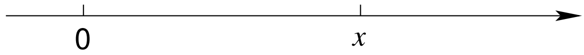


FIGURE 1. A real number $x \in \mathbb{R}$ corresponds to a point on the real line.

3.4. Real numbers and the real line. *Real numbers* are the set of rational and irrational numbers. We denote this set by the symbol '$\mathbb{R}$'. We will not define real numbers precisely, but a good thing to keep in mind is that real numbers are the natural environment to do *calculus*. Calculus is mainly based on the concept of *limits*, and the set of real numbers can be defined as the set of numbers that are limits of sequences of rational numbers. (We consider sequences and their limits later in the course.)

An alternative definition of real numbers uses the decimal representation of numbers. Every real number x can be written as

$$x = m.a_1a_2a_3a_4a_5\ldots \quad m \in \mathbb{Z},\ a_i \in \{0,1,2,\ldots 9\}.$$

For example

- $2/5 = 0.2$,
- $4/3 = 1.3333\ldots = 1.\overline{3}$ where the overline denotes a repeating sequence,
- $\pi = 3.14159265358979323846264 3\ldots$.

It is known that if x is rational, then the sequence of the a_i's is either finite or it starts to repeat itself at some point. Consequently, an irrational number has a decimal representation with a non-repeating, infinite sequence of digits a_i.

Similarly, we can illustrate the set of real numbers (or parts thereof) graphically, as the real line. This is a common graphical representation, where every number corresponds to a point on the line. Note that implicitly, the concept of the real line is used whenever you plot graphs of functions, etc.

4. Complex numbers

We now come to the main topic of this first part—complex numbers. Again, we can motivate the 'need' for complex numbers by considering equations without real solutions. It is not hard to find examples of such equations. Consider, for instance,

$$x^2 + 2x + 5 = 0. \tag{10}$$

Using the standard formula for quadratic equation, the solutions would be

$$x_{1,2} = -1 \pm \sqrt{-4} = -1 \pm 2\sqrt{-1},$$

and since there are no (real) square-roots of negative numbers, no real solutions exist. In a formal way, however, we can write down a solution as above or by defining the square-root of a negative number in a meaningful way. This idea leads to the introduction of complex numbers.

4.1. Definition of complex numbers.

DEFINITION 4.1. A *complex number* z is a number

$$z = x + yi, \quad x, y, \in \mathbb{R}$$

where i denotes the *imaginary unit* defined as a solution of the equation $i^2 = -1$.

We call x the *real part* of z and y the *imaginary part* of z. The corresponding notations are $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

REMARK 4.2.

(1) The set of all complex numbers is called $\mathbb{C}$.
(2) If $y = 0$, then $z = x \in \mathbb{R}$. So, the real numbers are a subset of the complex numbers, $\mathbb{R} \subset \mathbb{C}$. On the other hand, if $x = 0$, then $z = yi$. Such numbers are called (purely) *imaginary numbers*.
(3) In the engineering literature it is common to use 'j' instead of 'i' as a symbol for the imaginary unit.

We can now express the solutions of the quadratic equation $x^2 + 2x + 5 = 0$ from above as complex numbers. Indeed, we have

$$x_{1,2} = -1 \pm \sqrt{-4} = -1 \pm 2i,$$

Note the relation between the solutions x_1 and x_2. This gives rise to another definition.

DEFINITION 4.3. For a complex number $z = x + yi$, we define the *complex conjugate* number $\bar{z}$ as $\bar{z} = x - yi$.

4.2. Operations with complex numbers. Given a new set of numbers, we need to know how to calculate with them. Fortunately, most operations involving complex numbers are intuitive, that is satisfy (1)–(5), as long as we keep in mind that $i^2 = -1$.

They are defined as follows:

- **Addition:**

$$(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i. \tag{11}$$

For example: $(1 - 4i) + (2 + 3i) = 3 - 1i = 3 - i$.

- **Subtraction:**

$$(x_1 + y_1 i) - (x_2 + y_2 i) = (x_1 - x_2) + (y_1 - y_2)i. \tag{12}$$

For example: $(1 - 4i) - (2 + 3i) = -1 - 7i$.

- **Multiplication:**

$$\begin{aligned}(x_1 + y_1 i) \cdot (x_2 + y_2 i) &= x_1x_2 + y_1x_2 i + x_1y_2 i + y_1y_2 i^2 \\ &= (x_1x_2 - y_1y_2) + (y_1x_2 + x_1y_2)i. \end{aligned} \tag{13}$$

For example: $(1-4i) \cdot (2+3i) = 14 - 5i$.

- **Division:** This is best done using a little trick. For this note first, that for $z = x + yi$, we have
$$z \cdot \bar{z} = (x + yi) \cdot (x - yi) = x^2 + y^2 \in \mathbb{R}.$$
This can be used to compute z_1/z_2 in the following way:
$$\frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right) i$$
if $x_2^2 + y_2^2 > 0$.
For example: $(1-4i)/(2+3i) = \frac{1}{13}(1-4i) \cdot (2-3i) = \frac{1}{13}(-10-11i)$.

Remark 4.4.

(1) Recall that $\mathbb{R} \subset \mathbb{C}$. It is therefore important to note that the 'new' operations for complex numbers agree with the familiar ones, if both z_1 and z_2 are real.

(2) We use rules (1)–(5) for calculations with complex numbers, this means that equations for complex numbers can be transformed or simplified in the same way as equations for real numbers.

We finally introduce another concept, which already appeared in the division of complex numbers.

Definition 4.5. For a complex number $z = x + yi$ we define its *modulus* as
$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}.$$

Finally, we remark that complex numbers are *algebraically closed*, that means that any algebraic equation
$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$$
with complex coefficients $a_n, \ldots, a_0$ has a complex root (this will be proved in the course of complex analysis in the second year). Thus, we do not need to look for further extensions of numbers from the point of view of algebraic equations. But we will see another reason for this in the third chapter.

5. Geometry of complex numbers

Complex numbers shall not be viewed as purely abstract concept, in fact they are intimately connected with the geometry of the plane. Presentation in this section is greatly influenced by the booklet "Geometry of complex numbers, quaternions and spins" by V.I. Arnold (Moscow, 2002, in Russian) translated as [1, Part II].

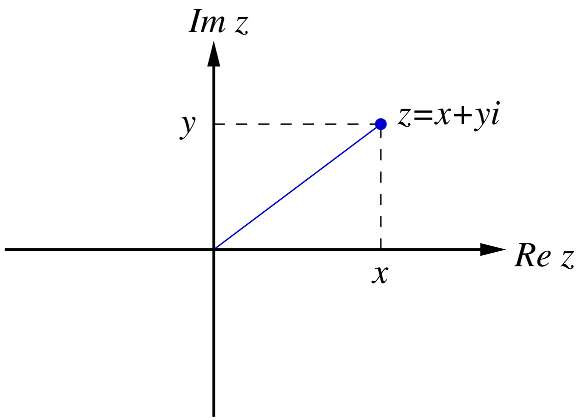


FIGURE 2. A complex number z can be represented as a point in the complex plane.

5.1. The complex plane. The graphical presentation is provided by the *complex plane* (or the *Argand diagram*). Recall that

$$\mathbb{C} = \{z = x + yi,\ x, y \in \mathbb{R}\},$$

such that each complex number is characterised by 2 real numbers x and y.

Draw *rectangular* (also known as *Cartesian*) coordinates on the plane. In the complex plane the real part x of a number z corresponds to its horizontal component (along the real axis), whereas the imaginary part y corresponds to its vertical component (along the imaginary axis), see Figure 2.

We can use the complex plane to interpret operations with complex numbers geometrically.

i) The complex conjugate of a number can be obtained by reflection in the real axis, see Figure 3.
ii) The addition of 2 complex numbers z_1 and z_2 has an easy geometric interpretation as a translation or using a parallelogram, see Figure 4. Therefore, complex numbers behave like *vectors* under addition. (We will discuss vectors in the last part of the course.)

Using the properties of the mirror reflection we note:

LEMMA 5.1. (1) *The conjugation of the complex conjugation returns the complex number:* $\bar{\bar{z}} = z$.

(2) *A complex number is equal to its complex conjugation if and only if it is real:* $z = \bar{z} \Leftrightarrow z \in \mathbb{R}$.

(3) *A complex number z satisfy the identity $z = -\bar{z}$ if and only if it is purely imaginary.*

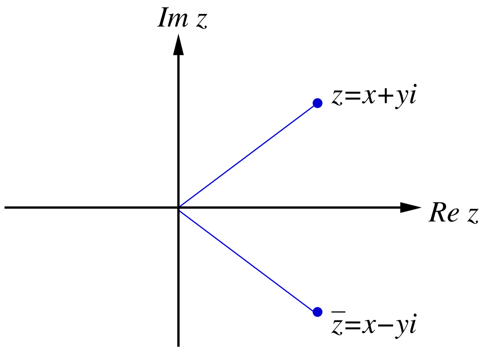


FIGURE 3. A complex number z and its complex conjugate $\bar{z}$ in the complex plane.

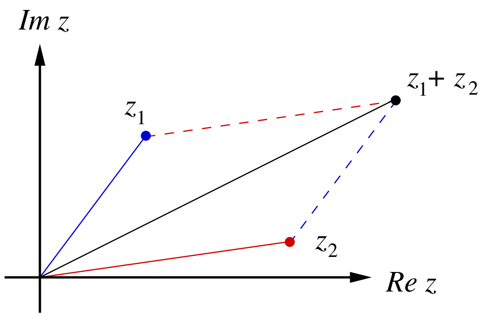


FIGURE 4. Addition of two complex numbers z_1 and z_2.

We will also need the following theorem.

THEOREM 5.2. (1) *Complex conjugation of the sum of two complex numbers is equal to the sum of their complex conjugates:*

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2.$$

(2) *Complex conjugation of the product of two complex numbers is equal to the product of their complex conjugates:*

$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2.$$

PROOF. the first statement follows from the above geometric interpretation of conjugation and sum. It can be also verified from the formula (11).

The second statement follows from the formula (12). □

In order to obtain a clear geometric interpretation for the multiplication and division of complex numbers, it turns out, that it is beneficial to introduce new coordinates in the complex plane; so-called polar coordinates.

5.2. The polar form of a complex number. Instead of using real and imaginary part of a complex number z to determine its position in the complex plane, we can also characterise z by its distance r from the origin 0, and the angle ϕ between the line connecting z and 0, and the positive real axis, see Figure 5. The coordinates (r, ϕ) are called *polar coordinates* in the (complex) plane.

First, we note the geometric meaning of modulus, which follows from the Pythagoras theorem (do you know its proof?):

LEMMA 5.3. *The distance from the point $z = x + yi$ to 0 is equal to the modulus $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$.*

COROLLARY 5.4. *A complex number z and its conjugation $\bar{z}$ have equal moduli: $|z| = |\bar{z}|$.*

Combining the previous lemma with the geometric meaning of addition of complex numbers we obtain the next

COROLLARY 5.5. *The distance between two complex numbers z and w is $|z - w|$.*

The known from geometry *triangle inequality* together with geometric interpretation of sum implies the following inequality for moduli of complex numbers:

$$|z + w| \leqslant |z| + |w|. \tag{14}$$

Thus for the polar coordinates we found that $r = |z|$, i.e. it is the modulus of z.

DEFINITION 5.6. The angle ϕ is called the *argument* of the complex number z, $\arg(z)$.

DEFINITION 5.7. A complex z number with the modulus equal 1, that is $|z| = 1$, is called *unimodular*. For an unimodular z with an argument $\arg(z) = \phi$ we have:

$$z = \cos\phi + i \cdot \sin\phi. \tag{15}$$

The identity (15) can be considered as a definition of *sine* and *cosine* functions.

THEOREM 5.8. *Multiplication by an unimodular complex number z is a rotation of the complex plane by the angle* $\arg(z)$.

PROOF. Take a complex number w. It is transformed by the multiplication to zw. We calculate the modulus:

$$|zw|^2 = zw\overline{zw} = (z\bar{z}) \cdot (w\bar{w}) = |w|^2.$$

So the lengths of a vector is preserved. Furthermore, the distance between points is preserved, cf. Cor. 5.5:

$$|zw_1 - zw_2| = |z(w_1 - w_2)| = |z| \cdot |(w_1 - w_2)| = |w_1 - w_2|.$$

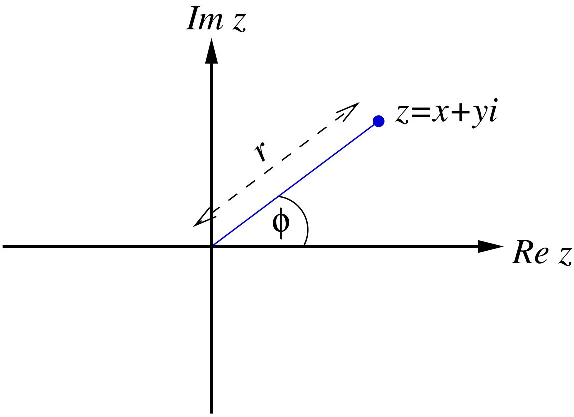


FIGURE 5. Polar coordinates can be used to describe a complex number z in the complex plane.

We note that the transformation $w \mapsto zw$ fixes point 0 (since $z \cdot 0 = 0$), so this transformation is a rotation around 0. To find the angle of rotation we note that $1 \mapsto 1 \cdot z = z$. That is, the complex number 1 with argument 0 is transformed to the number with argument $\arg(z)$. Therefore, multiplication by z increments the argument of any complex number by $\arg(z)$. □

The following simple result immediately follows from formula (13):

LEMMA 5.9. *If $a > 0$ is real then the transformation $w \mapsto aw$ is a scaling with the factor a. In particular, $|a \cdot w| = a \cdot |w|$.*

The lemma implies that if we scale the complex number z by the real factor $\frac{1}{|z|}$ then we get an unimodular complex number $\frac{z}{|z|}$. Thus we obtained the following:

COROLLARY 5.10. *Any complex number z is the product of the positive real number $|z|$ and the unimodular complex number $\frac{z}{|z|}$, that is: $z = |z| \cdot \frac{z}{|z|}$.*

This corollary together with the formula (15) suggest the following definition:

DEFINITION 5.11. For a complex number $z = x + yi$, we call

$$z = r(\cos\phi + i \cdot \sin\phi)$$

its *polar form*. Here, r denotes the modulus of z and ϕ its argument.

REMARK 5.12. The angle ϕ is measured in radians. Therefore, angles which differ by a multiple of 2π are identical. We use the convention that $\phi \in (-\pi, \pi]$.

EXAMPLE 5.13. i) For $z = 1 - i$, we compute $r = |z| = \sqrt{1+1} = \sqrt{2}$, and we find $\phi = -\pi/4$. Therefore,

$$z = \sqrt{2}\left(\cos(-\pi/4) + i \cdot \sin(-\pi/4)\right).$$

ii) For $z = i$, we have $r = 1$ and $\phi = \pi/2$, such that

$$z = i = 1 \cdot (\cos(\pi/2) + i \cdot \sin(\pi/2)),$$

which is obviously true.

We are ready to prove the following theorem:

THEOREM 5.14. *The product $z \cdot w$ has a modulus of $|z||w|$, and an argument $\arg(z) + \arg(w)$.*

PROOF. We can represent the number z as the product $|z|\frac{z}{|z|}$. We note that the number $\frac{z}{|z|}$ has the same argument as z (why?).

Thus, the multiplication by $z = |z| \cdot \frac{z}{|z|}$ is the composition of two transformations: the rotation by $\arg(z)$ (due to the multiplication by $\frac{z}{|z|}$) and scaling by $|z|$. Therefore the modulus of w is transformed to $|z||w|$ and the argument of w is transformed to $\arg(z) + \arg(w)$. □

Geometrically, we can interpret complex multiplication as a stretching and a rotation in the complex plane, see also Figure 6.

5.3. Application to trigonometry. We can use the above connection between arithmetic of complex numbers and geometry of plane to demonstrate main results in trigonometry.

First, we describe the transformation between the Cartesian and polar coordinates. That can be obtained from two two forms of writing of the same complex number and the observation that *two complex numbers are equal if and only if their real parts are equal and their complex parts are equal*. Then, we see that for a point $z = x + yi = r(\cos\phi + i\sin\phi)$ the equality of real and imaginary parts are:

$$x = r\cos\phi, \quad y = r\sin\phi,$$

and on the other hand

$$r = \sqrt{x^2 + y^2}, \quad \tan\phi = \frac{y}{x}.$$

The last formula suggests that $\phi = \arctan\frac{y}{x}$, but we need to be careful choosing the value of ϕ as it may differ by π from the actual. For example, two complex numbers $1 + i$ and $-1 - i$ has the common value $\frac{y}{x} = 1$, but $\arg(1 + i) = \frac{\pi}{4}$ and $\arg(-1 - i) = -\frac{3\pi}{4}$.

Let us now consider the multiplication of two complex numbers z_1 and z_2 in polar coordinates. For this, set

$$z_1 = r_1(\cos\phi_1 + i\sin\phi_1), \quad z_2 = r_2(\cos\phi_2 + i\sin\phi_2).$$

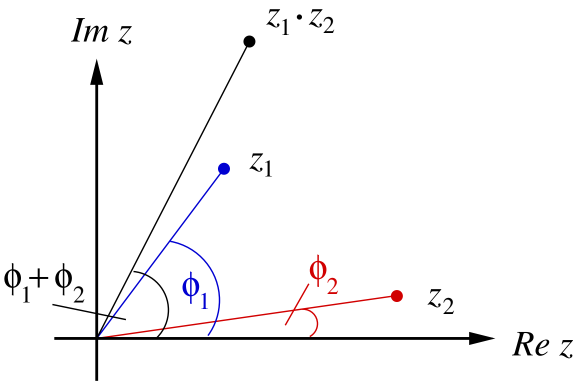


FIGURE 6. Multiplying two complex numbers in the complex plane.

Then using the theorem 5.14 we obtain:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2 \left(\cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2 + i(\cos\phi_1 \sin\phi_2 + \cos\phi_2 \sin\phi_1)\right) \\ &= r_1 r_2 \left(\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2)\right). \end{aligned}$$

This implies the following important trigonometric formulas of addition:

$$\begin{aligned} \cos(\phi_1 + \phi_2) &= \cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2, \\ \sin(\phi_1 + \phi_2) &= \cos\phi_1 \sin\phi_2 + \cos\phi_2 \sin\phi_1. \end{aligned}$$

In the special case $z_1 = z_2 = r(\cos\phi + i\sin\phi)$, we find that

$$\left(r(\cos\phi + i\sin\phi)\right)^2 = r^2 \left(\cos(2\phi) + i\sin(2\phi)\right),$$

which gives the doubling argument identities:

$$\cos(2\phi) = \cos^2\phi - \sin^2\phi, \qquad \sin(2\phi) = 2\cos\phi\sin\phi.$$

This results can be generalized and yields an important formula for complex numbers.

THEOREM 5.15 (De Moivre's Theorem). *For all complex numbers* $z = r(\cos\phi + i\sin\phi)$ *and all natural numbers* $n \in \mathbb{N}$, *we have*

$$\left(r(\cos\phi + i\sin\phi)\right)^n = r^n \left(\cos(n\phi) + i\sin(n\phi)\right).$$

This theorem can be proved using mathematical induction, and we will use it later to compute complex roots. Next, however, we will take a look at yet another representation of complex numbers, which comes from one of the most beautiful and surprising formulas in mathematics.

5.4. The exponential form and Euler's formula. We continue to explore the connection between geometry and arithmetic of complex numbers. The well-known proposition (Prop. I.5 in Euclid): *the base angles of an isosceles triangle are equal* (do you know its proof?). It is easy to show the following

PROPOSITION 5.16. *If in a triangle ABC, the vertex C belong to the (unique) circle with the diameter AB, then the angle ACB is right.*

PROOF. We denote by O the midpoint of AB (see the left drawing on Fig. 7), then O is the centre of the circle with the diameter AB. By the theorem assumption, the segments OA, OB and OC are equal and we have two pairs of equal angles in two isosceles triangles (see the illustration). Since the sum of all angles in ABC (as any other triangle) is $180°$, the angle ACB is exactly the half of this, that is it is a right angle. □

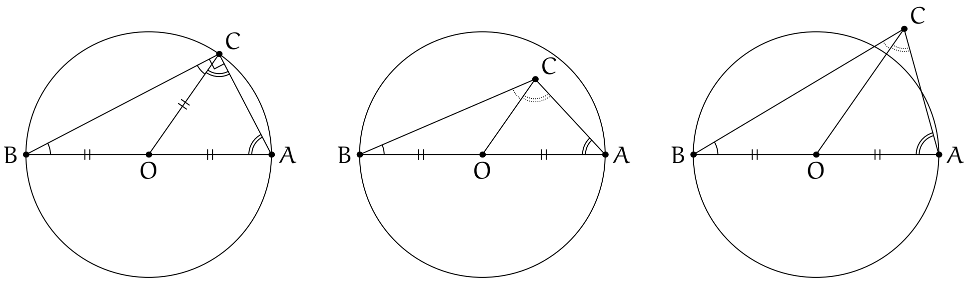


FIGURE 7. The diameter and the right angle.

Also, it is elementary to derive from the isosceles triangle theorem that *among two sides of a triangle, the bigger side is opposite to the larger angle* (this also follows from the sine rule). Using this statement we can show the converse of Prop. 5.16.

PROPOSITION 5.17. *If in a triangle ABC the angle C is right then the vertex C belong to the circle with the diameter AB.*

You can proof this theorem modifying the proof of Prop. 5.16, see the central and right drawing on Fig. 7.

The following result easily follows from the geometric meaning of multiplication of complex numbers.

LEMMA 5.18. *The following conditions are equivalent:*

(1) *Vectors z and w are orthogonal.*
(2) $\operatorname{Re}(z\bar{w}) = 0$ *(in other words: $z\bar{w}$ is purely imaginary).*

Moreover, if $w \neq 0$ the above conditions are equivalent to the following:

(3) $\mathrm{Re}(\frac{z}{w}) = 0$ *(in other words: $\frac{z}{w}$ is purely imaginary).*

The imaginary part has a geometric meaning as well:

LEMMA 5.19. *The following conditions are equivalent:*

(1) *Vectors z and w are co-linear.*
(2) $\mathrm{Im}(z\bar{w}) = 0$ *(in other words: $z\bar{w}$ is a real number).*

Moreover, if $w \neq 0$ the above conditions are equivalent to the following:

(3) $\mathrm{Im}(\frac{z}{w}) = 0$ *(in other words: $\frac{z}{w}$ is a real number).*

We already know that $\overline{zw} = \bar{z} \cdot \bar{w}$. We can show by mathematical induction that $\overline{z^n} = (\bar{z})^n$ for any natural n. Can we extend the meaning of an expression a^z for complex z and real $a > 0$? In view of our previous discussion it is naturally to request that $\overline{a^z} = a^{\bar{z}}$ on top of the usual law of exponents: $a^{z+w} = a^z \cdot a^w$. From these two requirements follows:

PROPOSITION 5.20. *Let $a > 0$ and ϕ be reals, then the number $a^{i\phi}$ shall be unimodular.*

PROOF. Consider the expression $z = (a^{i\phi} - 1)(a^{-i\phi} + 1)$, we claim that it is purely imaginary. To this end we will show that $\bar{z} = -z$:

$$\begin{aligned}
\bar{z} &= \overline{(a^{i\phi} - 1)(a^{-i\phi} + 1)} \\
&= (a^{-i\phi} - 1)(a^{i\phi} + 1) && \text{(by Thm. 5.2 and } \overline{a^z} = a^{\bar{z}}) \\
&= (a^{-i\phi} - 1)(a^{i\phi}a^{-i\phi})(a^{i\phi} + 1) && \text{(since } a^{i\phi}a^{-i\phi} = a^{i\phi - i\phi} = a^0 = 1) \\
&= ((a^{-i\phi} - 1)a^{i\phi})(a^{-i\phi}(a^{i\phi} + 1)) && \text{(by the associative law)} \\
&= (1 - a^{i\phi})(1 + a^{-i\phi}) && \text{(by the distributive law)} \\
&= -(a^{i\phi} - 1)(a^{-i\phi} + 1) \\
&= -z.
\end{aligned}$$

If $(a^{i\phi} - 1)(a^{-i\phi} + 1)$ is purely imaginary, then by Lem. 5.18 vectors $(a^{i\phi} - 1)$ and $(a^{i\phi} + 1)$ are orthogonal. But these vectors connect the point $a^{i\phi}$ with end-points 1 and -1 of the unit circle. By the Prop. 5.17 the orthogonality of vectors implies that $a^{i\phi}$ belong to the unit circle, i.e. $a^{i\phi}$ is an unimodular complex number. □

If $a^{i\phi}$ is unimodular, then as we already know $a^{i\phi} = \cos\psi + i\sin\psi$. For some angle ψ, which can be considered as function of ϕ. The law of exponents and the law of complex multiplication tell us that: $a^{in\phi} = \cos n\psi + i\sin n\psi$ for any natural n. Thus, ψ shall be a linear function of ϕ, that means that there is a constant α determined solely by a such that $\psi = \alpha\phi$. Clearly, the simplest situation occurs if $\alpha = 1$ and then $\psi = \phi$.

DEFINITION 5.21. The *Euler's constant* e is a real, which satisfies to *Euler's Formula*.

$$e^{i\phi} = \cos\phi + i\sin\phi. \tag{16}$$

We have arrived at one of the most remarkable formulas in the whole of mathematics. To hint its importance we mention that the number e also known as the *base of natural logarithms*. It is known that e is an irrational number approximately equal to $2.718281828\ldots$. Furthermore, like π, the Euler's constant is not a root of any algebraic equation with integer coefficients, such numbers are called *transcendental*. An example of an irrational numbers which is not transcendental is $\sqrt{2}$ since it is a root of the equation $x^2 - 2 = 0$ with integer coefficients.

Note that Euler's formula gives us a way to write the polar form of a complex number in a more compact way.

DEFINITION 5.22. For a complex number $z = x + yi = r(\cos\phi + i\sin\phi)$, we call

$$z = re^{i\phi}$$

its *exponential form*.

REMARK 5.23.

(1) We have discussed 3 equivalent representations of a complex number z, namely

$$\begin{aligned} z &= x + yi, \quad \text{with } x, y \in \mathbb{R} \\ &= r(\cos\phi + i\sin\phi), \quad \text{where } r = |z|,\ \phi = \arg(z) \\ &= re^{i\phi}. \end{aligned}$$

(2) De Moivre's Formula immediately follows from the exponential form. Indeed, the theorem simply follows from the fact that

$$z^n = \left(re^{i\phi}\right)^n = r^n \left(e^{i\phi}\right)^n = r^n e^{in\phi},$$

and converting this back into polar form.

(3) If we set $\phi = \pi$ in Euler's formula, we get $e^{i\pi} = \cos\pi + i\sin\pi = -1$ or

$$e^{i\pi} + 1 = 0.$$

This remarkable expression is known as *Euler's identity*. It gives a relation between the numbers $0, 1, i, e, \pi$ and is considered to be one of the most beautiful and important mathematical formulas.

5.5. Sketching sets of complex numbers in the complex plane. Before leaving this section we consider look at an example of a set of complex numbers sketched in the complex plane.

EXAMPLE 5.24. Sketch the following set in the complex plane

$$M = \{z \in \mathbb{C} \mid |z - 1 - i| > \sqrt{2}\}$$

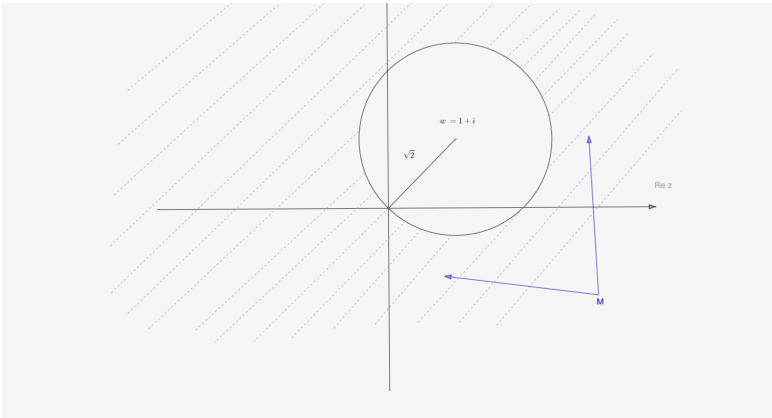


FIGURE 8. The set $M = \{z \in \mathbb{C} \mid |z - 1 - i| > \sqrt{2}\}$ in the complex plane.

Notice firstly that the complex number $z - 1 - i$ can be written $z - (1 + i)$, i.e. as the "difference" of two complex numbers. Write $z = x + yi$ with $x, y \in \mathbb{R}$. Thus $z - 1 - i = (x - 1) + (y - 1)i$ and so

$$\begin{aligned} |z - 1 - i| > \sqrt{2} \quad &\Leftrightarrow \quad \sqrt{(x-1)^2 + (y-1)^2} > \sqrt{2} \\ &\Leftrightarrow \quad (x-1)^2 + (y-1)^2 > 2 \end{aligned}$$

(where we have used the fact that $(x - 1)^2 + (y - 1)^2 \geqslant 0$). Now we note that $(x - 1)^2 + (y - 1)^2 = 2$ is the equation of the circle with radius $\sqrt{2}$ and centre $(1, 1)$ in the Cartesian x, y plane. Interpreting this remark in the context of the complex plane thus shows us that the set M is the set of complex numbers lying on the circle of radius $\sqrt{2}$ and centre $w = 1 + i$. See Figure 8.

6. Complex roots

In the last part of this chapter we discuss how to solve equations for complex numbers. Of particular interest here are n-th roots of complex numbers.

DEFINITION 6.1. For a number $w \in \mathbb{C}$ we define its *complex n-th roots* as the solutions z of the equation $z^n = w$.

EXAMPLE 6.2. i) The complex square-roots of $w = 2$ are given by $\{-\sqrt{2}, \sqrt{2}\}$.
ii) The complex 4th-roots of $w = 1$ are $\{1, -1, i, -i\}$ (you can easily check that the 4th power of all of these numbers gives $w = 1$).

So, complex square-roots form a set of complex numbers.

REMARK 6.3. We need to take care with the notation $z = \sqrt[n]{w}$ for complex roots. In this notes it is used for *the whole set of all roots*. Do not confuse this concept with the 'square-root function' $f(x) = \sqrt{x}$, which gives the unique number y (the only positive real from the set $\sqrt[2]{x}$) and is only defined for non-negative real numbers x.

6.1. Computation of complex roots. Complex roots of numbers are best computed using the exponential form. So, let $w = re^{i\phi}$, let $n \in \mathbb{N}$, and let $z = \rho e^{i\theta}$ be a solution of $z^n = w$. Since, by de Moivre's Formula

$$z^n = \rho^n e^{in\theta},$$

we must have

$$\rho = \sqrt[n]{r}$$

(where this root denotes the positive real solution of this equation), and there are n possible solutions for the angle θ

$$\theta \in \left\{ \frac{\phi + 2k\pi}{n}, \text{ with } 0 \leqslant k \leqslant n-1 \right\}.$$

In particular, we obtain the important result that every complex number (different from zero) has exactly n different n-th roots).

REMARK 6.4. Before considering an example let us look informally at why the last sentence holds (i.e. why every nonzero complex number has exactly n different n-th roots). Consider $w = re^{i\phi}$, then in fact $w = re^{i\cdot(\phi+2k\pi)}$ for *any* integer k (i.e. $k \in \mathbb{Z}$). In other words we have infinitely many ways of writing/representing the same complex number w in exponential (or polar) form. When we are looking for the arguments (angles) of the nth roots of w, we are in fact looking for θ such that $n\theta \in \{\phi + 2k\pi \mid k \in \mathbb{Z}\}$ since we are in effect trying to solve the equation $z^n = w$ with $z = \rho e^{i\theta}$ and $w = e^{i\cdot(\phi+2k\pi)}$ or, in other words, $\rho^n e^{in\theta} = e^{i\cdot(\phi+2k\pi)}$, (for any $k \in \mathbb{Z}$). But looking for θ such that $n\theta \in \{\phi + 2k\pi \mid k \in \mathbb{Z}\}$ of course means looking for $\theta \in \{\frac{\phi+2k\pi}{n} \mid k \in \mathbb{Z}\}$. Now if we let k range over the numbers $\{0, 1, \dots, n-1\}$ only we see that the possible values of θ in this range, i.e. the set

$$\left\{ \frac{\phi}{n}, \frac{\phi + 2\pi}{n} \dots, \frac{\phi + 2(n-1)\pi}{n} \right\} \tag{17}$$

are all distinct. On the other hand if we set $k = n$ and we let $\theta = \frac{\phi+2k\pi}{n}$ then, rewriting n for k we see that this is just $\theta = \frac{\phi+2n\pi}{n} = \frac{\phi}{n} + 2\pi = \frac{\phi}{n}$. But $\frac{\phi}{n}$ is already in the set of values picked out for θ by letting k range over the numbers $\{0, 1, \dots, n-1\}$. In effect, for $k \geqslant n$ (or $k < 0$) the values of θ just repeat some value already obtained in the set displayed in (17). So this set is precisely the set of arguments (angles) of the nth roots of $w = e^{i\phi}$.

EXAMPLE 6.5. Compute all $\sqrt[3]{1 + \sqrt{3}i}$. *(I.e. solve $z^3 = w$ where $w = 1 + \sqrt{3}i$.)*

Solution:

i) Transform the number w into polar form: $w = 1 + \sqrt{3}i$.
So, $|w| = 2$, and $\arg(w) = \arctan \frac{\sqrt{3}}{1} = \frac{\pi}{3}$, and we have $w = 2 \cdot e^{i\frac{\pi}{3}}$.

ii) Compute the roots for $\sqrt[3]{1+\sqrt{3}i}$. In other words solve the equation $z^3 = w = 1+\sqrt{3}i$. (Also written as $z = \sqrt[3]{1+\sqrt{3}i}$.) We expect to find 3 different solutions z_1, z_2, z_3. Using the formulas above we compute for[1] $k \in \{0, 1, 2\}$,

$$z_{k+1} = te^{\phi_{k+1}}$$

where (the modulus of z_{k+1})

$$t = \sqrt[3]{2}$$

and

$$\phi_{k+1} = \frac{\frac{\pi}{3} + 2k\pi}{3} = \frac{\pi + 6k\pi}{9}.$$

(To understand the notation[2] here, notice that with $k = 0$, $z_{k+1} = te^{\phi_{k+1}}$ rewrites as $z_1 = te^{\phi_1}$; with $k = 1$, $z_{k+1} = te^{\phi_{k+1}}$ rewrites as $z_2 = te^{\phi_2}$; and with $k = 2$ $z_{k+1} = te^{\phi_{k+1}}$ rewrites as $z_3 = te^{\phi_3}$.)

We thus find that

$$\begin{aligned}
z_1 &= \sqrt[3]{2} \cdot e^{i\frac{\pi}{9}}, \\
z_2 &= \sqrt[3]{2} \cdot e^{i\left(\frac{\pi}{9} + \frac{2}{3}\pi\right)} \\
&= \sqrt[3]{2} \cdot e^{i\frac{7}{9}\pi} \\
z_3 &= \sqrt[3]{2} \cdot e^{i\left(\frac{\pi}{9} + \frac{4}{3}\pi\right)} \\
&= \sqrt[3]{2} \cdot e^{i\frac{13}{9}\pi} \\
&= \sqrt[3]{2} \cdot e^{-i\frac{5}{9}\pi},
\end{aligned}$$

where for z_3 (in the last step) we have subtracted 2π such that the angle $\phi \in (-\pi, \pi]$.

iii) If required, we finally convert the roots $z_1, \ldots, z_3$ back into Cartesian coordinates, that is, into the standard form $z_k = x_k + y_k i$. For example, for $z_1 = x_1 + y_1 i$, we have

$$x_1 = \sqrt[3]{2}\cos(\pi/9) \approx 1.183938513, \quad y_1 = \sqrt[3]{2}\sin(\pi/9) \approx 0.4309183781,$$

and so $z_1 \approx 1.183938513 + 0.4309183781i$.

[1] The notation $k \in \{0, 1, 2\}$ means "k is in the set $\{0,1,2\}$", in other words "either $k = 0$ or $k = 1$ or $k = 2$".

[2] You might prefer (as I do) to name the solutions as z_0, z_1, z_2. In this case the notation becomes, for $k \in \{0, 1, 2\}$, $z_k = te^{\phi_k}$ where $\phi_k = \frac{\frac{\pi}{3}+2k\pi}{3} = \frac{\pi+6k\pi}{9}$. I have chosen not to use this notation here to maintain consistency with the rest of the notes (where solutions are denoted x_1, x_2 etc).

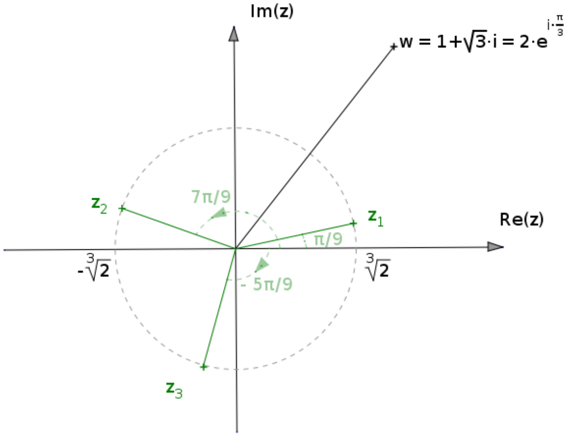


FIGURE 9. The three complex solutions z_1, z_2 and z_3 of $z^3 = w$ (with $w = 1 + \sqrt{3}i$).

REMARK 6.6. The last example can be represented in the complex plane as in Figure 9. Notice that geometrically the roots z_1, z_2 and z_3 lie on a circle. Also that the angle between the roots (in terms of the line connecting each root to the origin) is precisely $\frac{6\pi}{9}$, i.e. $\frac{2\pi}{3}$. More generally for $n \geqslant 1$ you will always find $\frac{2\pi}{n}$ separating the (lines to the origin) of the n-many n-th roots of w.

6.2. Solving polynomial equations. Complex roots are of importance for the solution of polynomial equations. We will only discuss one simple example here.

EXAMPLE 6.7. Find all complex solutions of $z^4 - 2z^2 + 1 = 3$.

Solution: For this example, let us first introduce $v = z^2$. Then we obtain an equation for v

$$v^2 - 2v + 1 = (v - 1)^2 = 3.$$

We immediately conclude that $v-1$ is either $\sqrt{3}$ or $-\sqrt{3}$, such that the two solutions for v are

$$v_1 = 1 + \sqrt{3}, \quad v_2 = 1 - \sqrt{3}.$$

Finally, recalling that $v = z^2$, we arrive at 4 solutions of the equation in z

$$\begin{aligned} z_1 &= \sqrt{v_1} &&= \sqrt{1+\sqrt{3}}\,, \\ z_2 &= -\sqrt{v_1} &&= -\sqrt{1+\sqrt{3}}\,, \\ z_3 &= \sqrt{v_2} &&= \sqrt{1-\sqrt{3}}\,, \\ & &&= \sqrt{(\sqrt{3}-1)\cdot(-1)}\,, \\ & &&= \left(\sqrt{(\sqrt{3}-1)}\right) i\,, \\ z_4 &= -\sqrt{v_2} &&= -\left(\sqrt{(\sqrt{3}-1)}\right) i\,. \end{aligned}$$

(Note that $v_2 < 0$.)

The computations in this example are pretty straightforward. This is not always the case. Indeed, there is no general method or formula for solving polynomial equations of degree greater than 4. (The degree of an equation is the highest power of z appearing in it.)

On the other hand, it is always true that for an equation of degree n, there exist n complex solutions. For example, we found 4 solutions for the 4-th order equation above. This important result is known as the *Fundamental Theorem of Algebra* (Gauss, Argand), which is most naturally proven in the course of *Complex Analysis*.

CHAPTER 2

Sequences and Series

Arithmetic of rational numbers, i.e. fractions, is performed according to explicit rules which return precise answers. How this can be extended to irrationals numbers. e.g. $\sqrt{2}$, π, e? The corresponding is easier to describe in terms of sequences and their limits.

In this part we discuss sequences and series of real numbers. We will mainly be concerned with the notion of a limit, which is one of the most important concepts in mathematics. We will discuss its rigorous definition, analyse properties of limits and derive conditions for sequences (and series) to have a limit.

1. Sequences of real numbers

1.1. Definition and Examples. A sequence of real numbers is simply an (infinite) ordered list of real numbers

$$(a_1, a_2, a_3, \ldots), \quad \text{with } a_n \in \mathbb{R} \text{ for all } n \in \mathbb{N}.$$

So, in a sequence, we simply have a real number a_n allocated to each natural number n. More precisely, we can define this as a function

Definition 1.1. A *sequence of real numbers* is a function $a : \mathbb{N} \to \mathbb{R}$. For the function values we write $a_n := a(n)$. The whole sequence will be denoted by (a_n) or $(a_n)_{n\in\mathbb{N}}$.

In the simplest cases we can define the function can be explicitly written.

Example 1.2. With the rule $a_n = 2n - 1$, we obtain the sequence of odd numbers

$$a_1 = 1,\ a_2 = 3,\ a_3 = 5,\ a_4 = 7, \ldots \quad \text{or} \quad (1, 3, 5, 7, \ldots).$$

Similarly, with $a_n = \frac{1}{n}$, we have

$$a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3},\ a_4 = \frac{1}{4}, \ldots$$

If a sequence is given by such a rule of the form $a_n = f(n)$, then we call the sequence *explicitly defined*.

An important example of explicitly defined sequence is the sequence of all prime numbers:

$$a_1 = 2, \quad a_2 = 3, \quad a_3 = 5, \quad a_4 = 7, \quad a_5 = 9, \quad a_6 = 11, \ldots .$$

Although the sequence is explicitly defined we do not know an analytic mathematical expression, which produces a_n for any n.

Often it is easy to define new elements of sequences from values of previous elements.

EXAMPLE 1.3. On the other hand, we have already come across the *Fibonacci numbers*, that is, the list of numbers

$$(1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$$

This sequence is defined by a recursion as follows:

$$a_1 = 1,\ a_2 = 1, \text{ and } \ a_n = a_{n-1} + a_{n-2} \ \text{ if } \ n > 2.$$

Hence, in order to compute a_{100}, we need a_{99} and a_{98} and all other elements of the sequence before that. Such sequences, for which $a_n = g(a_{n-1}, a_{n-2}, \dots, a_{n-k})$ are called *recursively or implicitly defined*. Sometimes they are also called difference equation, indicating their more difficult analysis. We also remark, that it is sometimes possible to convert recursive definition of a sequence into explicit, which has clear advantages. For example, this is possible for Fibonacci numbers (can you find an explicit formula?).

The next example starts from recursively defined sequence and provide an explicit formula for it.

EXAMPLE 1.4. Assume you pay £100 into an account at the beginning of the first year. At the end of each year, the bank pays 5% interest. How much money is there in the account after n years?

Solution: Let m_n denote the money in the account at the end of year n. Let $m_0 = 100$ (in pounds) denote the starting capital. Then

$$\begin{aligned}
m_1 &= 1.05 \cdot 100 = 105, \\
m_2 &= 1.05 \cdot 105 = 110.25, \\
m_3 &= 1.05 \cdot 110.25 = 115.76, \quad \text{etc.}
\end{aligned}$$

So, we can define the sequence (m_n) recursively by

$$m_n = 1.05 \cdot m_{n-1}.$$

On the other hand, a careful look at the elements shows that $m_2 = (1.05)^2 \cdot m_0$ and $m_3 = (1.05)^3 \cdot m_0$. We conclude that we have in general

$$m_n = (1.05)^n \cdot m_0.$$

This second formula is obviously more convenient, if you want to compute m_n for $n = 20, 40, 120, \dots$. It is easy to generalise this example to an arbitrary *geometric progression* $a_{n+1} = q \cdot a_n$, i.e. $a_{n+1} = q^n a_1$. An *arithmetic progression* $a_{n+1} = a_n + d$ can be treated similarly: $a_{n+1} = a_1 + nd$.

EXAMPLE 1.5. Assume that a bank agreed to pay 100% interest at the end of year on our deposit of £1. Thus, we will get 1+1=£2 at the end of the year. If the bank agreed to add the interest each month proportionally, then our deposit will be multiplied by $(1+\frac{1}{12})$ each month, and at the end of year we will get $(1+\frac{1}{12})^{12}$ (see the previous example). If the interest will be added every day then, the final sum will be $(1+\frac{1}{365})^{365}$. Motivated by this example, we are interested in the sequence

$$a_n = \left(1+\frac{1}{n}\right)^n. \tag{18}$$

Using the binomial formula:

$$\begin{aligned}
a_n &= \left(1+\tfrac{1}{n}\right)^n\\
&= 1+\tfrac{n}{1!}\tfrac{1}{n}+\tfrac{(n-1)n}{2!}\tfrac{1}{n^2}+\ldots+\tfrac{(n-k)(n-k+1)\ldots(n-1)n}{k!}\tfrac{1}{n^k}+\ldots+\tfrac{1}{n^n}\\
&= 1+1+\tfrac{1}{2!}\left(1-\tfrac{1}{n}\right)+\ldots+\tfrac{1}{k!}\left(1-\tfrac{k}{n}\right)\left(1-\tfrac{k-1}{n}\right)\ldots\left(1-\tfrac{1}{n}\right)+\ldots+\tfrac{1}{n^n} \qquad (19)\\
&\leqslant 1+1+\tfrac{1}{2!}+\tfrac{1}{3!}+\ldots+\tfrac{1}{k!}+\ldots+\tfrac{1}{n!} \qquad \text{(since } \left(1-\tfrac{m}{n}\right)\leqslant 1 \text{ for } m\leqslant n)\\
&\leqslant 1+1+\tfrac{1}{2}+\tfrac{1}{2^2}+\ldots+\tfrac{1}{2^{k-1}}+\ldots+\tfrac{1}{2^{n-1}} \qquad \text{(since } k!>2^{k-1} \text{ for all } k)\\
&\leqslant 1+2\\
&= 3. \qquad (20)
\end{aligned}$$

Thus $2 \leqslant a_n \leqslant 3$ for all n.

EXAMPLE 1.6. Let a be a positive number and we assume that $a \geqslant 1$, otherwise we can replace it by $\frac{1}{a} \geqslant 1$. We want to evaluate $\sqrt{a}$ from the following procedure. Put $x_1 = a$, and recursively define:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right). \tag{21}$$

We can show by induction that $x_{n+1} \leqslant x_n$ and the inequality between *arithmetic*

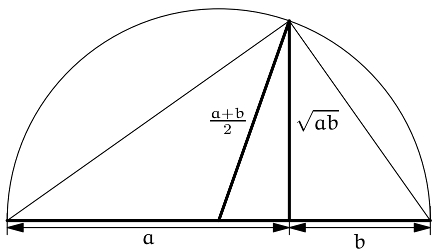


FIGURE 1. The arithmetic mean is no less than the geometric mean. The vertical interval is the geometric mean $\sqrt{ab}$ due to two similar right triangles.

and *geometric means* (see Fig. 1) implies that $x_{n+1} > \sqrt{a}$ (can you do this?), then $|x_{n+1} - \sqrt{a}| \leqslant |x_n - \sqrt{a}|$. That is, each next number in the sequence will be closer to $\sqrt{a}$. Will we get a right approximation from it?

Sometimes, we need some logical operations to define a sequence.

EXAMPLE 1.7. Here are two examples of sequences of *logically defined sequence*:

$$a_n = \begin{cases} n^2, & \text{if } n \text{ is even;} \\ -n^2, & \text{if } n \text{ is odd,} \end{cases} \qquad \text{and} \qquad b_n = \begin{cases} n, & \text{if } n \text{ is prime;} \\ n^2, & \text{if } n \text{ is composite.} \end{cases}$$

Note, that that the first sequence also admit an explicit definition: $a_n = (-n)^2$. Can you find an analytical expression for the sequence (b_n)?

There is no any conceptual difficulty to define sequences of complex numbers as well. However, some of the following results and definitions need to be amended accordingly.

1.2. Properties of sequences. Before we turn to limits of sequences, let us discuss two other, more basic, properties of sequences.

DEFINITION 1.8.
- A sequence (a_n) is *bounded below*, if there exists a number $c \in \mathbb{R}$, such that $a_n \geqslant c$ for all $n \in \mathbb{N}$.
- A sequence (a_n) is *bounded above*, if there exists a number $c \in \mathbb{R}$, such that $a_n \leqslant c$ for all $n \in \mathbb{N}$.
- A sequence (a_n) is *bounded*, if there exists a number $c \in \mathbb{R}$, such that $|a_n| \leqslant c$ for all $n \in \mathbb{N}$.

REMARK 1.9. Note that a sequence is bounded, if and only if it is bounded above and bounded below. The two properties—to be bounded above and below—are logically independent one from another as can be seen from the next example.

EXAMPLE 1.10.
(1) (a_n) with $a_n = \exp(n)$ is bounded below with $c = 0$, but not above.
(2) (a_n) with $a_n = -n$ is bounded above with $c = 0$, but not below.
(3) (a_n) with $a_n = (-1)^n e^n$ is not bounded below or above (it is unbounded).
(4) (a_n) with $a_n = 1/n$ is bounded.
To see this, note that $1/(n+1) < 1/n$, $1/1 = 1$ and $1/n > 0$ for all $n \in \mathbb{N}$. Thus $|a_n| \leqslant 1$ for all $n \in \mathbb{N}$, which shows the boundedness.
(5) The sequence (a_n) (18) is bounded below by 2 and above by 3 as shown in (20).
(6) The sequence (x_n) (21) is bounded above by a and below by $\sqrt{a}$, as discussed in that example.

DEFINITION 1.11.
- A sequence (a_n) is called *increasing (strictly increasing)*, if

$$a_{n+1} \geqslant a_n, \ (a_{n+1} > a_n) \quad \text{for all } n \in \mathbb{N}.$$

- A sequence (a_n) is *decreasing (strictly decreasing)*, if
$$a_{n+1} \leqslant a_n, \ (a_{n+1} < a_n) \quad \text{for all } n \in \mathbb{N}.$$
- If a sequence is either (strictly) increasing or (strictly) decreasing we call it (strictly) *monotone*.

As we can see from the next example

EXAMPLE 1.12. (1) (a_n) with $a_n = \exp(n)$ is strictly increasing.
(2) (a_n) with $a_n = 1/n$ is strictly decreasing.
(3) The constant sequence with $a_n = 3$ for all n is both increasing and decreasing.
(4) (a_n) with $a_n = \cos n$ is neither decreasing nor increasing (since $\cos n$ changes sign).
(5) The sequence (a_n) (18) shall be increasing, obviously we will get more money if the interest is added more frequently. Can you prove this rigorously? (Hint: if you compare the formula (19) for $n = m$ and $n = m + 1$ you will see that the later has one more positive term and each other term is bigger than the respective term for $n = m$.)
(6) The sequence (x_n) (21) is strictly decreasing, as discussed in that example.

The property of boundedness is not completely independent from the property to be monotone.

EXERCISE 1.13. Show that every monotone sequence is bounded at least either above or below.

We will later discuss how these properties are related to the property of a sequence to have a limit. Before, we do so, we first need to introduce this very important property.

2. Limits

2.1. Convergence of sequences. We now define what it means for a sequence to have a limit. The concept of a limit is the foundation of the whole of calculus. Whenever you compute a derivative, you compute a limit (of the difference quotient); whenever you compute an integral, you compute a limit (of a Riemann sum).

The goal in this part is to discuss this concept in detail, establish properties of sequences, which have a limit, and find conditions that ensure that a sequence has a limit.

DEFINITION 2.1. Let $(a_n) \subset \mathbb{R}$ be a sequence. We say that a_n tends (or converges) to the *limit* $l \in \mathbb{R}$ as n tends to infinity (or that a_n has the limit l), if for any $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$, such that $|a_n - l| < \varepsilon$ for all $n > N$.

If a sequence has a limit we call it *convergent* and say that the sequence *tends to* l. We write $a_n \to l$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = l$, if a_n has the limit l.

EXAMPLE 2.2. In order to see that this definition agrees with our expectations about what it means to have a limit, we will prove that the sequence $a_n = 1/n$ tends to $l = 0$ as $n \to \infty$, using the definition. This means, we have to show that for all $\varepsilon > 0$, we find a number N, such that $|a_n| < \varepsilon$ for all $n > N$.

For this, let us choose a fixed but arbitrary $\varepsilon > 0$. Then, for this ε, there exists a number N, such that $N > \frac{1}{\varepsilon}$ (for example, if $\varepsilon = 0.21$, we might choose $N = 5$). We claim that with this N, the condition in the definition is satisfied.

Indeed, for all $n > N$, we have $\frac{1}{n} < \frac{1}{N}$, and since $N > \frac{1}{\varepsilon}$, we also have $\frac{1}{N} < \varepsilon$.

So, we have that $a_n = |a_n - 0| < \frac{1}{\varepsilon}$ for all $n > N$. And since ε was arbitrary, this proves that $a_n \to 0$ as $n \to \infty$.

Discussion of the definition. In short we can write the definition for a sequence to have limit l as follows:

$$\lim_{n\to\infty} a_n = l \quad \Leftrightarrow \quad \forall\varepsilon > 0\ \exists N: \ |a_n - l| < \varepsilon,\ \forall n > N.$$

Let us discuss the separate parts of this in detail.

- '$\forall\varepsilon > 0$': This really means 'for all small positive numbers ε'. Indeed, if the inequality $|a_n - l| < \varepsilon$ is satisfied for one ε, then it is automatically satisfied for all larger numbers. So, this condition becomes stricter, the smaller the number ε is.
- '$\exists N \ldots \forall n > N$:' This means that the inequality is fulfilled for a whole 'tail' of the sequence (a_n).
- '$|a_n - l| < \varepsilon$': This means $l - \varepsilon < a_n < l + \varepsilon$, like an easy discussion of the inequality reveals.

Hence, we can illustrate the property of a sequence to have a limit l as in Figure 2: Give some number $\varepsilon > 0$, we find a number N, such that to the right of the line $n = N$ all elements of the sequence lie in a 'tube' of diameter 2ε around l. The dynamic process of convergence ('a sequence tends to a limit') has been translated into inequalities, which need to be satisfied for tails of sequences.

DEFINITION 2.3.
- We say that a sequence *diverges*, if it has no limit $l \in \mathbb{R}$.
- Moreover, a sequence (a_n) *diverges to* ∞ ($\lim_{n\to\infty} a_n = \infty$), if for all $A > 0$, there exists a number N, such that $a_n \geqslant A$ for all $n > N$.
- Similarly, a sequence (a_n) *diverges to* $-\infty$ ($\lim_{n\to\infty} a_n = -\infty$), if for all $B < 0$, there exists a number N, such that $a_n \leqslant B$ for all $n > N$.

REMARK 2.4. Comparing definition 2.1 of the limit and the definition 2.3 a sequence divergent to $\pm\infty$ we see many similarities. Thus, for those sequences we says that they *tend to* $\pm\infty$. The behaviour of divergent sequences (without any limit) is radically different, see for example $a_n = (-1)^n$.

Our first result tells us that a convergent sequence has a unique limit.

LEMMA 2.5. *A sequence* (a_n) *has at most one limit.*

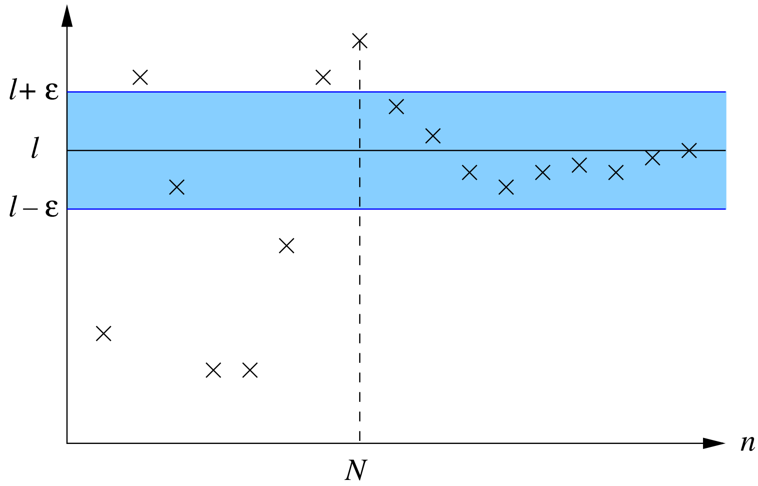


FIGURE 2. Convergence of a sequence to the limit l.

PROOF. We will prove this indirectly, so—looking for a contradiction—let us assume that there is a sequence (a_n), such that $\lim_{n\to\infty} a_n = l_1$ and $\lim_{n\to\infty} a_n = l_2$ with $l_1 \neq l_2$. Then, $|l_2 - l_1| > 0$, and we choose $\varepsilon = \frac{1}{2}|l_2 - l_1|$. Now,

$$\lim_{n\to\infty} a_n = l_1 \quad \Rightarrow \quad \exists N_1 : \ |a_n - l_1| < \varepsilon, \ \forall n > N_1$$
$$\lim_{n\to\infty} a_n = l_2 \quad \Rightarrow \quad \exists N_2 : \ |a_n - l_2| < \varepsilon, \ \forall n > N_2.$$

Therefore, for $n > \max\{N_1, N_2\}$, we find that $|a_n - l_1| < \varepsilon$ and $|a_n - l_2| < \varepsilon$. But this means,

$$\begin{aligned} |l_2 - l_1| &= |l_2 - a_n + a_n - l_1| \\ &\leqslant |l_2 - a_n| + |a_n - l_1| \\ &< \varepsilon + \varepsilon. \end{aligned}$$

Since $|l_2 - l_1| = 2\varepsilon$, we have arrived at $|l_2 - l_1| < |l_2 - l_1|$. This is impossible and the desired contradiction. We conclude that no sequence with two limits can exist. □

REMARK 2.6. In the above proof we have used the fact that

$$|x + y| \leqslant |x| + |y| \quad \forall x, y \in \mathbb{R}.$$

This important inequality is known as the *triangle inequality*.

2.2. Arithmetics of limits. One of the main tasks when faced with a sequence is to compute its limit (if it exists). In this part we will discuss several simple results which help us to achieve this.

THEOREM 2.7. *Let* (a_n) *and* (b_n) *be sequences with* $a_n \to a$ *and* $b_n \to b$ *as* $n \to \infty$. *Then*

i) $a_n + b_n \to a + b$,
ii) $a_n \cdot b_n \to a \cdot b$,
iii) $\frac{a_n}{b_n} \to \frac{a}{b}$, *if* $b \neq 0$,

as $n \to \infty$.

PROOF. We will only present the proof for part i) in order to give a flavour of the general method.

So, let $a_n \to a$ and $b_n \to b$ and choose a fixed, but arbitrary $\varepsilon > 0$. Then we find N_1 and N_2, such that

$$|a_n - a| < \frac{\varepsilon}{2} \qquad \forall n > N_1$$
$$|b_n - b| < \frac{\varepsilon}{2} \qquad \forall n > N_2.$$

Hence, for $n > \max\{N_1, N_2\}$ we have

$$a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}, \quad b - \frac{\varepsilon}{2} < b_n < b + \frac{\varepsilon}{2},$$

and therefore, after adding the inequalities,

$$a + b - \varepsilon < a_n + b_n < a + b + \varepsilon.$$

In summary, we have found an index $N = \max\{N_1, N_2\}$, such that for all $n > N$, we have $|(a_n + b_n) - (a + b)| < \varepsilon$. Since ε was arbitrary, this proves $a_n + b_n \to a + b$ as $n \to \infty$. $\square$

EXAMPLE 2.8. Compute $\lim_{n\to\infty} \frac{n^3-3n^2+1}{3n^3+4}$.

First note that both numerator and denominator of a_n diverge to infinity, as $n \to \infty$. This gives rise to a so-called indeterminate limit, and we need to simplify the expression, before we can apply the above result. This can be done as follows.

$$\begin{aligned} \lim_{n\to\infty} \frac{n^3 - 3n^2 + 1}{3n^3 + 4} &= \lim_{n\to\infty} \frac{n^3}{n^3} \cdot \frac{1 - 3\frac{1}{n} + \frac{1}{n^3}}{3 + \frac{4}{n^3}} \\ &= \frac{\lim_{n\to\infty} 1 - \lim_{n\to\infty} 3\frac{1}{n} + \lim_{n\to\infty} \frac{1}{n^3}}{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{4}{n^3}} \\ &= \frac{1}{3}. \end{aligned}$$

(Note that we have made use of the fact that $\lim_{n\to\infty} 1/n^k = 0$ for $k \in \mathbb{N}$. For $k = 1$ we have already discussed the proof. For general k, this will be proved in detail below.)

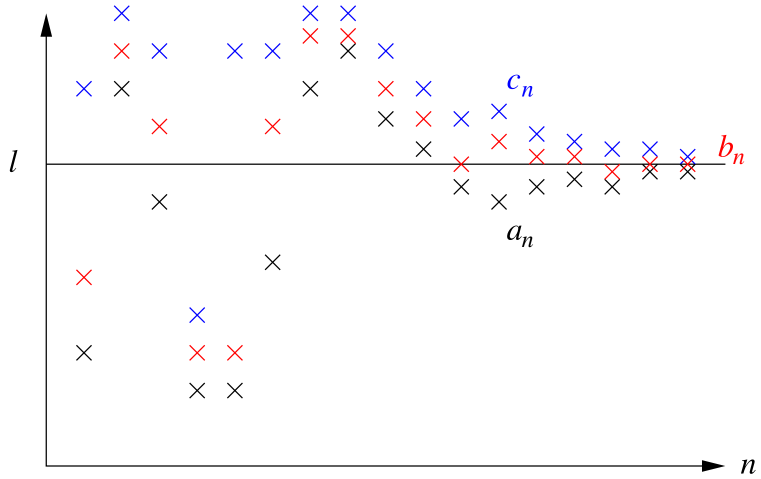


FIGURE 3. Illustration of the sandwich rule.

Another tool for the computation of limits is to compare a difficult sequence to a simpler one. For example, the following statement is true.

LEMMA 2.9. *Consider two sequences* (a_n) *and* (b_n) *with* $a_n \geqslant b_n$ *for all* $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then* $a \geqslant b$.

This lemma remains true in the case that b_n diverges to infinity, that is, if $b = \infty$.

The next result, known as the sandwich rule or squeeze rule, can be used in many examples to compute limits.

THEOREM 2.10 (Sandwich or Squeeze rule). *Let* (a_n) *and* (c_n) *be sequences with* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$. *Assume further, that for a sequence* (b_n), *we have*

$$a_n \leqslant b_n \leqslant c_n, \quad \forall n \in \mathbb{N}.$$

Then (b_n) *converges and* $\lim_{n\to\infty} b_n = l$.

PROOF. Let $\varepsilon > 0$. Then there exists numbers N_1 and N_2, such that

$$\begin{aligned} l - \varepsilon < a_n < l + \varepsilon, &\qquad \forall n > N_1 \\ l - \varepsilon < b_n < l + \varepsilon, &\qquad \forall n > N_2. \end{aligned}$$

Hence, both inequalities are satisfied for $n > \max\{N_1, N_2\}$, and in particular, we have

$$l - \varepsilon < a_n \leqslant b_n \leqslant c_n < l + \varepsilon \quad \forall n > \max\{N_1, N_2\}.$$

This implies $\lim_{n\to\infty} b_n = l$. An illustration of the theorem is given in Figure 3. □

2.3. Computing limits: Examples. One of the main tasks when given a sequence, is to compute its limit. In the simplest cases, we can apply Theorem 2.7. In this section, we will have a look at some important, more complicated examples.

i) $a_n = \frac{1}{n^\alpha}$ with $\alpha > 0$ as a parameter. Then $\lim_{n\to\infty} a_n = 0$.

For $\alpha = 1$ we already checked this, using the definition. If $\alpha > 1$, then we can use the sandwich rule to compute the limit, since

$$\frac{1}{n} \geqslant \frac{1}{n^\alpha} \geqslant 0.$$

Since $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} 0 = 0$, we conclude that $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.

For $0 < \alpha < 1$, we also find that $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$. This can be proved by checking the definition of convergence, or by using the general result that for a sequence (a_n) with $\lim_{n\to\infty} a_n = \pm\infty$, we have $\lim_{n\to\infty} 1/a_n = 0$. We write this as '$1/\infty = 0$'. (We will not prove this result.)

ii) $a_n = \beta^n$ with $\beta \in \mathbb{R}$ as a parameter. The sequence diverges for $\beta \leqslant -1$, for other values we have:

$$\lim_{n\to\infty} a_n = \begin{cases} \infty & \text{if } \beta > 1 \\ 1 & \text{if } \beta = 1 \\ 0 & \text{if } |\beta| < 1 \end{cases}.$$

There is nothing to prove for $\beta = 1$, so we concentrate on the other cases. Let us start with the case $\beta > 1$ and set $\beta = 1 + h$ with some $h > 0$. Then, using the binomial theorem,

$$\beta^n = (1+h)^n = 1 + nh + \ldots + nh^{n-1} + h^n > 1 + nh.$$

And therefore, $\lim_{n\to\infty} \beta^n \geqslant \lim_{n\to\infty} 1 + nh = \infty$.

If $|\beta| < 1$, then we set $\beta = 1/|\beta|$. Therefore $\beta > 1$ and by the above, we have $\lim_{n\to\infty} \beta^n = \infty$. Now we again apply '$1/\infty = 0$' to find $\lim_{n\to\infty} \beta^n = 0$.

iii) $a_n = \frac{n^\alpha}{\beta^n}$, where $\alpha > 0$ and $\beta > 1$. Then $\lim_{n\to\infty} a_n = 0$. (This is usually described as 'exponential growth beats polynomials growth'.)

We will not prove this here, but only note that the proof uses again the binomial theorem (see [2] for more details). Note, however, that the limit is of the form '$\frac{\infty}{\infty}$'. This is a so-called indeterminate limit. There are no general rules for limits of this type (see below for more examples), and each case needs to be discussed carefully.

As an example, let us compute the next limit:

$$\lim_{n\to\infty} \frac{n^{10} - 5n^2 + 2^n}{3 \cdot 2^n - n} = \lim_{n\to\infty} \frac{2^n}{2^n} \cdot \frac{\frac{n^{10}}{2^n} - \frac{5n^2}{2^n} + 1}{3 - \frac{n}{2^n}} = \frac{1}{3},$$

applying the above result.

iv) $a_n = \sqrt[n]{n} = n^{1/n}$. Then $\lim_{n\to\infty} a_n = 1$.

We note, that the limit is another indeterminate expression, now of the form '∞^0'. For the proof, we again employ the binomial theorem. Firstly note that

$a_1 = 1$. Consider $a_n = \sqrt[n]{n}$ for $n > 1$. Note that we can set $\sqrt[n]{n} = 1 + h_n$, for some real number $h_n > 0$. Then

$$n = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2}h_n^2 + \ldots + nh_n^{n-1} + h_n^n.$$

Thus, $n > \frac{n(n-1)}{2}h_n^2$, which after some transformations leads to $h_n < \sqrt{\frac{2}{n-1}}$. Since $\lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0$, we can apply the sandwich rule to see that $\lim_{n\to\infty} h_n = 0$. (Formally let (c_n) and (b_n) be the sequences defined by setting $c_n = 0$ for all $n \in \mathbb{N}$ and $b_1 = 1$ and $b_n = \sqrt{\frac{2}{n-1}}$ for $n > 1$. Then $\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = 0$ whereas $c_n \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$. So by the Sandwich rule $\lim_{s\to\infty} a_n = 0$.) This in turn implies that $\sqrt[n]{n} = 1$.

v) $a_n = \left(1 + \frac{1}{n}\right)^n$ introduced in Example 1.5. We will demonstrate the existence of the limit in Example 2.16, its value $e = \lim_{n\to\infty} a_n$ is called *Euler's constant*, which is $e \approx 2.718281828\ldots$.

This limit is of the form '1^∞'. In this case, it is not easy to see that the sequence should converge at all. It is even more surprising, that the sequence converges to Euler's number e. We will also derive a different formula for e:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

vi) $a_n = \sqrt{n+5} - \sqrt{n+3}$. Then $\lim_{n\to\infty} a_n = 0$. This limit is of the form '$\infty - \infty$'. Again, for limits of this type, no general rule exists, and each of them needs to be discussed separately. Especially, when square-roots are involved, the following method is often successful:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n+5} - \sqrt{n+3} &= \lim_{n\to\infty} \sqrt{n+5} - \sqrt{n+3} \cdot \frac{\sqrt{n+5} + \sqrt{n+3}}{\sqrt{n+5} + \sqrt{n+3}} \\
&= \frac{(n+5) - (n+3)}{\sqrt{n+5} + \sqrt{n+3}} \\
&= \frac{2}{\sqrt{n+5} + \sqrt{n+3}} \\
&= 0.
\end{aligned}$$

To summarise this section, let us recall the main points to observe, when computing limits of sequences.

- If necessary, split the expression for your sequence into as simple *convergent* parts.
- If possible, apply Theorem 2.7.
- Use the rules '$1/\infty = 0$' or '$1/0 = \infty$'.
- If the limit is of one of the following (indeterminate) types

  '∞/∞', '$0/0$', '$0 \cdot \infty$', '$\infty - \infty$', '1^∞', '∞^0', then no general rules are available. You need to simplify the expression as much as possible (and maybe use rules like 'exponential growth beats polynomial growth').

REMARK 2.11. If $\mathbb{I} \subseteq \mathbb{R}$ is an interval (i.e. of the real line), f is a *continuous* function over the interval $\mathbb{I}$, and (a_n) and (b_n) are sequences such that

(i) $a_n = f(b_n)$, for all $n \in \mathbb{N}$,

(ii) $b_n \in \mathbb{I}$, for all $n \in \mathbb{N}$,

(iii) (b_n) converges and $\lim_{n\to\infty} b_n = b$,

then

$$\begin{aligned} \lim_{n\to\infty} a_n &:= \lim_{n\to\infty} f(b_n) \\ &= f\left(\lim_{n\to\infty} b_n\right) \\ &= f(b) \end{aligned}$$

The point here for this course is that you are not expected to be conversant with the notion of a continous function. However, in exercises you may come across examples of functions that are continous over an interval of the real line and you can apply this remark. For example you may come across the following cases:

(1) $\mathbb{I} = \mathbb{R}$ and $f(x) = \sin(x)$, $\cos(x)$ or $\exp(x)$.

(2) $\mathbb{I} = [0, +\infty)$ and $f(x) = \sqrt{x}$.

REMARK 2.12. If the sequence (a_n) has limit a and (c_n) is a (infinite) subsequence of (a_n) then (c_n) also has limit a. For example, if

(i) $c_n = a_{2n}$ for all $n \in \mathbb{N}$,
(ii) $c_n = a_{n+K}$ for some fixed $K \in \mathbb{N}$, e.g. $K = 1$ or $K = 10^7$.

2.4. Convergence criteria. We have introduced the properties of monotonicity and boundedness at the beginning of this part. We will now relate them to the property of a sequence to have a limit. The first result is the following.

LEMMA 2.13. *Every convergent sequence is bounded.*

PROOF. Assume the sequence (a_n) has a limit l. Then for all $\varepsilon > 0$ we find a number N, such that $|a_n - l| < \varepsilon$ for all $n > N$. Now, let us choose $\varepsilon = 1$. Then we find a number N, such that $|a_n - l| < 1$ for all $n > N$. Hence the subsequence $(a_n)_{n>N}$ is bounded. On the other hand, there are only finitely many elements $a_1, a_2, \ldots, a_N$ left, and there must exist a maximum or minimum of those finitely many numbers. Therefore, the sequence (a_n) is bounded. $\square$

So far we have focussed on ways to compute the limit of a sequence. In some cases, however, this computation is very difficult or even impossible to achieve. Then, the next best thing is show that a sequence actually has a limit. (In that case, it makes sense to look for ways to approximate this limit.) We will state one result, which will be of particular importance in the next chapter, when we discuss *series* of real numbers.

THEOREM 2.14 (Monotone Convergence). *Consider a sequence (a_n), which is decreasing and bounded below. Then (a_n) has a limit.*

Similarly, consider a sequence (b_n), which is increasing and bounded above. Then (b_n) has a limit.

REMARK 2.15.

(1) In the above theorem, no statement about the location of the limit is made. This is characteristic for a convergence criterion. It only gives us conditions for a sequence to converge.
(2) We can state the theorem in a different way: A monotone sequence either diverges to $\pm\infty$ or it has a limit l. (This gives us a connection between monotonicity and convergence.)
(3) We can also ask what happens if a sequence (a_n) is bounded (but not necessarily monotone). In this case it can be shown that a subsequence converges, but in general not the whole sequence. (This is known as the *Bolzano–Weierstrass Theorem*, but it is beyond the scope of this course.)

We conclude this part with an application of Theorem 2.14.

EXAMPLE 2.16. We have seen that the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ considered in Example 1.5 is monotonically increasing and bounded above by 3. Thus there is a limit, which is called *Euler's constant* e.

EXAMPLE 2.17. Consider the sequence x_n defined in Example 1.6. We have seen that the sequence is monotonically decreasing: $x_{n+1} \leqslant x_n$. Yet, the sequence is bounded: $x_n \geqslant \sqrt{a}$. Thus, the sequence has a limit $\lim_{n\to\infty} x_n = l$. Take the recurrence relation (21) and pass to the limit there:

$$\lim_{n\to\infty} x_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} x_n + \frac{a}{\lim_{n\to\infty} x_n}\right).$$

Thus the limit l has to satisfy the relation $l = \frac{1}{2}\left(l + \frac{a}{l}\right)$, that is $l^2 = a$ or $l = \pm\sqrt{a}$. However, since all $x_n \geqslant \sqrt{a}$ are positive, the limit l have to be positive as well. Therefore $l = \sqrt{a}$.

EXAMPLE 2.18. Let us consider (a_n), defined by

$$a_{n+1} = \frac{1}{2}a_n(1 - a_n), \quad a_0 = \frac{1}{2}.$$

It is easy to check the following:

a) If $a_n \in [0,1]$, then $a_{n+1} \in [0,1]$.
b) If $a_n \in [0,1]$, then $a_{n+1} \leqslant a_n$.

Indeed consider (a). Firstly $a_1 = \frac{1}{2}$. Suppose that $n \geqslant 1$ and $a_n \in [0,1]$. If $a_n = 0$ or 1 then $a_{n+1} = 0$. Otherwise $0 < a_n < 1$ and so $0 < 1 - a_n < 1$. But this means that

$$0 < \frac{1}{2} a_n(1 - a_n) < \frac{1}{2} a_n .$$

Thus $a_{n+1} \in [0,1]$. So by Mathematical Induction we can conclude that $a_n \in [0,1]$ for all $n \in \mathbb{N}$.

Now consider (b). As we have already seen, if $a_n = 0$ or 1 then $a_{n+1} = 0$. So $a_{n+1} \leqslant a_n$ (by (a)). Otherwise $0 < 1 - a_n < 1$ (as we have already noted) so that

$$a_{n+1} = \frac{1}{2} a_n(1 - a_n) < \frac{1}{2} a_n < a_n .$$

Thus $a_{n+1} \leqslant a_n$ for all $n \geqslant 0$.

Therefore, the sequence (a_n) is decreasing, and it is bounded below by $c = 0$. Using Theorem 2.14, we conclude that (a_n) has a limit, say a.

In this case, we can also compute the limit a. For this, note first that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ (this follows from Rem. 2.12, which is easy to show from the definition of the limit). Hence, we can take the limit in the defining equation to find (for the second "=" in the equation below)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{1}{2} a_n(1 - a_n) \right)$$

But then by applying Theorem 2.7 we get that

$$\lim_{n\to\infty} a_n = \frac{1}{2} \lim_{n\to\infty} a_n (1 - \lim_{n\to\infty} a_n)$$

and so since we know that the sequence (a_n) has a limit, which we denote a (i.e. $a = \lim_{n\to\infty} a_n$) we get the equation

$$a = \frac{1}{2} a(1 - a).$$

This equation has the solutions $a_1 = 0$ and $a_2 = -1$, and the only possible limit for (a_n) is therefore $a = a_1 = 0$. We have thus shown that (a_n) converges to 0.

Example 2.19. We shall be careful when using the above technique to evaluate limits of recursively defined sequences. For example, take the sequence defined by $a_{n+1} = -a_n$. If we put $\lim_{n\to\infty} a_{n+1} = -\lim_{n\to\infty} a_n$ we shall conclude $\lim_{n\to\infty} a_n = 0$. However, the sequence $1, -1, 1, -1, \ldots$ does not have a limit at all.

3. Infinite Series

Given a sequence, (a_k), we have so far investigated the behaviour of (a_k) as $k \to \infty$. In some examples, however, we are not interested in the sequence (a_k) itself, but in what happens when we add up the elements a_k. Adding up, the first n terms, $a_1, \ldots, a_n$, we can define

$$s_n = a_1 + \ldots + a_n = \sum_{k=1}^{n} a_k.$$

The number s_n is called a *partial sum*. If we are interested in the sum of all numbers a_k, then—since adding an infinite quantity of numbers is not possible—we will again consider a limit process, namely the limit of the sequence of partial sums (s_n). This leads to the idea of an infinite series.

3.1. Definition and Examples.

DEFINITION 3.1. Let (a_k) be a sequence, and let (as above) $s_n = \sum_{k=1}^{n} a_k$ denote the n-*th partial sum*. The *infinite series* $\sum_{k=1}^{\infty} a_k$ is said to converge, if the sequence of partial sums (s_n) *converges*. We call $s = \lim_{n\to\infty} s_n$ the sum of the series and write

$$s = \sum_{k=1}^{\infty} a_k.$$

(To denote that $\sum_{k=1}^{\infty} a_k$ converges, we also use the notation $\sum_{k=1}^{\infty} a_k < \infty$.)

If the sequence (s_n) diverges to $\pm\infty$, then we say that the infinite series *diverges* to $\pm\infty$ and write $\sum_{k=1}^{\infty} a_k = \pm\infty$.

Let us take a look at a few examples.

i) Let us take the interval $[0, 1]$ and colour the left half $[0, \frac{1}{2}]$ in blue. Then we colour in blue the left half $[\frac{1}{2}, \frac{3}{4}]$ of the reminder $[\frac{1}{2}, 1]$ and so on. On the n-th step the uncoloured part of the interval has the length $\frac{1}{2^n}$ and the coloured has the length $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}$. In the limit $n \to \infty$ the whole interval will be coloured thus we shall agree that:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^k}.$$

The same Fig. 4 can be used to illustrate one of Zeno's paradoxes: *Achilles and the Tortoise*.

ii) More generally, if $a_k = q^k$ for some fixed real number $q \in \mathbb{R}$, then the corresponding series $\sum_{k=0}^{\infty} q^k$ is called a *geometric series*. The geometric series is one of the rare examples, for which the sum can be computed explicitly. To see this, recall that

$$\sum_{k=0}^{n} q^k = 1 + q + q^2 + \ldots + q^n = \frac{1 - q^{n+1}}{1 - q}, \quad \text{if } q \neq 1.$$

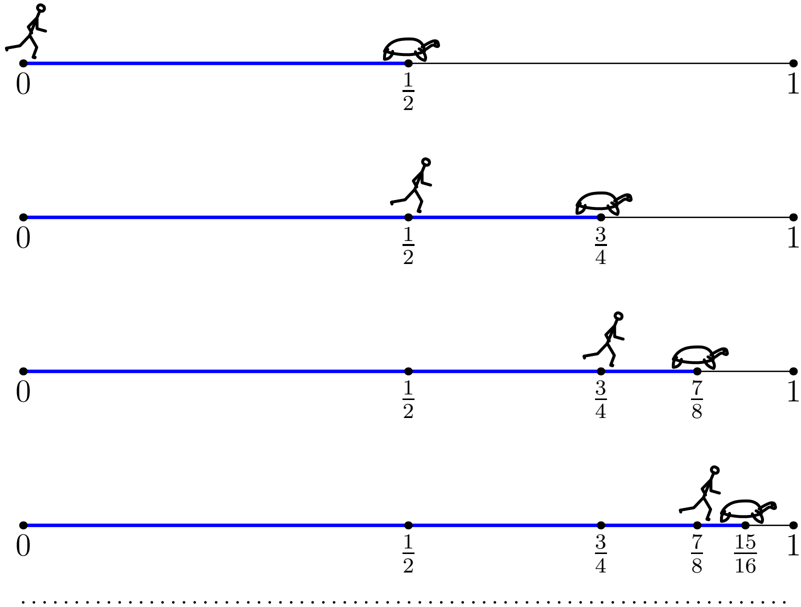


FIGURE 4. Achilles and Tortoise

(If $q = 1$, then $\sum_{k=0}^{n} q^k = n + 1$.)

Since we have an explicit formula for the n-th partial sum, we can now easily investigate its behaviour as n tends to infinity. Since $q^n \to 0$ for $|q| < 1$, we obtain the following important result

$$\sum_{k=0}^{\infty} q^k = \begin{cases} \frac{1}{1-q}, & \text{if } |q| < 1; \\ \infty, & \text{if } q > 1. \end{cases}$$

(Note that no statement is made for the range $q < 1$. In this case the series diverges, but does not approach infinity or minus infinity.)

iii) For $a_k = 1/k$, we find $s_1 = 1$, $s_2 = 1 + \frac{1}{2}$, $s_3 = 1 + \frac{1}{2} + \frac{1}{3}$, etc. Obviously, for the sequence (a_k) we have $a_k \to 0$, as $k \to \infty$. But what happens to the sequence of partial sum (s_n) as n tends to infinity?

To understand this, we group terms in[1] $\sum a_k$ in a clever way:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k} &= 1 && (> \tfrac{1}{2}) \\
&+ \frac{1}{2} + \frac{1}{3} && (> \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}) \\
&+ \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} && (> \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{2}) \\
&+ \frac{1}{8} + \frac{1}{9} + \ldots + \frac{1}{14} + \frac{1}{15} && (> \tfrac{1}{16} + \tfrac{1}{16} + \ldots + \tfrac{1}{16} + \tfrac{1}{16} = \tfrac{1}{2}) \\
&+ \frac{1}{16} + \frac{1}{17} + \ldots + \frac{1}{30} + \frac{1}{31} && (> \tfrac{1}{32} + \tfrac{1}{32} + \ldots + \tfrac{1}{32} + \tfrac{1}{32} = \tfrac{1}{2}) \\
&+ \ldots
\end{aligned}$$

So, we can always group terms in the sum, such that they add up to more than 1/2. In particular, if we add up all terms in the sum on the right-hand side, we have to add up '1/2's infinitely many times. We therefore conclude, that the sequence of partial sum is unbounded and diverges. But this means that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ also diverges.

The infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

is called the *harmonic series*, and is an important example of an infinite series, for which the corresponding sequence (a_k) tends to 0, but the series $\sum a_k$ still diverges.

REMARK 3.2. The divergence of the harmonic series is extremely slow. For example, to reach a sum of 4, we have to add up the first 30 terms in the series. But to reach a sum of 20, we would have to add up 275 million terms.

EXAMPLE 3.3. As an application, we want to use the geometric series to prove that

$$0.99999\ldots \quad = \quad 0.\overline{9} \quad = \quad 1$$

For this, note that the decimal representation $0.9999\ldots$ describes the number

$$0.99999\ldots \quad = \quad 0 + \frac{9}{10^1} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots \quad = \quad 9 \cdot \sum_{k=1}^{\infty} \frac{1}{10^k}.$$

[1] Note the use of $\sum a_k$ as shorthand for $\sum_{k=1}^{\infty} a_k$.

Now, using the formula for the geometric sum and observing that

$$\frac{1}{10^0} + \sum_{k=1}^{\infty} \frac{1}{10^k} = 1 + \sum_{k=1}^{\infty} \frac{1}{10^k} = \sum_{k=0}^{\infty} \frac{1}{10^k},$$

we indeed find

$$0.99999\ldots = 9 \cdot \left(\frac{1}{1 - \frac{1}{10}} - 1 \right) = 9 \cdot \frac{1}{9} = 1.$$

As noted above, it is an exception that the sum of an infinite series can be computed explicitly. Here is some remarkable series with intriguing sums:

$$e = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k!};$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1};$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Usually, the main tasks when given an infinite series is to decide whether it converges or not. We will discuss a variety of criteria for this.

Remark 3.4. It is helpful to remember that the convergence of an infinite series $\sum_{k=1}^{\infty} a_k$ is equivalent to the convergence of the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$. Implication in one way is explicitly required by Definition 3.1. To see the opposite connection, take an arbitrary sequence (s_n) and define a new sequence (a_n) by the relations:

$$a_1 = s_n, \qquad \text{and} \qquad a_n = s_n - s_{n-1}, \qquad \text{for } n > 1.$$

We can directly check that (s_n) is the sequence of partial sums for the series $\sum_{k=1}^{\infty} a_k$. Thus we can use all results for sequences obtained earlier to derive convergence of series and wise-verse, all new results for series can be used to investigate convergence of sequences.

3.2. Properties of (convergent) series. It is not easy to decide whether a series converges. For the example of the harmonic series we have noted above that the divergence is so slow, that it is almost impossible to detect in numerical experiments. We will learn about different tests that can be used for infinite series. The first one is a fairly easy to check condition for divergence.

Lemma 3.5 (Vanishing Test). *Consider* $\sum_{k=1}^{\infty} a_k$. *If the infinite series converges, then* $\lim_{k\to\infty} a_k = 0$.

PROOF. Let $s_n = \sum_{k=1}^{n} a_k$ denote the n-th partial sum. By the assumption in the lemma, the infinite series converges, and therefore $\lim_{n\to\infty} s_n$ exists. But then—using a standard trick from the theory of convergent sequences—the limit $\lim_{n\to\infty} s_{n-1}$ must exist, too, and we have $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1}$. Therefore,

$$\begin{aligned} 0 &= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= \lim_{n\to\infty} (s_n - s_{n-1}) \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k \right) \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n-1} a_k + a_n - \sum_{k=1}^{n-1} a_k \right) \\ &= \lim_{n\to\infty} a_n. \end{aligned}$$

□

REMARK 3.6. Important: You can only use this lemma in the following way

If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ cannot converge.

(That is why the result is called the vanishing test.)

The fact that $\lim_{k\to\infty} a_k = 0$ does not imply the convergence of the series. Indeed, for the harmonic series $\sum_{k=1}^{\infty} a_k$, the sequence $a_k = 1/k$ tends to zero, but the series diverges to infinity.

In order to discuss further tests, we will first introduce the concept of *absolute convergence*.

DEFINITION 3.7. An infinite series $\sum_{k=1}^{\infty} a_k$ is called *absolutely convergent*, if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Obviously, there is only a difference between the two series in the definition, if (a_k) contains negative elements (because otherwise $|a_k| = a_k$ for all k). But in this case, we can expect the sum $\sum_{k=1}^{\infty} a_k$ to be smaller than $\sum_{k=1}^{\infty} |a_k|$, since both positive and negative numbers are added. In particular, the absolute convergence of a series implies its convergence. This is exactly what the next lemma states.

LEMMA 3.8. *If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges, and we have*

$$\left| \sum_{k=1}^{\infty} a_k \right| \leqslant \sum_{k=1}^{\infty} |a_k|.$$

In the following we will discuss several criteria for the absolute convergence of an infinite series. They are useful, because we have just learned that if a series satisfies a condition for absolute convergence, then it will also converge.

Similar to the methods we used for sequences, a main technique is to compare the terms in an infinite series with terms in a series that is known to converge (or to diverge). Indeed, we have the following theorem.

THEOREM 3.9 (Comparison test). *Let $\sum_{k=1}^{\infty} b_k$ be a series of non-negative numbers, and assume that $\sum_{k=1}^{\infty} b_k$ converges (i.e. $\sum_{k=1}^{\infty} b_k < \infty$). Furthermore, let $\sum_{k=1}^{\infty} a_k$ be an infinite series, such that $|a_k| < c \cdot b_k$, for all k with some $c \in \mathbb{R}$. Then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.*

PROOF. We consider the partial sums for the series $(|a_k|)$. So, let $s_n = \sum_{k=1}^{n} |a_k|$ denote the n-th partial sum. Then, since obviously all $|a_k| \geqslant 0$, the sequence (s_n) is increasing. Moreover,

$$\begin{aligned} \sum_{k=1}^{n} |a_k| &\leqslant \sum_{k=1}^{n} c \cdot b_k \\ &= c \sum_{k=1}^{n} b_k \\ &\leqslant c \sum_{k=1}^{\infty} b_k. \end{aligned}$$

Hence, the sequence (s_n) is bounded, and therefore, by Theorem 2.14, the sequence (s_n) converges. But this implies $\sum_{k=1}^{n} |a_k| < \infty$, that is, the infinite series formed using the (a_k) is absolutely convergent. □

Before we use this result, let us state an immediate consequence

LEMMA 3.10. *For the sequence (c_k) with $c_k \geqslant 0$ for all $k \in \mathbb{N}$, assume that $\sum_{k=1}^{\infty} c_k = \infty$ (i.e. $\sum_{k=1}^{\infty} c_k$ diverges to $+\infty$). Furthermore, assume that for the series $\sum_{k=1}^{\infty} a_k$, we have $a_k \geqslant b \cdot c_k$ for all $k \in \mathbb{N}$ for some $b \in \mathbb{R}$ such that $b > 0$. Then $\sum_{k=1}^{\infty} a_k$ diverges, that is, $\sum_{k=1}^{\infty} a_k = \infty$.*

The comparison test leads us to a convergence result for a very important type of infinite series.

EXAMPLE 3.11. i) Consider

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}.$$

We can use the comparison test with the sequence $c_k = 1/k$. Indeed, we have

$$\frac{1}{\sqrt{k}} \geqslant \frac{1}{k}, \quad \forall k \in \mathbb{N},$$

and $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$. Therefore, Lemma 3.10 implies that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \infty.$$

ii) Now, consider

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

To decide about the convergence of this infinite series, we consider the *telescopic series*

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

We have already discussed, that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1},$$

and this implies

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \frac{n}{n+1} = 1\,.$$

Now, we also have

$$\left|\frac{1}{(k+1)^2}\right| = \frac{1}{(k+1)^2} \leqslant \frac{1}{k(k+1)},$$

and so, applying the comparison test (i.e. Theorem 3.9), we get

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)^2} < \infty$$

However

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)^2}$$

We thus conclude that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

(i.e. that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges).

The last 2 examples were special cases of a more general result, the proof of which is beyond the scope of this course. Nevertheless, the result is so important, that it cannot be left out.

LEMMA 3.12. *Consider the infinite series*

$$\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}, \quad \textit{with } \alpha > 0.$$

Then the series converges, if $\alpha > 1$*, and it diverges for* $\alpha \leqslant 1$*.*

REMARK 3.13. Note that in the same way as for sequences, the convergence of a series is—by the very definition—an asymptotic property. So, if any the conditions above is only satisfied for a *tail* of the sequence (a_k), that is, for all k greater than some number K, then we can still use this test to decide about convergence. In other words, the first $1, 2, \dots, K$ terms of the sequence (a_k) play no role.

3.3. The ratio test. In the last part of this chapter we will discuss the most important test for the convergence of an infinite series—the ratio test.

THEOREM 3.14 (Ratio Test or D'Alembert's Test). *Consider the infinite series $\sum_{k=1}^{\infty} a_k$, and assume that the limit*

$$L = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$$

exists. Then,

- *if $L < 1$, the series $\sum_{k=1}^{\infty} a_k$ converges absolutely;*
- *if $L > 1$, the series diverges.*

PROOF. We first consider the case $L < 1$: The idea of the proof is to compare the series with an appropriate geometric series. For this, let us choose a number q, such that $L < q < 1$. Since $\frac{|a_{k+1}|}{|a_k|} \to L$, and letting $\varepsilon = q - L$, we know (by the definition of convergence) that there exists a number $\widehat{K} \in \mathbb{N}$, such that

$$L - \varepsilon \; < \; \frac{|a_{k+1}|}{|a_k|} \; < \; L + \varepsilon \,,$$

for all $k > \widehat{K}$. In particular, as $L + \varepsilon = L + q - L = q$, and setting $K = \widehat{K} + 1$, we thus know that

$$\frac{|a_{k+1}|}{|a_k|} \; < \; q \,,$$

for all $k \geqslant K$. In other words

$$|a_{k+1}| \; < q|a_k| \,,$$

for all $k \geqslant K$. Thus we see that $|a_{K+1}| < q|a_K|$, and $|a_{K+2}| < q|a_{K+1}| < q^2|a_K|$, and more generally that, for any number $n > 2$,

$$|a_{K+n}| < q|a_{K+n-1}| < q^2|a_{K+n-2}| < \ldots < q^{n-1}a_{K+1} < q^n|a_K|.$$

Since $q < 1$, we can use the comparison test for the series $\sum_{k=K}^{\infty} |a_k|$ with the convergent geometric series $\sum_{k=1}^{\infty} q^k$. Indeed note that $\sum_{k=K}^{\infty} |a_k|$ can be rewritten as $\sum_{k=0}^{\infty} |a_{K+k}|$ and we see that

$$|a_{K+k}| \; \leqslant \; |a_K| \cdot q^k$$

for all $k \geqslant 0$. Hence by the comparison test Theorem 5 (with the constant $c = |a_K|$) and the fact that $\sum_{k=1}^{\infty} q^k$ converges, we know that the series $\sum_{k=K}^{\infty} |a_k| = \sum_{k=0}^{\infty} |a_{K+k}|$ converges. But this implies that the series $\sum_{k=1}^{\infty} |a_k|$ converges (see the above remark). I.e. $\sum_{k=1}^{\infty} a_k$ converges absolutely.

For the second case, $L > 1$, we use a similar argument as above to see that there is a number K, such that $|a_{k+1}| > |a_k|$ for all $k > K$. But then $a_k \not\to 0$, and therefore, by the vanishing test, the series does not converge. □

EXAMPLE 3.15.

a) For $\sum_{k=1}^{\infty} \frac{1}{k!}$ we have $a_k = \frac{1}{k!}$, and therefore

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)!} = \frac{1}{k+1}.$$

Therefore, $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 0 < 1$, and we conclude that the series converges.

b) For $\sum_{k=1}^{\infty} \frac{k^4}{4^k}$, we have $a_k = \frac{k^4}{4^k}$, and thus

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)^4 \cdot 4^k}{k^4 \cdot 4^{k+1}} = \frac{1}{4}\frac{(k+1)^4}{k^4}.$$

Therefore, $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \frac{1}{4} < 1$, and the series converges.

c) For $\sum_{k=1}^{\infty} \frac{(k+1)^2}{k^4+1}$ we have $a_k = \frac{(k+1)^2}{k^4+1}$. In this case

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+2)^2 \cdot (k^4+1)}{((k+1)^4+1)\cdot(k+1)^2}$$

Now, $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1$. In this case the ratio test does not allow us to make a statement about whether the series converges or diverges. Here we need to use a different test. For example, observing that $a_k = \frac{(k+1)^2}{k^4+1} \approx \frac{1}{k^2}$ for large values of k, we could use the comparison test.

3.4. Alternating series. So far, all tests have been for the *absolute convergence* of a series. We can use them to decide about the convergence of a series, since every absolutely convergent series must be convergent. A natural question is, if there are series that are convergent, but not absolutely convergent.

The answer is yes, and an example can be found in the class of *alternating series*.

DEFINITION 3.16. An infinite series $\sum_{k=1}^{\infty} a_k$ is called *alternating*, if $a_k \cdot a_{k+1} < 0$.

So, in an alternating series, the terms a_k always change sign. Examples are given by series of the form

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}, \qquad \sum_{k=1}^{\infty} (-1)^k \cdot \frac{k^2}{k^4+1}, \qquad \text{etc.}$$

A convergence test for alternating series goes back to the German mathematician Leibniz.

LEMMA 3.17 (Alternating Test or Leibniz's Test). *Let $\sum_{k=1}^{\infty} a_k$ be an alternating series. Assume that*

(1) $|a_k| \to 0$ *as* $k \to \infty$*; and*
(2) $|a_k|$ *is decreasing, i.e.* $|a_{k+1}| < |a_k|$ *for all* k.

Then the series $\sum_{k=1}^{\infty}$ *converges.*

To prove the Lemma we notice that the sequence of partial sums (s_n) is such that the intervals $[s_n, s_{n+1}]$ are nested, that is $[s_{n+1}, s_{n+2}] \subset [s_n, s_{n+1}]$, and their length $s_{n+1} - s_n = a_{n+1}$ tends to zero. Such sequence of intervals has the only common point which is the limit for the sequence of intervals' endpoints s_n.

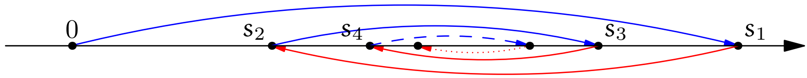


FIGURE 5. Partial sums of an alternating series.

EXAMPLE 3.18 (Alternating harmonic series). We will discuss the *alternating harmonic series*, using this lemma. This series is defined as

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \dots$$

Hence, for this series $a_k = \frac{(-1)^{k+1}}{k}$ and $|a_k| = \frac{1}{k}$. Obviously, $|a_k| \to 0$ as $k \to \infty$, and moreover, $|a_k|$ is a decreasing sequence. Hence, by the alternating series test this series converges.

We know already that the series is not absolutely convergent, because the series $\sum_{k=1}^{\infty} |a_k|$ is just the harmonic series, which diverges.

REMARK 3.19. The Alternating Test inherited the first condition $\lim_{k to \infty} |a_k| \to 0$ from the Vanishing Test, see Lem. 3.5, and it alone is not sufficient for convergence of an alternating series as the next example shows.

EXAMPLE 3.20. Define

$$a_k = \begin{cases} -\dfrac{1}{2^m}, & \text{if } k = 2m,\ m \in \mathbb{N}. \\[2ex] \dfrac{1}{m}, & \text{if } k = 2m+1,\ m \in \mathbb{N}. \end{cases}$$

Then the series is alternating and $\lim_{k\to\infty} a_k = 0$—consider separately when k is even and odd. However, for the series $\sum_{k=1}^{n} a_k$ the partial sums are:

$$s_{2m} = \sum_{k=1}^{m} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{2^m} > \sum_{k=1}^{m} \frac{1}{k} - 1.$$

They tend to infinity because the partial sums f the harmonic series do. The series $\sum_{k=1}^{\infty} a_k$ diverges in the absence of the monotone condition $|a_{k+1}| < |a_k|$.

3.5. Arithmetics of convergent series. We will only briefly list some rules for computations with convergent series. The first 2 of these have already been used in calculations earlier.

LEMMA 3.21. *If the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent, then*

- $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k,$
- $\displaystyle\sum_{k=1}^{\infty} \alpha \cdot a_k = \alpha \sum_{k=1}^{\infty} a_k,$ *for any* $\alpha \in \mathbb{R}$.

Furthermore, if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are absolutely convergent, then

- $\displaystyle\left(\sum_{k=1}^{\infty} a_k\right) \cdot \left(\sum_{l=1}^{\infty} b_l\right) = \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} a_l \cdot b_{k-l}.$

Note that the last double sum is a way to write down all possible products of any term from the first sum and any term from the second sum. Furthermore, any such product will appear at the double sum only once.

4. Power series

As an application of the theory presented here, we now have a look at power series. Power series have been introduced in MATH1050, and we want to recall the most important results here and put them in relation to the general theory we have considered so far.

4.1. Radius and Interval of Convergence.

DEFINITION 4.1. A *power series* is a series of the form

$$\sum_{k=1}^{\infty} c_k x^k,$$

where (c_k) is a given sequence of coefficients and $x \in \mathbb{R}$.

So, for a power series the terms are of the form $a_k = c_k x^k$. If x is varied, then the value of the series changes, and we can think of the series as describing a function $f(x) = \sum_{k=1}^{\infty} c_k x^k$.

EXAMPLE 4.2. If $c_k = 1/k!$, then a power series can be formed as $\sum_{k=0}^{\infty} \frac{x^k}{k!}$, and we have seen already that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

We will not discuss what the relation between a function f and its power series is, here. This is part of the MATH1050 course. We concentrate on finding out conditions for the convergence of a given power series.

To decide about the convergence of a power series we can apply the ratio test. With $a_k = c_k x^k$ (so that $|a_k| = |c_k| \cdot |x|^k$) we have

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|c_{k+1}|}{|c_k|} \cdot \frac{|x|^{k+1}}{|x|^k} = \frac{|c_{k+1}|}{|c_k|} \cdot |x|.$$

Thus, assuming $\lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|} = l$ exists, we find that

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = l|x|.$$

Therefore,

- the series converges for $|x| < \frac{1}{l}$,
- the series diverges for $|x| > \frac{1}{l}$.

Introducing $R = 1/l$, then the series diverges for $x \in (-R, R)$, and it diverges for $x < -R$ or $x > R$.

Note that $R = \lim_{k\to\infty} \frac{|c_k|}{|c_{k+1}|}$. This motivates our next definition.

DEFINITION 4.3. The number

$$R = \lim_{k\to\infty} \frac{|c_k|}{|c_{k+1}|}$$

is called the *radius of convergence* of the power series $\sum_{k=1}^{\infty} c_k x^k$.

The ratio test does not give any information about what happens for $x = \pm R$. Indeed, these 2 cases have to be studied for each series individually. After such an investigation we may establish the *interval of convergence*. In fact, convergence may occur either at both end-points, or only one of them or neither. Thus, in general there are four possibilities for the interval of convergence: $[-R, R]$, $[-R, R)$, $(-R, R]$ or $(-R, R)$ depending on the convergence at the end-points.

EXAMPLE 4.4. As an example we consider the series

$$\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

Here, $c_k = 1/k$ and we see that

$$\frac{|c_k|}{|c_{k+1}|} = \frac{k}{k+1} \to 1 \quad \text{as } k \to \infty.$$

I.e. the radius of convergence of the series is $R = 1$ and we conclude that the series converges if $|x| < 1$, that is, if $x \in (-1, 1)$.

$\Big($ Note *that reasoning directly via the ratio test gives*

$$\frac{|a_{k+1}|}{|a_k|} = \frac{k}{k+1}|x| \to 1 \cdot |x| \quad \textit{as } k \to \infty.$$

Hence the series converges if $1 \cdot |x| < 1$*, i.e. if* $|x| < 1$*.* $\Big)$

In order to find out what happens for $x = \pm 1$, we substitute these 2 values into the series:

- $x = 1$: In this case the series is $\sum_{k=1}^{\infty} \frac{1}{k}$, i.e. it is the harmonic series which diverges.
- $x = -1$: Now the series is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$.Note that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = (-1) \cdot \sum_{k=1}^{\infty} \frac{(-1)^k}{k}.$$

  Thus the series converges as the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges (by application of the alternating series test).

In summary, the power series $\sum_{k=1}^{\infty} \frac{x^k}{k}$ converges, if $x \in [-1, 1)$.

4.2. Illustration: the Euler formula. Let us for the moment return to complex numbers. The theory of sequences and series can be used for complex numbers with minimal adjustments. Already know that *Euler's number* $e = 2.71828\ldots$ can be defined as the limit of a series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

In a similar way we can define the exponential $\exp(x)$ of a number $x \in \mathbb{R}$ as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots.$$

In particular, we can use this series to evaluate $e^{i\pi}$. Plotting partial sums s_n on the complex plane we can visualise Euler's identity $e^{i\pi} + 1 = 0$ as on Fig. 6.

Expanding $e^{i\phi}$ into the power series we obtain:

$$\begin{aligned}
\exp(i\phi) = \sum_{n=0}^{\infty} \frac{(i\phi)^n}{n!} &= 1 + \frac{(i\phi)^1}{1!} + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^3}{3!} + \frac{(i\phi)^4}{4!} + \frac{(i\phi)^5}{5!} \ldots \\
&= 1 + i\frac{\phi^1}{1!} - \frac{\phi^2}{2!} - i\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + i\frac{\phi^5}{5!} \ldots \\
&= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} \mp \ldots + i\left(\frac{\phi^1}{1!} - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} \mp \ldots\right) \qquad (22)
\end{aligned}$$

where we have used $i^2 = -1$, $i^3 = -i$, etc., and where we have changed the order of terms in the sum. Comparing (22) with Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$ we equate the real and imaginary parts of both equations. This gives power series for

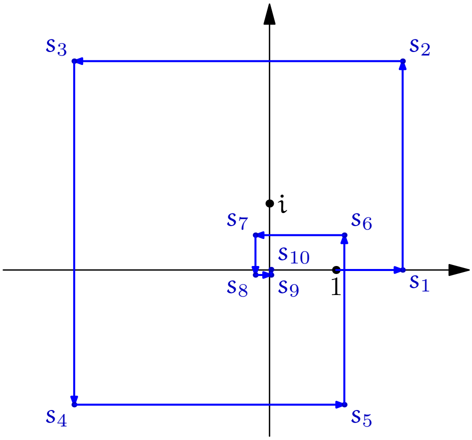


FIGURE 6. Euler's identity $e^{i\pi} + 1 = 0$ graphically represented through the exponent series.

sine and cosine functions:

$$\begin{aligned}
\sin(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \dots, \\
\cos(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \dots
\end{aligned}$$

CHAPTER 3

Vectors

1. Introduction

We often meet a quantity, which has a *direction* and a *magnitude*. Standard examples can be found in mechanics: a force, for example. Think of a force acting on (=pushing) a ball. Then the outcome of this push will depend both on the direction of the force and on its strength. Another example from physics are velocities. In geometry, translations of a plane or space are defined by the direction and distance as well.

Such quantities are described by mathematical object called a *vectors*. The main properties of vectors that they can be added and multiplied by a number. In this context numbers are called *scalars*.

Graphically, vectors are usually represented by arrows (on the plane or in space). The direction of the arrow corresponds to the direction of the vector, and its length to the vector's magnitude.

We will indicate vectors in this script by bold letters, i.e, like $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$, etc. In writing, there are several options. It is common to denote vectors as $\vec{a}$, $\vec{b}$ or $\vec{v}$, or as $\underline{a}$, $\underline{b}$ or $\underline{v}$. It does not matter, what method is chosen, but it is important (and very helpful) to distinguish vectors from numbers (scalars), in order to avoid confusion, when such quantities are mixed.

We have already met vectors on the plane in Section 5, Chap. 1 and seen that complex numbers are helpful to describe geometrical properties, cf. Lem. 5.18 and 5.19, Chap. 1. To treat vectors in 3D space, which we denote by $\mathbb{R}^3$, we will use *quaternions*. For a geometrical introduction of quaternions see also [1, Part II, p. 52].

In this (elementary) introduction to vectors we will define the main operations with vectors and discuss their relations to the equations for lines, planes and spheres.

2. Quaternions and vectors

To define complex numbers we used the imaginary unit i, such that $i^2 = -1$. To describe vectors in $\mathbb{R}^3$ we need three imaginary units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, which will be called *base quaternions*.

2.1. Quaternion: introduction. A *quaternion* is an expression of the form:

$$u = u_0 + u_1 i + u_2 j + u_3 k, \qquad \text{where } u_0, u_1, u_2, u_3 \in \mathbb{R}. \tag{23}$$

The real number $u_0 = \mathrm{Re}(u)$ is called the *real part* (or *scalar part*) of the quaternion u. The expression $u_1 i + u_2 j + u_3 k = \mathrm{Im}(u)$ is called the *imaginary part* of the quaternion u. We keep this name to stress the similarity to complex numbers. It may be also called the *vector part* of the quaternion u. We will show that it is convenient to identify vectors in $\mathbb{R}^3$ with expressions $bi + cj + dk$, that is with quaternions with vanishing real part.

Similarly to complex case we define the *conjugated quaternion*:

$$\bar{u} = u_0 - u_1 i - u_2 j - u_3 k, \qquad \text{where } u = u_0 + u_1 i + u_2 j + u_3 k. \tag{24}$$

We define *addition* of quaternions component-wise, that is by the formula:

$$a + b = (a_o + b_0) + (a_1 + b_1) i + (a_2 + b_2) j + (a_3 + b_3) k, \tag{25}$$

where

$$\begin{aligned} a &= a_0 + a_1 i + a_2 j + a_3 k, \\ b &= b_0 + b_1 i + b_2 j + b_3 k. \end{aligned}$$

We can easily check that such addition is commutative and associative, see (1) and (2). Also we define a multiplication of a quaternion by a real number (scalar) by component-wise multiplication:

$$ta = ta_0 + ta_1 i + ta_2 j + ta_3 k, \quad \text{where } a = a_0 + a_1 i + a_2 j + a_3 k \quad \text{and} \quad t \in \mathbb{R}. \tag{26}$$

We can again verify the commutativity $tu = ut$, $stu = tsu$ and associativity $s(tu) = (st)u$ of such multiplication.

To make a connection between quaternions and vectors we introduce the concept of coordinates and position vectors.

2.2. Position vectors and coordinates. Let us consider the 3-dimensional space $\mathbb{R}^3$ and denote the coordinate axes as usual by x, y and z. (All of the following works in $\mathbb{R}^2$ in the same way; we just do not have the z-direction.) Every point $P \in \mathbb{R}^3$, can be characterised by its coordinates $P = (p_1, p_2, p_3)$, where p_1 denotes the (signed) distance of P to the y-z plane, p_2 is the distance of P to the x-z plane, and p_3 is the distance of P to the x-y plane, see Figure 1.

We can associate points in $\mathbb{R}^3$ with vectors in $\mathbb{R}^3$ by introducing the following vectors

- i as the vector with magnitude $\|i\| = 1$ in the direction of the x-axis;
- j as the vector with magnitude 1 in the direction of the y-axis;
- k as the vector with magnitude 1 in the direction of the z-axis.

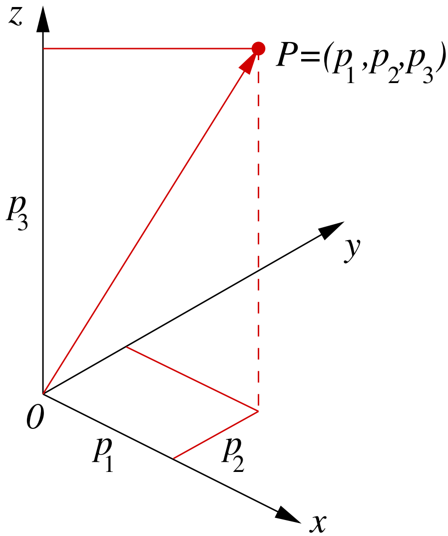


FIGURE 1. Coordinates of a point $P \in \mathbb{R}^3$ and the corresponding position vector.

Using the vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, a point in $\mathbb{R}^3$ can be associated with its *position vector*, that is the vector connecting P to the origin $\mathcal{O}$. This vector is denoted by $\overrightarrow{\mathcal{O}P}$, and if $P = (p_1, p_2, p_3)$, we have

$$\overrightarrow{\mathcal{O}P} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}.$$

We also write $\overrightarrow{\mathcal{O}P} = (p_1, p_2, p_3)$ as an abbreviation for the position vector. (It does not really matter, whether to write the position vector as a row vector or as a column vector. In this course, we use row vectors.)

Interpreting vectors as position vectors of points in $\mathbb{R}^3$, it is now straightforward to express formulas for the addition of vectors by the triangle rule and addition of the respective quaternions (25), see the left drawing on Fig. 2. It is worth to notice that addition of vectors can be done either by the triangle rule or the parallelogram rule—with the same result, see Fig. 3.

Similarly, we can visualise the multiplication by scalar, see the right drawing on Fig. 2. So, by multiplying a vector with a real number, we only change its magnitude, but not its direction. More precisely, let $\|\mathbf{a}\|$ denote the magnitude of the vector $\mathbf{a}$. Then

$$\|r\mathbf{a}\| = |r|\|\mathbf{a}\|, \quad \text{for all } r \in \mathbb{R}.$$

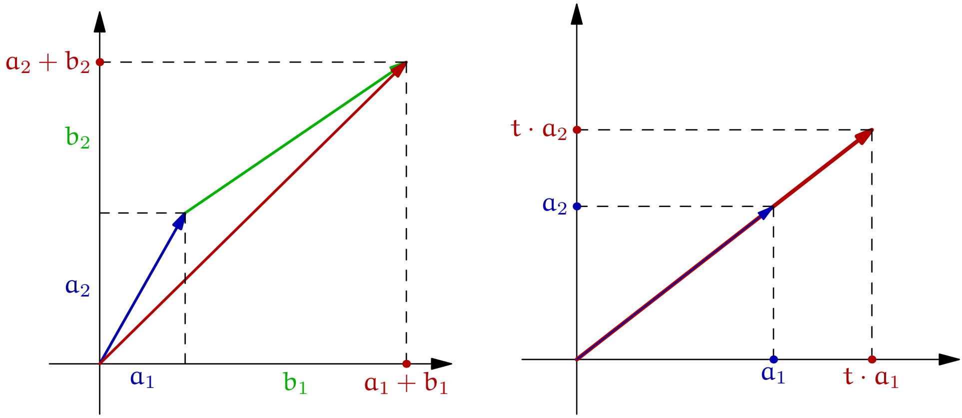


FIGURE 2. Addition and multiplication of vectors through coordinates.

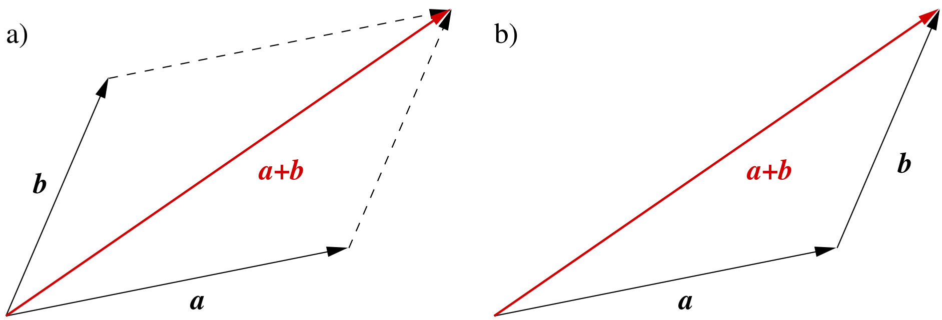


FIGURE 3. Addition of 2 vectors using the parallelogram law in on the left in a) or, equivalently, the triangle law in panel b).

EXAMPLE 2.1. For $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k} = (1, 3, -1)$ and $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} = (2, 4, -3)$, we have

$$2\mathbf{a} - \mathbf{b} = (2, 6, -2) - (2, 4, -3) = (0, 2, 1) = 2\mathbf{j} + \mathbf{k}.$$

REMARK 2.2. All of the above applies in a straightforward manner to vectors in $\mathbb{R}^2$ (or more generally to vectors in $\mathbb{R}^n$). In the case of a 2-dimensional vector we can simply forget about the $\mathbf{k}$-coordinate and set it to 0 in all formulas.

3. Scalar product and vector product

We have seen in Lem. 5.18 and 5.18, Chap. 1 that product of complex numbers can express the property of vectors to be parallel and orthogonal. We want to introduce a product of quaternions which will be acting in a similar way.

3.1. Multiplication of quaternions. If we assume that product is associative and has the distributive law then it will be enough to define product of base quaternions. Following the complex number case we put

$$\mathbf{i}^2 = -1, \qquad \mathbf{j}^2 = -1, \qquad \mathbf{k}^2 = -1. \tag{27}$$

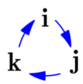
Furthermore we need to put

$$\mathbf{i}\mathbf{j} = \mathbf{k}. \tag{28}$$

The motivation for this can be found in rotations: after we rotate by 180° around x-axis and by 180° around y-axis, the result will be the same as of a single rotation by 180° around z-axis. Another way to remember the quaternion multiplication is through the formula:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1. \tag{29}$$

We can also use the following mnemonic diagram:



The product of two base quaternion is the third one, it is come with the sign $+$ if we pass from the first factor to the second along the blue arrows and the sign is $-$ if we go in the opposite direction.

It is an important distinction of quaternion from all other numbers we studied so far is that quaternion multiplication is *not* commutative:

$$\mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i}, \qquad \mathbf{i}\mathbf{k} = -\mathbf{k}\mathbf{i}, \qquad \mathbf{k}\mathbf{j} = -\mathbf{j}\mathbf{k}. \tag{30}$$

From now on we have enough information for the multiplication of quaternions. First, we multiply both sides of (28) *from the left* by $\mathbf{i}$ and get $\mathbf{i}\mathbf{i}\mathbf{j} = \mathbf{i}\mathbf{k}$, or $-\mathbf{j} = \mathbf{i}\mathbf{k}$ using (27). Thus, $\mathbf{k}\mathbf{i} = \mathbf{j}$ by (30). Similarly we get $\mathbf{j}\mathbf{k} = \mathbf{i}$.

To start, we calculate the product $\mathbf{a}\bar{\mathbf{a}}$ of a quaternion and its conjugated (24):

$$\bar{\mathbf{a}}\mathbf{a} = \mathbf{a}\bar{\mathbf{a}} = a_0^2 + a_1^2 + a_2^2 + a_3^2, \qquad \text{where } \mathbf{a} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}. \tag{31}$$

We obtained the real number. In the special case of vectors:

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})(-a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}) = a_1^2 + a_2^2 + a_3^2.$$

Furthermore, this is a simple application of Pythagoras' theorem to see that the last expression gives the square of vector's length. We will also call this magnitude the *norm* of the vector.

Definition 3.1. The *norm* $\|\mathbf{a}\|$ of the vector $\mathbf{a} = (a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is defined as

$$\|\mathbf{a}\| = \sqrt{-\mathbf{a}\mathbf{a}} = \sqrt{\mathbf{a}\bar{\mathbf{a}}} = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{32}$$

Note, the following property of the norm: $\|\lambda \mathbf{a}\| = |\lambda|\|\mathbf{a}\|$ for any real λ. Such a scaling can be used to produce a vector with the same direction and any new norm. In particular, for any non-zero vector $\mathbf{a}$ the vector $\mathbf{b} = \frac{1}{\|\mathbf{a}\|}\mathbf{a}$ has the same direction and *unit norm*: $\|\mathbf{b}\| = \frac{\|\mathbf{a}\|}{\|\mathbf{a}\|} = 1$.

In general the product of two vectors as quaternions is:

$$\mathbf{ab} = -(a_1 b_1 + a_2 b_2 + a_3 b_3) \tag{33}$$

$$\qquad + (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}, \tag{34}$$

where

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$

As we see, the product of two vectors has both the real and imaginary parts. They both deserve a special attention. To begin with, we notice the following consequences of (33) which split the non-commutative product into commuting and anti-commuting parts:

$$\operatorname{Re}(\mathbf{ab}) = \operatorname{Re}(\mathbf{ba}), \qquad \operatorname{Im}(\mathbf{ab}) = -\operatorname{Im}(\mathbf{ba}). \tag{35}$$

3.2. The scalar product.

Definition 3.2. The *scalar product* (or *dot product*) $\mathbf{a} \cdot \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = -\operatorname{Re}(\mathbf{ab}) = -\frac{1}{2}(\mathbf{ab} + \mathbf{ba}) = a_1 b_1 + a_2 b_2 + a_3 b_3, \tag{36}$$

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

Remark 3.3. The scalar product of two vectors is a real number, which is also called *scalar* in opposition to vectors.

From the definition (36) and properties (35) we have $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. Also, because of quaternions multiplication law we have the following properties

(1) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$;
(2) $(r\mathbf{a}) \cdot \mathbf{b} = r(\mathbf{a} \cdot \mathbf{b})$

Geometrical meaning of the scalar product is reviled by the following statement.

Lemma 3.4. *The scalar product can be geometrically presented as:*

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos\theta, \tag{37}$$

where θ denotes the angle between $\mathbf{a}$ and $\mathbf{b}$.

Proof. We calculate:

$$\begin{aligned} \|\mathbf{a} + \mathbf{b}\|^2 &= -(\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b}) \\ &= -\mathbf{aa} - \mathbf{ba} - \mathbf{ab} - \mathbf{bb} \\ &= \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2. \end{aligned}$$

Thus:

$$\begin{aligned} 2\mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 \\ &= 2\|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos\theta, \end{aligned}$$

by the Cosine Rule: $\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos\theta$. □

COROLLARY 3.5. (1) *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are perpendicular vectors, then* $\theta = \pi/2$, *and therefore* $\cos\theta = 0$. *We conclude that* $\mathbf{a} \cdot \mathbf{b} = 0$.

(2) *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel vectors, with* $\theta = 0$, *we have* $\cos\theta = 1$, *and thus* $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\|$. *In particular, for every vector* $\mathbf{a}$ *we find that* $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.

Obviously, formula (37) can be used to compute the angle between vectors.

EXAMPLE 3.6. Compute the angle θ of the vector $\mathbf{a} = (1, 2, 3)$ with the x-axis.

A general vector along the x-axis is of the form $\mathbf{b} = (x, 0, 0)$. Then, if θ denotes the angle between $\mathbf{a}$ and $\mathbf{b}$, we have

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|} = \frac{(1,2,3) \cdot (x,0,0)}{\|(1,2,3)\| \cdot \|(x,0,0)\|} = \frac{x}{\sqrt{14}x} = \frac{1}{\sqrt{14}}.$$

We conclude that $\theta = \arccos(1/\sqrt{14})$. (In particular, we see that—as expected—the angle does not depend on the value of x.)

EXAMPLE 3.7. In a triangle, an *altitude* is a segment of the line through a vertex perpendicular to the opposite side. We use the scalar product to show that three altitudes are concurrent[1], that is all three meet at a point—the *orthocenter* of the triangle, see Fig. 4.

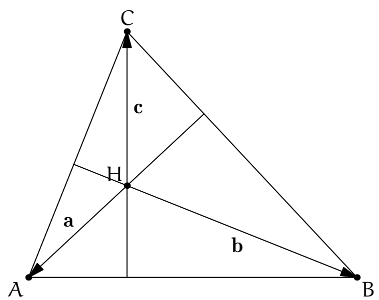


FIGURE 4. Three altitudes of a triangle meet at its orthocenter.

Let ABC be a triangle and H be the intersection of *two* altitudes from vertices A and B. Denote by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ vectors from H to A, B and C respectively. Using

[1] See http://www.cut-the-knot.org/triangle/altitudes.shtml for the surprising history and various proofs of this result.

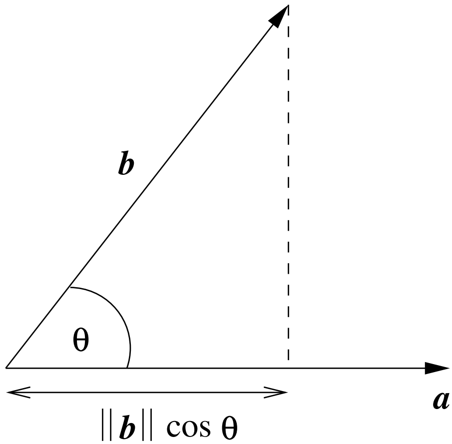


FIGURE 5. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ with angle θ between them, the projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by $\|\mathbf{b}\| \cos\theta$.

the scalar product, we write the given orthogonality as

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0 \qquad \text{and} \qquad \mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = 0.$$

Opening the brackets we obtain:

$$\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} = 0 \qquad \text{and} \qquad \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{c} = 0.$$

Subtracting one from another we conclude

$$\mathbf{b} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{c} = 0 \quad \text{or, equivalently} \quad (\mathbf{b} - \mathbf{a}) \cdot \mathbf{c} = 0.$$

Thus the line from H to C is the third altitude of the triangle ABC and H is its orthocenter.

Apart from its connection to the angle between vectors, there is a second geometric interpretation of the scalar product. This one is particularly useful for computing distances between geometrical objects (like the distance from a point to a plane or the distance between 2 straight lines). For this, consider the 2 vectors in Figure 5. Using simple trigonometry, we see that the proportion of $\mathbf{b}$ in the direction of $\mathbf{a}$ (i.e. the length of the perpendicular projection) is given by $\|\mathbf{b}\| \cos\theta$, where θ denotes the angle between $\mathbf{a}$ and $\mathbf{b}$. Comparing this with the formula for the scalar product, we find that

$$\text{'projection of } \mathbf{b} \text{ onto } \mathbf{a}\text{'} \ = \ \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}.$$

REMARK 3.8. So far, all the formulas have been stated for vectors in $\mathbb{R}^3$. As usual, if we deal with 2-dimensional vectors, then we simply ignore the third component and find that $(v_1, v_2) \cdot (w_1, w_2) = v_1 w_1 + v_2 w_2$.

3.3. The vector product in $\mathbb{R}^3$. Next, we discuss the imaginary part of the product (34). In contrast to the scalar product, which is defined for vectors in all dimensions, this *vector product* is only defined for three-dimensional vectors.

DEFINITION 3.9. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be three-dimensional vectors. The *vector product* (or *cross product*) $\mathbf{a} \times \mathbf{b}$ is defined by:

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \operatorname{Im}(\mathbf{a}\mathbf{b}) = \frac{1}{2}(\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a}) \\
&= (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k},
\end{aligned} \tag{38}
$$

In contrast to the scalar product, the vector product of two vectors is another vector (hence the name).

LEMMA 3.10. $\mathbf{a} \times \mathbf{b}$ *is a vector, which is perpendicular to both* $\mathbf{a}$ *and* $\mathbf{b}$, *and which has norm*

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \sin\theta,$$

where θ *denotes the angle between* $\mathbf{a}$ *and* $\mathbf{b}$.

PROOF. To establish orthogonality we use the scalar product:

$$
\begin{aligned}
\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) &= \frac{1}{2}\mathbf{a} \cdot (\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a}) \\
&= -\frac{1}{4}(\mathbf{a}(\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a}) + (\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a})\mathbf{a}) \\
&= -\frac{1}{4}(\mathbf{a}\mathbf{a}\mathbf{b} - \mathbf{a}\mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b}\mathbf{a} - \mathbf{b}\mathbf{a}\mathbf{a}) \\
&= -\frac{1}{4}(-\|\mathbf{a}\|^2\mathbf{b} + \mathbf{b}\|\mathbf{a}\|^2) \\
&= 0.
\end{aligned}
$$

We used that the multiplication by the real number $\|\mathbf{a}\|^2$ is commutative. Since the scalar product vanishes, $\mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ are orthogonal.

We will calculate the vector product in the 2D position. The formula $\|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \sin\theta$ gives the area S of parallelogram spanned by vectors $\mathbf{a}$ and $\mathbf{b}$. This area can be also calculated if we will subtract from the area of the rectangle the area of shaded region on the Fig. 6:

$$
\begin{aligned}
S &= (a_1 + b_1)(a_2 + b_2) - a_1 a_2 - b_1 b_2 - 2a_2 b_1 \\
&= a_1 a_2 + b_1 a_2 + a_2 + a_1 b_2 + b_1 b_2 - a_1 a_2 - b_1 b_2 - 2a_2 b_1 \\
&= a_1 b_2 - a_2 b_1
\end{aligned}
$$

□

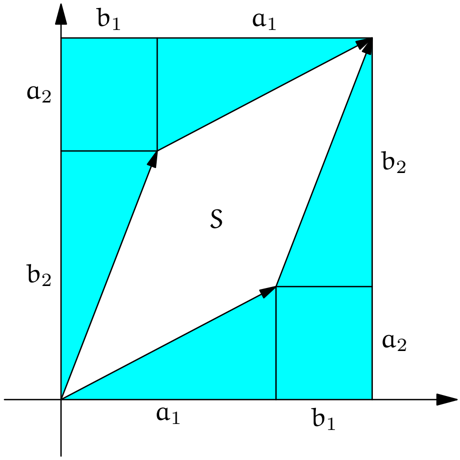


FIGURE 6. Area of the parallelogram

REMARK 3.11. The direction of $\mathbf{a} \times \mathbf{b}$ is not completely determined by the orthogonality condition, since there are 2 options for the vector to be perpendicular to the two given vectors. In order to remove this ambiguity, the *right-hand rule* is used to determine the direction of $\mathbf{a} \times \mathbf{b}$.

REMARK 3.12. Another way to express this formula is to make use of determinants. If you are familiar with the concept of a determinant, then you can also remember that

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

EXAMPLE 3.13. Find a vector $\mathbf{v}$, which is perpendicular to the 2 vectors $\mathbf{a} = (3, 0, 1)$ and $\mathbf{b} = (1, 1, -2)$ and which has norm $\|\mathbf{v}\| = 1$.

We first compute a vector $\mathbf{c}$ that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and adjust the length in a second step. We use the vector product and set

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = (3, 0, 1) \times (1, 1, -2) = (-1, 7, 3).$$

(To make sure we have not made a mistake, we can easily check that $\mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b} = 0$.)

Having computed a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, we still need to adjust the norm. So, let $\mathbf{v} = r\mathbf{c}$, and let us compute r. We want to have $\|\mathbf{v}\| = 1$, and so need to solve

$$\|\mathbf{v}\| = \|r\mathbf{c}\| = |r| \cdot \|\mathbf{c}\| = 1.$$

Therefore, we choose $r = 1/\|\mathbf{c}\|$. A simple computation shows $\|\mathbf{c}\| = \sqrt{59}$, and so the desired vector is

$$\boldsymbol{v} = \frac{1}{\sqrt{59}}(-1, 7, 3).$$

REMARK 3.14. It worth to highlight the relation between orthogonality and two types of products we considered:

(1) To *check* orthogonality of two vectors $\mathbf{a}$ and $\mathbf{b}$ we use their *scalar* product: $\mathbf{a} \cdot \mathbf{b} = 0$.
(2) To *build* a vector orthogonal to vectors $\mathbf{a}$ and $\mathbf{b}$ we use their *vector* product: $\mathbf{a} \times \mathbf{b} = 0$.

4. Lines, Planes and Spheres

In the last part of this chapter, we will discuss the equations for (straight) lines, planes and spheres using vectors.

4.1. Equations for lines. Before deriving equations, let us think about what (geometrical) data we need to characterise a line. There are several options:

i) A point P, and a vector $\boldsymbol{v}$ determine a unique line through P in the direction of $\boldsymbol{v}$.
ii) Two points P_1 and P_2 determine the line connecting them.
iii) In $\mathbb{R}^2$, a line is determined by its slope and a point P, through which it goes.

Let us now see how this data can be turned into equations.

i) For the first case, let Q be an arbitrary point on the line. Then we have

$$\overrightarrow{OQ} = \overrightarrow{OP} + s\boldsymbol{v}, \tag{39}$$

for some $s \in \mathbb{R}$. (As usual, $\overrightarrow{OQ}$ denotes the position vector of the point Q.) If the value of the parameter s is changed, then different points on the line are reached. Because of this, the equation (39) is called the parametric form of a (straight) line.

If, in $\mathbb{R}^3$, we have $P = (p_1, p_2, p_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$, and if we set $Q = (x, y, z)$ for the unknown point Q, then the parametric from of the line can also be written as

$$(x, y, z) = (p_1, p_2, p_3) + s \cdot (v_1, v_2, v_3).$$

This reads in coordinates

$$\begin{aligned} x &= p_1 + sv_1 \\ y &= p_2 + sv_2 \\ z &= p_3 + sv_3, \end{aligned}$$

from which it follows that

$$\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3} \quad (= s).$$

This is the second form of the equation of a straight line.

REMARK 4.1. If $v_i = 0$ for some $1 \leqslant i \leqslant 3$ you should deduce the second form of the equation(s) of the line appropriately. E.g. (i) If $v_1 = 0$ (and $v_2 \neq 0$, $v_3 \neq 0$) then the appropriate equations are: $x = p_1$, $\frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}$. (ii) If $v_1 = v_3 = 0$ (and $v_2 \neq 0$) then the appropriate equations are $x = p_1$, $z = p_3$. This is the line parallel to the y axis that passes through the point (p_1, p_2, p_3) (or, more succinctly, the point $(p_1, 0, p_3)$).

ii) If we know 2 points P_1 and P_2 on the line, then we can easily convert this data into the first case by observing that the connecting vector $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$ is parallel to the line. Therefore, we have the parametric form of the line as

$$\overrightarrow{OQ} = \overrightarrow{OP_1} + s \cdot \overrightarrow{P_1P_2}, \quad s \in \mathbb{R}.$$

iii) The last case is specific to $\mathbb{R}^2$. Here the slope of a line is defined as the ratio between the change in the y-direction and the change in the x-direction. More precisely, if $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ are 2 given points on a straight line (and $p_1 \neq q_1$), then the slope of the line is

$$m = \frac{q_2 - p_2}{q_1 - p_1}.$$

Given the slope m of a line, we can easily construct the direction vector $\mathbf{v}$ as $\mathbf{v} = (1, m)$. In particular, the parametric form is $(x, y) = (p_1, p_2) + s \cdot (1, m)$ or

$$x = p_1 + s, \quad y = p_2 + s \cdot m.$$

But from this we easily obtain $y = p_2 + (x - p_1)m = mx + p_2 - p_1 m$. This is the familiar form of a linear function, whose graph is (of course) a straight line.

REMARK 4.2. In the special case when $q_1 = p_1$ (so that $q_1 - p_1 = 0$), this means that P and Q lie on the same vertical line (i.e. parallel to the y axis) in the xy-plane. Hence in the this case the direction vector $\mathbf{v} = \mathbf{j} = (0, 1)$ and the parametric form of the line is: $(x, y) = (p_1, p_2) + s \cdot (0, 1)$.

EXAMPLE 4.3. Find the equation of the line through $P_1 = (1, 2)$ and $P_2 = (3, 4)$ and its slope. Where does the line intersect the x-axis?

To solve this problem, we first compute a vector $\boldsymbol{v}$ in the direction of the line as $\boldsymbol{v} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (3-1, 4-2) = (2,2)$. We therefore have for a point $Q = (x, y)$ on the line, that

$$(x, y) = (1,2) + s \cdot (2,2), \quad s \in \mathbb{R}.$$

This implies $(x-1)/2 = (y-2)/2$ or $y = x+1$. These are the different possibilities of stating the equations of the line. From the parametric form we find that the slope is $m = 2/2 = 1$, and of course this is confirmed, since $y = x+1$.

For the second part, observe that intersections with the x-axis occur, if $y = 0$. Using the parametric form, this means, we have to solve $y = 2 + 2s = 0$. The solution is $s = -1$, which —substituted into the equation for x gives $x = 1 + (-1) \cdot 2 = -1$. Therefore, the intersection occurs at the point $(-1, 0)$. Of course, this result can also (and probably more easily) obtained from the equation $y = x + 1$.

As a summary, we note that there are several options to write down equations for lines in $\mathbb{R}^2$ or $\mathbb{R}^3$. The most general (and usually quite easily derived) form is the parametric form. All information about the line is contained in this equation, and so all further questions can be answered from this form alone. For certain questions, however, it might be useful to transform this into the second form of the equation for a line, as some computations are easier there.

4.2. Planes in $\mathbb{R}^3$. We continue with the discussion of planes in $\mathbb{R}^3$. Again, we start by establishing the amount of data we need to characterise a plane, and then think about, how to turn this data into an equation.

Planes can be characterised in the following ways:

i) A point P, and 2 vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ determine a plane, containing P and parallel to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, as long as $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ point in different directions. This means $\boldsymbol{v}_2 \neq a\boldsymbol{v}_1$ for some $a \in \mathbb{R}$.
ii) Three points P_1, P_2 and P_3 determine a plane that contains all 3 points, as long as the points do not lie in a straight line.
iii) A point P and a vector $\boldsymbol{n}$ determine the plane which goes through P and for which all vectors in the plane are perpendicular to $\boldsymbol{n}$.

The last option might seem a bit contrived, but we will see that it results in a quite useful equation for the plane. An illustration can be found in Figure 7

Let us derive equations for planes with the given data:

i) Similar to the case of a straight line, we can immediately turn this data into the *parametric form* of a plane by observing that for any point Q in the plane, we must have

$$\overrightarrow{OQ} = \overrightarrow{OP} + s\boldsymbol{v}_1 + t\boldsymbol{v}_2, \quad s, t \in \mathbb{R}.$$

So, for a plane we have two parameters, rather than only one.

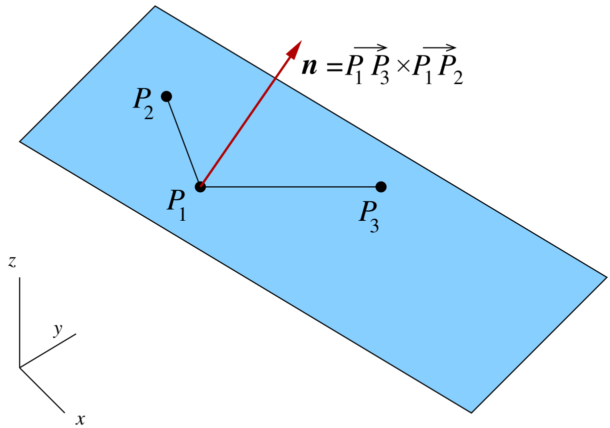


FIGURE 7. A plane can be characterised by 3 points on the plane or by it's normal vector.

ii) When given 3 points P_1, P_2, P_3, it is easy to write down the parametric form of the plane these points determine. We have

$$\overrightarrow{\mathcal{O}Q} = \overrightarrow{\mathcal{O}P_1} + s \cdot \overrightarrow{P_1P_2} + t \cdot \overrightarrow{P_1P_3}, \quad s, t \in \mathbb{R}.$$

iii) If the vector $\mathbf{n}$ is perpendicular to the plane, then this means it is perpendicular to all vectors in the plane. So, let $Q = (x, y, z)$ be a point on the plane. Then the vector $\overrightarrow{PQ}$ lies in the plane, and we therefore have

$$\overrightarrow{PQ} \cdot \mathbf{n} = 0.$$

But since, $\overrightarrow{PQ} = \overrightarrow{\mathcal{O}Q} - \overrightarrow{\mathcal{O}P}$, this equation can be written as

$$(\overrightarrow{\mathcal{O}Q} - \overrightarrow{\mathcal{O}P}) \cdot \mathbf{n} = 0,$$

or

$$\overrightarrow{\mathcal{O}Q} \cdot \mathbf{n} = \overrightarrow{\mathcal{O}P} \cdot \mathbf{n}.$$

Hence, if we know $\mathbf{n} = (n_1, n_2, n_3)$ and $P = (p_1, p_2, p_3)$, then the equation for the plane reads

$$n_1x + n_2y + n_3z = n_1p_1 + n_2p_2 + n_3p_3.$$

This is sometimes called the *Cartesian form* of the equation for a plane.

As for the case of lines, we illustrate these general comments using an example.

EXAMPLE 4.4. Find the equation of the plane through $A = (1, 0, 0)$, $B = (1, 1, 0)$ and $C = (0, 2, -1)$, and compute its line of intersection with the plane whose Cartesian form is $2x + y = 4$.

We start by deriving the equation of the first plane. Two vectors in this plane are given by $\overrightarrow{AB} = (0,1,0)$ and $\overrightarrow{AC} = (-1,2,-1)$. We use them to compute a normal vector by letting

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (0,1,0) \times (-1,2,-1) = (1,0,-1).$$

Now, computing $\mathbf{n} \cdot \overrightarrow{\mathcal{OA}} = (1,0,-1) \cdot (1,0,0) = 1$, we obtain as the equation of the plane

$$(1,0,-1) \cdot (x,y,z) = 1, \quad \text{or} \quad x - z = 1.$$

In order to find the intersection with the second plane, we note that points in the intersection need to satisfy both equations describing the planes. This means we can find those points by solving the equations

$$x - z = 1 \quad \text{and} \quad 2x + y = 4$$

simultaneously. The first equation implies that $z = x - 1$, whereas the second equation gives $y = 4 - 2x$. Therefore, point on the line of intersection are of the form

$$(x,y,z) = (x, 4-2x, x-1),$$

or, setting $x = s$ as a parameter

$$(x,y,z) = (0,4,-1) + s \cdot (1,-2,1).$$

This is the parametric form of the line of intersection of the 2 planes.

Notation. If a vector $\mathbf{n}$ is perpendicular to a plane S we say that $\mathbf{n}$ is a *normal vector* for the plane S.

Note. Given any plane S whose equation in Cartesian form is

$$a_1x + a_1y + a_3z = b$$

for some $a_1, a_2, a_3, b \in \mathbb{R}$, the vector (a_1, a_2, a_3) is a normal vector for S.

Why? Consider any (fixed) point $P = (p_1, p_2, p_3)$ in the plane. Then

$$a_1p_1 + a_2p_2 + a_3p_3 = b \tag{40}$$

Now $\mathbf{n}$ is a normal vector for S if and only if

$$\overrightarrow{PQ} \cdot \mathbf{n} = 0$$

for any point $Q = (x,y,z)$ in the plane S, if and only if

$$(\overrightarrow{OQ} - \overrightarrow{OP}) \cdot \mathbf{n} = 0$$

i.e. if and only if

$$(x,y,z) \cdot \mathbf{n} = (p_1, p_2, p_3) \cdot \mathbf{n}.$$

i.e. if and only if

$$n_1x + n_2y + n_3z = b$$

by (40). Hence we can take $n_i = a_i$ for $1 \leqslant i \leqslant 3$. In other words

$$\mathbf{n} = (a_1, a_2, a_3)\,.$$

Example (i) A normal vector for the plane

$$3x - 4y + 5z = 7$$

is $\mathbf{n} = (3, -4, 5)$.

(ii) A normal vector for the plane

$$2x + 3z = 0$$

is $\mathbf{m} = (2, 0, 3)$.

4.3. Spheres in $\mathbb{R}^3$. Let us start by recalling the equation for a circle in $\mathbb{R}^2$. If the circle has centre (x_1, y_1) and radius r, then points on the circle satisfy

$$(x - x_1)^2 + (y - y_1)^2 = r^2.$$

Using the 2-dimensional vectors (x, y) and (x_1, y_1), we can rewrite this as

$$\|(x, y) - (x_1, y_1)\|^2 = r^2.$$

Hence—as expected—a circle is the set of points that are at distance r from the centre (x_1, y_1).

In the same way, a *sphere* in $\mathbb{R}^3$ is the set of points at a certain distance r from a centre (x_1, y_1, z_1). Thus, these points have to satisfy the equation

$$\|(x, y, z) - (x_1, y_1, z_1)\|^2 = r^2,$$

which means

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2.$$

Example 4.5. Where does the sphere with centre $P = (2, -1, 1)$ and radius $r = 4$ intersect the z-axis?

The equation of the sphere is

$$(x - 2)^2 + (y + 1)^2 + (z - 1)^2 = 16.$$

Points on the z-axis are of the form $(0, 0, z)$, so we need to solve

$$4 + 1 + (z - 1)^2 = 16$$

This means $(z - 1)^2 = 11$, which has solutions $z_\pm = 1 \pm \sqrt{11}$. The 2 points of intersection are therefore

$$P_\pm = (0, 0, z_\pm) = (0, 0, 1 \pm \sqrt{11}).$$

Example 4.6. A sphere with its centre at the origin just touches the plane S whose Cartesian form is

$$2x - y - 3z = -1\,.$$

What is the radius of the sphere? State its equation.

Solution. A normal vector for S is

$$\mathbf{n} = (2, -1, -3)\,.$$

Hence the line L through $\mathcal{O} = (0, 0, 0)$ in the direction of $\mathbf{n}$ is given by the equation (in parametric form)

$$\begin{aligned}(x, y, z) &= (0, 0, 0) + s \cdot (2, -1, -3) \\ &= (2s, -s, -3s)\,.\end{aligned}$$

Thus L intersects S at the point satisfying

$$2(2s) - (-s) - 3(-3s) = -1$$

i.e. where $s = -\frac{1}{14}$. Hence the intersection is obtained by substituting $-\frac{1}{14}$ for s in the equation of the line L. I.e. the intersection is at the point

$$(x, y, z) = -\frac{1}{14}(2, -1, -3)\,.$$

The distance from the plane to the origin is thus $\frac{1}{14}\sqrt{2^2 + 1^2 + 3^2} = \frac{1}{\sqrt{14}}$. Hence the equation of the sphere is

$$x^2 + y^2 + z^2 = \frac{1}{14}\,.$$

(i.e. $(x - 0)^2 + (y - 0)^2 + (z - 0)^2 = (\frac{1}{\sqrt{14}})^2$) and the radius of the sphere is $r = \frac{1}{\sqrt{14}}$.

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