# Calculus II

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CHAPTER 1

# General Information

This is an online manual is designed for students. The manual is available at the moment in HTML with frames (for easier navigation), HTML without frames and PDF formats. Each from these formats has its own advantages. Please select one better suit your needs.

There is on-line information on the following courses:

- Calculus I.
- Calculus II.
- Geometry.

## 1. Web page

There is a Web page which contains this course description as well as other information related to this course. Point your Web browser to

`http://v-v-kisil.scienceontheweb.net/courses/math152.html`

## 2. Course description and Schedule

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## 3. Warnings and Disclaimers

Before proceeding with this interactive manual we stress the following:

- These Web pages are designed in order to help students as a source of *additional information*. They are **NOT** an obligatory part of the course.
- The main material introduced during *lectures* and is contained in *Textbook*. This interactive manual is **NOT** a substitution for any part of those primary sources of information.
- It is **NOT** required to be familiar with these pages in order to pass the examination.

- The entire contents of these pages is continuously improved and updated. Even for material of lectures took place weeks or months ago changes are made.

# Contents

CHAPTER 9

# Infinite Series

## 5. A brief review of series

We refer to the chapter Infinite Series of the course **Calculus I** for the review of the following topics.

(i) Sequences of numbers
(ii) Convergent and Divergent Series
(iii) Positive Term Series
(iv) Ratio and Root Test
(v) Alternating Series and Absolute Convergence

## 6. Power Series

It is well known that polynomials are simplest functions, particularly it is easy to differentiate and integrate polynomials. It is desirable to use them for investigation of other functions. Infinite series reviewed in the previous sections are very important because they allow to represent functions by means of power series, which are similar to polynomials in many respects. An example of such representations is harmonic series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

DEFINITION 6.1. Let $x$ be a variable. A *power series in* $x$ is a series of the form

$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots ,$$

where each $b_k$ is real number.

A power series turns to be infinite (constant term) series if we will substitute a constant $c$ instead of the variable $x$. Such series could converge or diverge. All power series converge for $x = 0$. The convergence of power series described by the following theorem.

THEOREM 6.2. (i) *If a power series $\sum b_n x^n$ converges for a nonzero number $c$, then it is absolutely convergent whenever $|x| < |c|$.*
(ii) *If a power series $\sum b_n x^n$ diverges for a nonzero number $d$, then it diverges whenever $|x| > |d|$.*

PROOF. The proof follows from the Basic Comparison Test of the power series for $|x|$ and convergent geometric series with $r = \left|\frac{x}{c}\right|$. □

From this theorem we could conclude that

THEOREM 6.3. *If* $\sum b_n x^n$ *is a power series, then exactly one of the following true:*

(i) *The series converges only if* $x = 0$.
(ii) *The series is absolutely convergent for every* $x$.
(iii) *There is a number* $r$ *such that the series is absolutely convergent if* $x$ *is in open interval* $(-r, r)$ *and divergent if* $x < -r$ *or* $x > r$.

The number $r$ from the above theorem is called *radius of convergence*. The totality of numbers for which a power series converges is called its *interval of convergence*. The interval of convergence may be any of the following four types: $[-r, r]$, $[-r, r)$, $(-r, r]$, $(-r, r)$.

There is a more general type of power series

DEFINITION 6.4. Let $b$ be a real number and $x$ is a variable. A *power series in* $x - d$ is a series of the form

$$\sum_{n=0}^{\infty} b_n (x-d)^n = b_0 + b_1(x-d) + b_2(x-d)^2 + \cdots + b_n(x-d)^n + \cdots ,$$

where each $b_n$ is a real number.

This power series is obtained from the series in Definition 6.1 by replacement of $x$ by $x - d$. We could obtain a description of convergence of this series by replacement of $x$ by $x - d$ in Theorem 6.3.

The following exercises should be solved in the following way:

(i) Determine the radius $r$ of convergence, usually using Ratio test or Root Test.
(ii) If the radius $r$ is finite and nonzero determine if the series is convergent at points $x = -r$, $x = r$. Note that the series could be alternating at one of them and apply Alternating Test.

EXERCISE 6.5. Find the interval of convergence of the power series:

$$\sum \frac{1}{n^2+4} x^n; \qquad \sum \frac{1}{\ln(n+1)} x^n;$$
$$\sum \frac{10^{n+1}}{3^{2n}} x^n; \qquad \sum \frac{(3n)!}{(2n)!} x^n;$$
$$\sum \frac{10^n}{n!} x^n; \qquad \sum \frac{1}{2n+1} (x+3)^n;$$
$$\sum \frac{n}{3^{2n-1}} (x-1)^{2n}; \qquad \sum \frac{1}{\sqrt{3n+4}} (3x+4)^n;$$

## 7. Functional Sequences and Series, Uniform Convergence

Power series are a particular example of a wider concept.

DEFINITION 7.1. Let us consider an infinite *sequence of functions* or *functional sequence* $\{f_n(x)\}$ with a common domain D:

$$f_1(x), f_2(x), f_3(x), \dots, f_n(x), \dots.$$

The functional sequence $\{f_n(x)\}$ for each particular value $x_0 \in D$ defines a sequence of numbers $\{f_n(x_0)\}$.

DEFINITION 7.2. Let for any $x_0 \in D$ the sequence of numbers $\{f_n(x_0)\}$ be convergent and have a limit denoted by $f(x_0)$, then the the function $f(x)$ on D is called the *limit of functional sequence*. We write it as

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{or} \qquad f_n(x) \to f(x), n \to \infty.$$

Although a convergence $f_n(x) \to f(x)$ implies all convergences $f_n(x_0) \to f(x_0)$ for any $x_0 \in D$, the rate of convergence of numerical sequences $f_n(x_0)$ may vary at different points. Thus the following notion plays an important rôle:

DEFINITION 7.3. We say that a function $f(x)$ is a *uniform limit* of a functional sequence $f_n(x)$ on a domain D, or equivalently a functional sequence $f_n(x)$ *uniformly converges* to $f(x)$ on a domain D if for any $\epsilon > 0$ there is such $N \in \mathbb{N}$ that

$$|f_n(x_0) - f(x_0)| < \epsilon \qquad \text{for all } n > N \text{ and } x_0 \in D.$$

In the opposite case:

DEFINITION 7.4. A function $f(x) = \lim_{n\to\infty} f_n(x)$ is a *non-uniform limit* of a functional sequence $f_n(x)$ if there is $\epsilon > 0$ such that for any $N \in \mathbb{N}$ there exist $x_0 \in D$ and $n > N$ such that

$$|f_n(x_0) - f(x_0)| \geqslant \epsilon.$$

EXAMPLE 7.5. The functional sequences

$$f_n(x) = \frac{1}{1+n^2x^2} \qquad \text{and} \qquad g_n(x) = \frac{nx}{1+n^2x^2}$$

both converge to the function $f(x) \equiv 0$ on the interval $[0, 1]$. However $f_n(x)$ *uniformly* converges and $g_n(x)$ converges in a *non-uniform* way (prove it!)

Similarly we can define these notions for series.

DEFINITION 7.6. Let a series have functions $f_n(x)$ as its terms:

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \dots + f_n(x) + \cdots,$$

then it is called *functional series*.

DEFINITION 7.7. A functional series $\sum_{n=1}^{\infty} f_n(x)$ is called *convergent* if the functional sequence $S_k(x) = \sum_{n=1}^{k} f_n(x)$ of its partial sum is convergent.

DEFINITION 7.8. A functional series $\sum_{n=1}^{\infty} f_n(x)$ is called *uniformly convergent* if the functional sequence $S_k(x) = \sum_{n=1}^{k} f_n(x)$ of its partial sum is uniformly convergent.

EXAMPLE 7.9. (i) Any power series $\sum_{1}^{\infty} a_n x^n$ converges *uniformly* within the interval of convergence.

(ii) The series

$$\sum_{1}^{\infty} \frac{\sin nx}{n!}$$

*uniformly* converges on $\mathbb{R}$ (prove it!).

(iii) The series

$$\sum_{n=1}^{\infty} x^n(1-x^n)$$

converges on {[0,1]} in *non-uniform* way.

## 8. Power Series Representations of Functions

As we have seen in the previous section a power series $\sum b_n x^n$ could define a convergent infinite series $\sum b_n c^n$ for all $c \in (-r, r)$ which has a sum $f(c)$. Thus the power series define a function $f(x) = \sum b_n x^n$ with domain $(-r, r)$. We call it the *power series representation of* $f(x)$. Power series are used in calculators and computers.

EXAMPLE 8.1. Find function represented by $\sum(-1)^k x^k$.

The following theorem shows that integration and differentiations could be done with power series as easy as with polynomials:

THEOREM 8.2. *Suppose that a power series* $\sum b_n x^n$ *has a radius of convergence* $r > 0$*, and let* $f$ *be defined by*

$$f(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$$

*for every* $x \in (-r, r)$. *Then for* $-r < x < r$

$$f'(x) = b_1 + b_2 x + b_3 x^2 + \cdots + n b_n x^{n-1} + \cdots = \sum_{n=1}^{\infty} n b_n x^{n-1}; \tag{8.1}$$

$$\int_0^x f(t)\,dt = b_0 x + b_1 \frac{x^2}{2} + b_2 \frac{x^3}{3} + \cdots + b_n \frac{x^{n+1}}{n+1} + \cdots = \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}. \tag{8.2}$$

EXAMPLE 8.3. Find power representation for

(i) $\frac{1}{(1+x)^2}$.
(ii) $\ln(1+x)$ and calculate $\ln(1.1)$ to five decimal places.
(iii) $\arctan x$.

THEOREM 8.4. *If $x$ is any real number,*

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

PROOF. The proof follows from observation that the power series $f(x) = \sum \frac{x^n}{n!}$ satisfies to the equation $f'(x) = f(x)$ and the only solution to this equation with initial condition $f(0) = 1$ is $f(x) = e^x$. □

COROLLARY 8.5.

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

EXAMPLE 8.6. Find a power series representation for $\sinh x$, $xe^{-2x}$.

EXERCISE 8.7. Find a power series representation for $f(x)$, $f'(x)$, $\int_0^x f(t)\,dt$.

$$f(x) = \frac{1}{1+5x}; \qquad f(x) = \frac{1}{3-2x}.$$

EXERCISE 8.8. Find a power series representation and specify the radius of convergence for:

$$\frac{x}{1-x^4}; \qquad \frac{x^2-3}{x-2}.$$

EXERCISE 8.9. Find a power series representation for

$$f(x) = x^2 e^{(x^2)}; \qquad f(x) = x^4 \arctan(x^4).$$

## 9. Maclaurin and Taylor Series

We find several power series representation of functions in the previous section by a variety of different tools. *Could it be done in a regular fashion?* Two following theorem give the answer.

THEOREM 9.1. *If a function $f$ has a power series representation*

$$f(x) = \sum_{k=0}^{\infty} b_n x^n$$

*with radius of convergence $r > 0$, then $f^{(k)}(0)$ exists for every positive integer $k$ and*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

THEOREM 9.2. *If a function* $f$ *has a power series representation*

$$f(x) = \sum_{k=0}^{\infty} b_n (x-d)^n$$

*with radius of convergence* $r > 0$*, then* $f^{(k)}(d)$ *exists for every positive integer* $k$ *and*

$$f(x) = f(d) + \frac{f'(d)}{1!}(x-d) + \frac{f''(d)}{2!}(x-d)^2 + \cdots + \frac{f^{(n)}(d)}{n!}(x-d)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(d)}{n!}(x-d)^n$$

EXERCISE 9.3. Find Maclaurin series for:

$$f(x) = \sin 2x; \qquad f(x) = \frac{1}{1-2x}.$$

REMARK 9.4. It is easy to see that linear approximation formula is just the Taylor polynomial $P_n(x)$ for $n = 1$.

The last formula could be split to two parts: the $n$*th-degree Taylor polynomial* $P_n(x)$ of $f$ at $d$:

$$P_n(x) = f(d) + \frac{f'(d)}{1!}(x-d) + \frac{f''(d)}{2!}(x-d)^2 + \cdots + \frac{f^{(n)}(d)}{n!}(x-d)^n$$

and the *Taylor remainder*

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-d)^{n+1},$$

where $z \in (d, x)$. Then we could formulate a sufficient condition for the existence of power series representation of $f$.

THEOREM 9.5. *Let* $f$ *have derivatives of all orders throughout an interval containing* $d$*, and let* $R_n(x)$ *be the Taylor remainder of* $f$ *at* $d$*. If*

$$\lim_{n\to\infty} R_n(x) = 0$$

*for every* $x$ *in the interval, then* $f(x)$ *is represented by the Taylor series for* $f(x)$ *at* $d$*.*

EXAMPLE 9.6. Let $f$ be the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0, \end{cases}$$

then $f$ cannot be represented by a Maclaurin series.

EXERCISE 9.7. Show that for function $f(x) = e^{-x}$

$$\lim_{n\to\infty} R_n(x) = 0$$

and find the Maclaurin series.

The important Maclaurin series are:

| Function | Maclaurin series | Convergence |
|---|---|---|
| $e^x$ | $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ | $(-\infty, \infty)$ |
| $\ln(1+x)$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ | $(-1, 1]$ |
| $\sin x$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ | $(-\infty, \infty)$ |
| $\cos x$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ | $(-\infty, \infty)$ |
| $\sinh x$ | $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ | $(-\infty, \infty)$ |
| $\cosh x$ | $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ | $(-\infty, \infty)$ |
| $\arctan x$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ | $[-1, 1]$ |

EXERCISE 9.8. Find Maclaurin series for $\sin^2 x$.

EXERCISE 9.9. Find a series representation of $\ln x$ in powers of $x-1$.

EXERCISE 9.10. Find first three terms of the Taylor series for $f$ at $d$:

$$f(x) = \arctan x, \quad d = 1; \qquad f(x) = \csc x, \quad d = \pi/3.$$

## 10. Applications of Taylor Polynomials

We could use the Taylor polynomial $P_n(x)$ for an approximation of a function $f(x)$ in a neighborhood of point $x_0$. The important observation is: to keep amount of calculation on a low level we prefer to consider polynomials $P_n(x)$ with small $n$. But for such $n$ the obtained accuracy is tolerable only for a small neighborhood of $x_0$. If $x$ is remote from $x_0$ to obtain a reasonably good approximation with $P_n(0)$ for a small $n$ we need to take the Taylor expansion in another point $x_0'$ which is closer to $x$.

EXERCISE 10.1. Find the Maclaurin polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$ for $f(x)$, sketch their graphs. Approximate $f(a)$ to four decimal places by means of $P_3(x)$ and estimate $R_3(x)$ to estimate the error.

$$f(x) = \ln(x+1) \qquad a = 0.9.$$

EXERCISE 10.2. Find the Taylor formula with remainder for the given $f(x)$, $d$ and $n$.

$$f(x) = e^{-1}; \qquad d = 1, \quad n = 3.$$

$$f(x) = \sqrt[3]{x}; \qquad d = -8, \quad n = 3.$$

CHAPTER 11

# Vectors and Surfaces

## 2. Vectors in Three Dimensions

Similarly to Cartesian coordinates on the Euclidean plane we could introduce *rectangular coordinate system* or $xyz$*-coordinate system* in three dimensions. The origin is usually denoted by $O$ and three axises are $OX$, $OY$, $OZ$. The positive directions are selected in the way to form the *right-handed coordinate system*. In this system the *coordinate of a point* is an ordered triple of real numbers $(a_1, a_2, a_3)$. Points with all three coordinates being positive form the *first octant*.

Similarly to two dimensional case we have the following formulas

THEOREM 2.1. (i) *The* distance *between* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *is*

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

(ii) *The midpoint of the line segment* $P_1(x_1, y_1, z_1)$ *to* $P_2(x_2, y_2, z_2)$ *is*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

(iii) *An equation of a sphere of radius* $r$ *and center* $P_0(x_0, y_0, z_0)$ *is*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

We define *vector* $\mathbf{a} = (a_1, a_2, a_3)$ in the three dimensional case as a transformation which maps point $(x, y, z)$ to $(x + a_1, y + a_2, z + z_3)$. Vectors could be added and multiplied by a scalar according to the rules:

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (a_1 + b_1, a_2 + b_2, a_3 + b_3); \\ c\mathbf{a} &= (ca_1, ca_2, ca_3); \end{aligned}$$

There is a special *null vector* $\mathbf{0} = (0, 0, 0)$ and *inverse vector* $-\mathbf{a} = (-a_1, -a_2, -a_3)$ for any vector $a$.

We have the following properties:

(i) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
(ii) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.
(iii) $\mathbf{a} + \mathbf{0} = \mathbf{a}$.
(iv) $\mathbf{a} + -\mathbf{a} = \mathbf{0}$.
(v) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + a\mathbf{b}$.

(vi) $(c+d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$.
(vii) $(cd)\mathbf{a} = c(d\mathbf{a}) = d(c\mathbf{a})$.
(viii) $1\mathbf{a} = \mathbf{a}$.
(ix) $0\mathbf{a} = \mathbf{0} = c\mathbf{0}$.

We define *subtraction of vectors* (or *difference of vectors*) by the rule:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

DEFINITION 2.2. Nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ have
(i) the *same direction* if $\mathbf{b} = c\mathbf{a}$ for some scalar $c > 0$.
(ii) the *opposite direction* if $\mathbf{b} = c\mathbf{a}$ for some scalar $c < 0$.

DEFINITION 2.3. We define vectors:

$$\mathbf{i} = (1,0,0), \qquad \mathbf{j} = (0,1,0), \qquad \mathbf{k} = (0,0,1).$$

It is obvious that

$$\mathbf{a} = (a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

The *magnitude of vector* is defined to be

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

## 3. Dot Product

Besides addition of vectors and multiplication by the scalar there two different operation which allows to multiply vectors.

DEFINITION 3.1. The *dot product* (or *scalar product*, or *inner product*) $\mathbf{a} \cdot \mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

THEOREM 3.2. Properties of the dot product *are:*
(i) $\mathbf{a} \cdot \mathbf{a} = \|a\|^2$.
(ii) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
(iii) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
(iv) $(m\mathbf{a}) \cdot \mathbf{b} = m\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$.
(v) $\mathbf{0} \cdot \mathbf{a} = 0$.

DEFINITION 3.3. Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero vectors.
(i) If $\mathbf{b} \neq c\mathbf{a}$ then *angle* $\theta$ *between* $a$ *and* $b$ is the angle of triangle defined by them.
(ii) If $\mathbf{b} = c\mathbf{a}$ then $\theta = 0$ if $c > 0$ and $\theta = \pi$ if $c < 0$.

Vectors are *orthogonal* or *perpendicular* if $\theta = \pi/2$. By a convention $\mathbf{0}$ is orthogonal and parallel to any vector.

THEOREM 3.4. *For nonzero* $\mathbf{a}$ *and* $\mathbf{b}$*:*

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta.$$

COROLLARY 3.5. *For nonzero* $\mathbf{a}$ *and* $\mathbf{b}$*:*

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}.$$

COROLLARY 3.6. *Two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal if and only if* $\mathbf{ab} = 0$.

COROLLARY 3.7 (Cauchy-Schwartz-Bunyakovskii Inequality).

$$|\mathbf{a}\cdot\mathbf{b}| \leqslant \|\mathbf{a}\|\,\|\mathbf{b}\|$$

THEOREM 3.8 (Triangle Inequality).

$$\|\mathbf{a}+\mathbf{b}\| \leqslant \|\mathbf{a}\| + \|\mathbf{b}\|.$$

We define *component of* $\mathbf{a}$ *along* $\mathbf{b}$

$$\mathbf{comp_b a} = \mathbf{a}\cdot\frac{1}{\|\mathbf{b}\|}\mathbf{b}$$

DEFINITION 3.9. The *work done by a constant force* $\mathbf{a}$ as its point of application moves along the vector $\mathbf{b}$ is $\mathbf{a}\cdot\mathbf{b}$.

## 4. Vector Product

DEFINITION 4.1. A *determinant of order 2* is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

A *determinant of order 3* is defined by

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_1.$$

DEFINITION 4.2. The *vector product* (or *cross product*) $\mathbf{a}\times\mathbf{b}$ is

$$\begin{aligned} \mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}. \end{aligned}$$

THEOREM 4.3. *The vector* $\mathbf{a}\times\mathbf{b}$ *is orthogonal to both* $\mathbf{a}$ *and* $\mathbf{b}$.

THEOREM 4.4. *If* $\theta$ *is the angle between nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$*, then*

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta.$$

COROLLARY 4.5. *Two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel if and only if* $\mathbf{a}\times\mathbf{b} = \vec{0}$.

EXERCISE 4.6. Compile the multiplication table for vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Be careful, because:

$$\begin{aligned} \mathbf{i}\times\mathbf{j} &\neq \mathbf{j}\times\mathbf{i} \\ (\mathbf{i}\times\mathbf{j})\times\mathbf{j} &\neq \mathbf{i}\times(\mathbf{j}\times\mathbf{j}). \end{aligned}$$

THEOREM 4.7. Properties of the vector product *are*

(i) $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$.
(ii) $(m\mathbf{a})\times\mathbf{b} = m(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(m\mathbf{b})$.
(iii) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$.
(iv) $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$.
(v) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.
(vi) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Dot and vector products related to geometric properties.

EXERCISE 4.8. Prove that the distance from a point $R$ to a line $l$ is given by

$$d = \frac{\left\|\vec{PQ}\times\vec{PR}\right\|}{\left\|\vec{PQ}\right\|}.$$

EXERCISE 4.9. Prove that the volume of the oblique box spanned by three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$.

## 5. Lines and Planes

THEOREM 5.1. *Parametric equation for the line through* $P_1(x_1, y_1, z_1)$ *parallel to* $\mathbf{a} = (a_1, a_2, a_3)$ *are*

$$x = x_1 + a_1 t, \quad y = y_1 + a_2 t, \quad z = z_1 + a_3 t; \qquad t \in \mathbb{R}^{\cdot}$$

Note that we obtain the same line if we use any vector $\mathbf{b} = c\mathbf{a}$, $c \neq 0$.

COROLLARY 5.2. *Parametric equation for the line through* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *are*

$$x = x_1+(x_2-x_1)t, \quad y = y_1+(y_2-y_1)t, \quad z = z_1+(z_2-z_1)t; \qquad t \in \mathbb{R}^{\cdot}$$

EXERCISE 5.3. Find equations of the lines:

(i) $P(1, 2, 3)$; $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
(ii) $P_1(2, -2, 4)$, $P_2(2, -2, -3)$.

EXERCISE 5.4. Determine whether the lines intersect: $x = 2 - 5t$, $y = 6 + 2t$, $z = -3 - 2t$; $x = 4 - 3v$, $y = 7 + 5v$, $z = 1 + 4v$.

DEFINITION 5.5. Let $\theta$ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ and let $l_1$ and $l_2$ be lines that are parallel to the position vectors of $\mathbf{a}$ and $\mathbf{b}$.

(i) The *angles between lines* $l_1$ and $l_2$ are $\theta$ and $\pi - theta$.
(ii) The lines $l_1$ and $l_2$ are *parallel* iff $\mathbf{b} = c\mathbf{a}$ for $c \in \mathbb{R}$.

(iii) The lines $l_1$ and $l_2$ are *orthogonal* iff $\mathbf{a} \cdot \mathbf{b} = 0$ for $c \in \mathbb{R}$.

The *plane* through $P_1$ with normal vector $\vec{P_1P_2}$ is the set of all points P such that $\vec{P_1P}$ is orthogonal to $\vec{P_1P_2}$.

THEOREM 5.6. *An equation of the plane through* $P_1(x_1, y_1, z_1)$ *with normal vector* $\mathbf{a} = (a_1, a_2, a_3)$ *is*

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0.$$

THEOREM 5.7. *The graph of every linear equation* $ax+by+cz+d = 0$ *is a plane with normal vector* $(a, b, c)$.

EXERCISE 5.8. Find an equation of the plane through $P(4, 2, -6)$ and normal vector $\vec{OP}$.

EXERCISE 5.9. Sketch the graph of the equation
(i) $y = -2$;
(ii) $3x - 2z - 24 = 0$;

DEFINITION 5.10. Two planes with normal vectors **a** and **b** are
(i) *parallel* if **a** and **b** are parallel;
(ii) *orthogonal* if **a** and **b** are orthogonal;

EXERCISE 5.11. Find an equation of the plane through $P(3, -2, 4)$ parallel to $-2x + 3y - z + 5 = 0$.

THEOREM 5.12 (Symmetric Form for a Line).

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}.$$

EXERCISE 5.13. Show that *distance* from a point $P_0(x_0, y_0, z_0)$ to the plane $ax + by + cz + d = 0$ is

$$h = \left|\mathbf{comp}_{\mathbf{n}} \vec{P_0P_1}\right|,$$

where $\mathbf{n} = (a, b, c)$ and $P_1$—any point on the plane.

EXERCISE 5.14. Show that planes $3x + 12y - 6z = -2$ and $5x + 20y - 10z = 7$ are parallel and find distance between them.

EXERCISE 5.15. Find an equation of the plane that contains the point $P(4, -3, 0)$ and line $x = t + 5$, $y = 2t - 1$, $z = -t + 7$.

EXERCISE 5.16. Show that *distance* between two lines defined by points $P_1$, $Q_1$ and $P_2$, $Q_2$ is given by the formula

$$d = \left|\mathbf{comp}_{\mathbf{n}} \vec{P_1P_2}\right|, \qquad \mathbf{n} = \frac{\vec{P_1Q_1} \times \vec{P_2Q_2}}{\left\|\vec{P_1Q_1} \times \vec{P_2Q_2}\right\|}.$$

EXERCISE 5.17. Find the distance between point $P(3, 1, -1)$ and line $x = 1 + 4t$, $y = 3 - t$, $z = 3t$.

## 6. Surfaces

It is important to represent different surfaces (not only planes) from 3d space into our two dimensional drawing. Some useful technique is given by *trace on a surface* $S$ in a plane, namely by intersection of $S$ an the plane.

There are several classic important types of surfaces. To follows given examples you need to remember equations of conics in Cartesian coordinates.

EXAMPLE 6.1. $z = x^2 + y^2$ define *circular paraboloid* or *paraboloid of revolution*.

DEFINITION 6.2. Let $C$ be a curve in a plane, and let $l$ be a line that is not in a parallel plane. The set of points on all lines that are parallel to $l$ and intersect $C$ is a *cylinder*. The curve $C$ called is called *directrix of the cylinder*.

EXAMPLE 6.3. The *right circular cylinder* is given by the equation $x^2 + y^2 = r^2$.

Similarly to quadratic equations equations defining conics the equation

$$Ax^2 + By^2 + cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

defines *quadric surface*. We consider simplest cases with $D = E = F = G = H = I = 0$.

DEFINITION 6.4. *Ellipsoid*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

DEFINITION 6.5. The *hyperboloid of one sheet*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

DEFINITION 6.6. The *hyperboloid of two sheets*:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

DEFINITION 6.7. The *cone*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

DEFINITION 6.8. The *paraboloid*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz.$$

DEFINITION 6.9. The *hyperbolic paraboloid*:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = cz.$$

CHAPTER 12

# Vector-Valued Functions

DEFINITION 0.1. Let $D$ be a set of real numbers. A *vector-valued function* $\mathbf{r}$ with domain $D$ is a correspondence that assigns to each number $t$ in $D$ exactly one vector $\mathbf{r}(t)$ in $\mathbb{R}^3$.

THEOREM 0.2. *If* $D$ *is a set of real numbers, then* $\mathbf{r}$ *is a vector-valued function with domain* $D$ *if and only if there are scalar function* $f$, $g$, *and* $h$ *such that*

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}.$$

EXERCISE 0.3. Sketch the two vectors

$$\mathbf{r}(t) = t\,\mathbf{i} + 3\sin t\mathbf{j} + 3\cos t\,\mathbf{k}, \quad \mathbf{r}(0),\ \mathbf{r}(\pi/2).$$

Set of *endpoints* of all vectors $\vec{OP} = \mathbf{r}(t)$ define a *space curve* $C$. A *parameter equation* of the curve $C$ is

$$x = f(t), \quad y = g(t), \quad z = z(t).$$

The *orientation* of $C$ is the direction determined by increasing values of $t$.

EXERCISE 0.4. Sketch the curve and indicate orientation:

$$\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + 3\,\mathbf{k}; \quad 0 \leqslant t \leqslant 4.$$

The following theorem is completely analogous to arc length of a plane curve:

THEOREM 0.5. *If a curve* $C$ *has a smooth parameterization*

$$x = f(t), \quad y = g(t), \quad z = z(t), \quad a \leqslant t \leqslant b$$

*and if* $C$ *does not intersect itself, except possibly for* $t = a$ *and* $t = b$, *then the length* $L$ *of* $C$ *is*

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt.$$

EXERCISE 0.6. Find the arc length:

$$x = e^t\cos t, \qquad y = e^t, \qquad z = e^t\sin t; \quad 0 \leqslant t \leqslant 2\pi;$$

$$x = 2t, \quad y = 4\sin 3t, \quad z = 4\cos 3t; \quad 0 \leqslant t \leqslant 2\pi;$$

## 1. Limits, Derivatives and Integrals of Vector-valued Functions

All definitions and results in this section are in close relation with the theory of scalar-valued function **Calculus I**. We advise to refresh Chapters on Limits and Derivative from **Calculus I** course.

DEFINITION 1.1. Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. The *limit* $\mathbf{r}(t)$ as $t$ approaches $a$ is

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k}.$$

provides $f$, $g$, and $h$ have limits as $t$ approaches $a$.

The next definition coincides with definition of continuity for scalar-valued function:

DEFINITION 1.2. A vector valued function $\mathbf{r}$ is *continuous* at $a$ if

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a).$$

Particularly $\mathbf{r}(t)$ is continuous iff $f(t)$, $g(t)$, and $h(t)$ are continuous. Similarly we define derivative

DEFINITION 1.3. Let $\mathbf{r}$ be a vector-valued function. The *derivative* is the vector-valued function $\mathbf{r}'$ defined by

$$\mathbf{r}'(t) = \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)]$$

for every $t$ such that the limit exists.

EXERCISE 1.4. Find the domain, first and second derivatives of the functions:

$$\begin{aligned} \mathbf{r}(t) &= \sqrt[3]{t}\,\mathbf{i} + \frac{1}{t}\mathbf{j} + e^{-t}\,\mathbf{k}; \\ \mathbf{r}(t) &= \ln(1-t)\,\mathbf{i} + \sin t\,\mathbf{j} + t^2\,\mathbf{k}. \end{aligned}$$

THEOREM 1.5. *Let* $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ *and* $f$, $g$, *and* $h$ *are differentiable, then*

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

The *geometric meaning* is as expected—this is tangent vector to the curve defined by $\mathbf{r}$.

EXERCISE 1.6. Find parameter equation for the tangent line to $C$ at $P$:

$$x = e^t, \quad y = te^t, \quad z = t^2 + 4; \qquad P(1, 0, 4).$$

The properties of the derivative are as follows:

THEOREM 1.7. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are differentiable vector-valued functions and* $c$ *is a scalar, then*

(i) $[\mathbf{u}(t) + \mathbf{v}(t)]' = \mathbf{u}'(t) + \mathbf{v}'(t)$;

(ii) $[c\mathbf{u}(t)]' = c\mathbf{u}'(t)$;
(iii) $[\mathbf{u}(t)\cdot\mathbf{v}(t)]' = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$;
(iv) $[\mathbf{u}(t)\times\mathbf{v}(t)]' = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$;

As a consequence of these properties we could easily prove the following

THEOREM 1.8. *If* $\mathbf{r}$ *is differentiable and* $\|\mathbf{r}\|$ *is constant, then* $\mathbf{r}'$ *is orthogonal to* $\mathbf{r}'(t)$ *for every* $t$ *in the domain of* $\mathbf{r}'$.

Finally we define integrals of vector-valued functions using integrals of scalar-valued functions:

DEFINITION 1.9. Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ and $f$, $g$, and $h$ are integrable, then

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$

If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an *antiderivative* of $\mathbf{r}(t)$.

THEOREM 1.10. *If* $\mathbf{R}(t)$ *is an antiderivative of* $\mathbf{r}(t)$ *on* $[a, b]$, *then*

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

EXERCISE 1.11. Find $\mathbf{r}(t)$ subject to the given conditions:

$$\mathbf{r}'(t) = 2\mathbf{i} - 4t^3\,\mathbf{j} + 6\sqrt{t}\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}.$$

CHAPTER 13

# Partial Differentiation

## 1. Functions of Several Variables

It is common that real-world quantities depend from many different parameters. Mathematically we describe them as functions of several variables. We start from definition of functions of two variables.

DEFINITION 1.1. Let $D$ be a set of ordered pairs of real numbers. A *function of two variables* $f$ is a correspondence that assigns to each pair $(x, y)$ in $D$ exactly one real number, denoted by $f(x, y)$. The set $D$ is the *domain* of $f$. The *range* of $f$ consists of all real numbers $f(x, y)$, where $(x, y) \in D$.

EXERCISE 1.2. Describe domain of $f$ and find its values:

$$\begin{aligned} f(r, s) &= \sqrt{1-r} - e^{r/s}; \quad f(1,1), f(0,4), f(-3,3) \\ f(x, y, z) &= 2 + \tan x + y \tan z; \quad f(\pi/4, 4, \pi/6), f(0,0,0). \end{aligned}$$

EXERCISE 1.3. Sketch graph of $f$:

$$f(x, y) = \sqrt{2 - 2x - x^2 - y^2}, \qquad f(x, y) = 3 - x - 3y.$$

EXERCISE 1.4. Sketch the *level curves* for $f$:

$$f(x, y) = xy, \qquad k = -4, 1, 4.$$

EXERCISE 1.5. (i) Find the equation of *level surface* of $f$ that contains the point $P$.

$$f(x, y, z) = z^2 y + x; \quad P(1, 4, -2).$$

(ii) Describe the level surface of $f$ for given $k$:

$$f(x, y, z) = z + x^2 + 4y^2, \qquad k = -6, 6, 12.$$

## 2. Limits and Continuity

The fundamental notion of limit could be introduced for a function of two variables as follows

DEFINITION 2.1. Let a function $f$ of two variables be defined throughout the interior of a circle with center $(a, b)$, except possibly at $(a, b)$ itself. The statement

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \quad \text{or} \quad f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta, \text{ then } |f(x,y) - L| < \epsilon.$$

EXERCISE 2.2. Find limits

$$\lim_{(x,y)\to(2,1)} \frac{4+x}{2-y}, \qquad \lim_{(x,y)\to(-1,3)} \frac{y^2+x}{(x-1)(y+2)}.$$

THEOREM 2.3 (Two-Path Rule). *If two different paths to a point* $P(a,b)$ *produce two different limiting values for* $f$*, then* $\lim_{(x,y)\to(a,b)} f(x,y)$ *does not exist.*

EXERCISE 2.4. Show that the limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 2xy + 5y^2}{3x^2 + 4y^2}, \qquad \lim_{(x,y)\to(0,0)} \frac{3xy}{5x^4 + 2y^4}.$$

DEFINITION 2.5. A function $f$ of two variables is *continuous* at an interior point $(a,b)$ of its domain if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

EXERCISE 2.6. Describe the set of all points at which $f$ is continuous

$$f(x,y) = \frac{xy}{x^2 - y^2}, \qquad f(x,y) = \sqrt{xy}\tan z.$$

DEFINITION 2.7. Let a function $f$ of two variables be defined throughout the interior of a circle with center $(a,b,c)$, except possibly at $(a,b,c)$ itself. The statement

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L \quad \text{or} \quad f(x,y,z) \to L \text{ as } (x,y,z) \to (a,b,c)$$

means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if

$$0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta, \text{ then } |f(x,y,z) - L| < \epsilon.$$

THEOREM 2.8 (Composition of Continuous Functions). *If a function* $f$ *of two variables is continuous at* $(a,b)$ *and a function* $g$ *of one variables is continuous at* $f(a,b)$*, then the function* $h(x,y) = g(f(x,y))$ *is continuous at* $(a,b)$*.*

EXERCISE 2.9. Use Theorem on Composition of Continuous Functions to determine where $h$ is continuous.

$$f(x,y) = 3x + 2y - 4, \qquad g(t) = \ln(t+5).$$

## 3. Partial Derivatives

For functions of several variables the concept of derivative could modified as follows:

DEFINITION 3.1. Let $f$ be a function of two variables. The *first partial derivatives* of $f$ with respect to $x$ and $y$ are functions $f'_x$ and $f'_y$ such that

$$\frac{\partial}{\partial x} f(x, y) = f'_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h},$$
$$\frac{\partial}{\partial x} f(x, y) = f'_x(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

EXERCISE 3.2. Find first partial derivatives of $f$

$$f(x, y) = (x^3 - y^2)^5; \qquad f(x, y) = e^x \ln xy;$$
$$f(r, s, v, p) = r^3 \tan s + \sqrt{s} e^{(v^2)} - v \cos 2p; \qquad f(x, y, z) = xyz\, e^{xyz}.$$

This notion has a geometrical meaning which is very close to geometrical meaning of usual derivative derivative.

THEOREM 3.3. *Let $S$ be the graph of $z = f(x, y)$, and let $P(a, b, f(a, b))$ be a point on $S$ at which $f'_x$ and $f'_y$ exists. Let $C_1$ and $C_2$ be the traces of $S$ on the planes $x = a$ and $y = b$, respectively, and let $l_1$ and $l_2$ be the tangent lines to $C_1$ and $C_2$ at $P$.*

(i) *The slope of $l_1$ in the plane $x = a$ is $f'_y(a, b)$.*
(ii) *The slope of $l_1$ in the plane $y = b$ is $f'_x(a, b)$.*

We could define *second partial derivatives* by repetition. There are four of them:

$$f''_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right);$$
$$f''_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right);$$
$$f''_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right);$$
$$f''_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

EXERCISE 3.4. If $v = y \ln(x^2 + z^2)$, find $v'''_{zzy}$.

THEOREM 3.5. *Let $f$ be a function of two variables $x$ and $y$. If $f$, $f'_x$, $f'_y$, $f''_{xy}$, and $f''_{yx}$ are continuous on an open region $R$, then $f''_{xy} = f''_{yx}$ through $R$.*

EXERCISE 3.6. Verify that $f''_{xy} = f''_{yx}$.

$$f(x, y) = \frac{x^2}{x + y}; \qquad f(x, y) = \sqrt{x^2 + y^2 + z^2}.$$

### Review

EXERCISE 3.7. Find the interval of convergence of the power series:

$$\sum(-1)^n\frac{3^n}{n!}(x-4)^n; \qquad \sum(-1)^n\frac{e^{n+1}}{n^n}(x-1)^n.$$

EXERCISE 3.8. Obtain a power series representation for the function

$$f(x) = x^2\ln(1+x^2); \qquad f(x) = \arctan\sqrt{x}.$$

EXERCISE 3.9. Find all values of $c$ such that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j} + c\mathbf{k}$, and $\mathbf{b} = \mathbf{i} + 22\mathbf{j} - 3c\mathbf{k}$.

EXERCISE 3.10. Find the volume of the box having adjacent sides $AB$, $AC$, $AD$: $A(2,1,-1)$, $B(3,0,2)$, $C(4,-2,1)$, $D(5,-3,0)$.

EXERCISE 3.11. Find an equation of the plane through $P(-4,1,6)$ and having the same trace in $xz$-plane as the plane $x + 4y - 5z = 8$.

EXERCISE 3.12. Find arc length of the curve: $x = 2t$, $y = 4\sin 3t$, $z = 4\cos 3t$; $0 \leqslant t \leqslant 2\pi$.

EXERCISE 3.13. Find a parametric Al equation of the tangent line to curve $x = t\sin t$, $y = t\cos t$, $z = t$; at $P(\pi/2, 0, \pi/2)$.

EXERCISE 3.14. Show that limit does not exist.

$$\lim_{(x,y,z)\to(2,0,0)} \frac{(x-2)yz^2}{(x-2)^4 + y^4}.$$

## 4. Increments and Differentials

DEFINITION 4.1. Let $w = f(x,y)$, and let $\Delta x$ and $\Delta$ be increments of $x$ and $y$, respectively. The *increment of function* $w$ is

$$\Delta w = f(x + \Delta x, y + \Delta y) - f(x,y).$$

THEOREM 4.2. *Let* $w = f(x,y)$, *where the function* $f$ *is defined on a rectangular region* $R = \{(x,y) :\ a < x < b,\ c < y < d\}$. *Suppose* $f'_x$ *and* $f'_y$ *exist throughout* $R$ *and are continuous at* $(x_0, y_0)$. *Then*

$$\Delta w = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y.$$

A function $w$ is *differentiable* if its increment could be represented as above.

DEFINITION 4.3. The *differential of function* $w$ is

$$dw = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta x.$$

## 5. Chain Rules

Among different rules of derivation most powerful is the

THEOREM 5.1 (Chain rules). *If* $w = f(u, v)$, *with* $u = g(x, y)$, $v = h(x, y)$, *and if* $f$, $g$, *and* $h$ *are differentiable, then*

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x};$$
$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}.$$

PROOF. It follows from the Theorem on Increment. □

This formulas could be better understood and remembered if we will draw a tree representing dependence of variables.

EXERCISE 5.2. Find $\partial w/\partial x$, $\partial w partialy$ if $w = uv + v^2$, $u = x \sin y$, $v = y \sin x$.

Similar formulas are true for different number of variables

EXERCISE 5.3. Find $\partial z/\partial x$, $\partial z/\partial y$ if $z = pq + qw$, $p = 2x - y$, $q = x - 2y$, $w = -2x + 2y$.

Chain rules could be used to derive already known formulas in a new way.

EXERCISE 5.4. Derive formula $(uv)' = u'v+uv'$ using chain rules.

EXERCISE 5.5. Derive from chain rules the following formula for implicit derivatives of $y$ defined by $F(x, y) = 0$:

$$y' = -\frac{F'_x(x, y)}{F'_y(x, y)}.$$

## 6. Directional Derivatives

We could give a definition generalizing partial derivatives.

DEFINITION 6.1. Let $w = f(x, y)$ and $\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}$ be a unit vector. The *directional derivative* of $f$ at $P(x, y)$ in the direction $\mathbf{u}$, denoted $D_\mathbf{u} f(x, y)$, is

$$D_\mathbf{u} = \lim_{s\to 0} \frac{f(x + su_1, y + su_2) - f(x, y)}{s}.$$

Partial derivatives are particular cases of directional derivatives: $\partial/\partial x = D_\mathbf{i}$ and $\partial/\partial y = D_\mathbf{j}$. It is interesting that we could calculate any directional derivative if we know only partial ones.

THEOREM 6.2. *If* $f$ *is a differentiable function of two variables, then*

$$D_\mathbf{u} f(x, y) = f'_x(x, y)u_1 + f'_y(x, y)u_2.$$

PROOF. It is follows from the Chain Rules. □

EXERCISE 6.3. Find directional derivative

$$f(x,y) = x^3 - 3x^2y - y^3, \quad P(1,-2), \quad \mathbf{u} = \frac{1}{2}(-\mathbf{i} + \sqrt{3}\mathbf{j}).$$

DEFINITION 6.4. Let f be a function of two variables. The *gradient* of f is the vector valued function

$$\nabla f(x,y) = f'_x(x,y)\mathbf{i} + f'_y(x,y)\mathbf{j}.$$

Directional derivative in *gradient form* is

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

EXERCISE 6.5. Find gradient

$$f(x,y) = e^{3x} \tan y, \quad P(0, \pi/4).$$

From gradient form of directional derivative easily follows the following theorem:

THEOREM 6.6. *Let* $f$ *be a function of two variables that is differentiable at the point* $P(x,y)$.

(i) *The maximum value of* $D_{\mathbf{u}}$ *is* $\|\nabla f(x,y)\|$.
(ii) *The maximum rate of increase of* $f(x,y)$ *occurs in direction of* $\nabla f(x,y)$.
(iii) *The minimum value of* $D_{\mathbf{u}}$ *is* $-\|\nabla f(x,y)\|$.
(iv) *The minimum rate of increase of* $f(x,y)$ *occurs in direction of* $-\nabla f(x,y)$.

Similarly directional derivatives and gradients could be defined for functions of three variables.

EXERCISE 6.7. Find directional derivative at P in the direction to Q. Find directions of maximal and minimal increase of f.

$$f(x,y,z) = \frac{x}{y} - \frac{y}{z}, \quad P(0,-1,2), \quad Q(3,1,-4).$$

## 7. Tangent Planes and Normal Lines

THEOREM 7.1. *Suppose that* $F(x,y,z)$ *has continuous first partial derivatives and that* $S$ *is the graph of* $F(x,y,z) = 0$. *If* $P_0$ *is a point on* $S$ *and if* $F'_x$, $F'_y$, $F'_z$ *are not all* 0 *at* $P_0$, *then the vector* $\nabla F]_{P_0}$ *is normal to the tangent plane to* $S$ *at* $P_0$. *And equation of the tangent plane is*

$$F'_x(x_0,y_0,z_0)(x-x_0) + F'_y(x_0,y_0,z_0)(y-y_0) + F'_z(x_0,y_0,z_0)(z-z_0) = 0.$$

THEOREM 7.2. *An equation for the tangent plane to the graph of* $z = f(x,y)$ *at the point* $(x_0,y_0,z_0)$ *is*

$$z - z_0 = f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$

EXERCISE 7.3. Find equation for the tangent plane and normal line to the graph.

$$9x^2 - 4y^2 - 25z^2 = 40; \quad P(4,1,-2).$$

## 8. Extrema of Functions of Several Variables

The definition of *local maximum*, *local minimum*, which are *local extrema*, are the same as for function of one variable.

DEFINITION 8.1. Let $f$ be a function of two variables. A pair $(a, b)$ is a *critical point* of $f$ if either

(i) $f'_x(a, b) = 0$ and $f'_y(a, b) = 0$, or
(ii) $f'_x(a, b)$ or $f'_y(a, b)$ does not exist.

DEFINITION 8.2. Let $f$ be a function of two variables that has continuous second partial derivatives. The *discriminant* $D$ of $f$ is given by

$$D(x, y) = f''_{xx}f''_{yy} - [f''_{xy}]^2 = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{vmatrix}.$$

The following result is similar to Second Derivative Test.

TEST 8.3 (Test for Local Extrema). Let $f$ be a function of two variables that has continuous second partial derivatives throughout an open disk $R$ containing a critical point $(a, b)$. If $D(a, b) > 0$, then $f(a, b)$ is

(i) a local maximum of $f$ if $f''_{xx}(a, b) < 0$.
(ii) a local minimum of $f$ if $f''_{xx}(a, b) > 0$.

If a critical point with existent partial derivatives is not a local extrema then it is called *saddle point*. We could determine them by determinant:

THEOREM 8.4. *Let $f$ have continuous second partial derivatives throughout an open disk $R$ containing an critical point $(a, b)$ with existent derivatives. If $D(a, b)$ is negative, then $(a, b)$ is a saddle point.*

EXERCISE 8.5. Find extrema and saddle points.

$$\begin{aligned} f(x, y) &= x^2 - 2x + y^2 - 6y + 12 \\ f(x, y) &= -2x^2 - 2xy - \frac{3}{2}y^2 - 14x - 5y \\ f(x, y) &= -\frac{1}{3}x^3 + xy + \frac{1}{2}y^2 - 12y. \end{aligned}$$

EXERCISE 8.6. Find the max and min of $f$ in $R$.

$$f(x, y) = x^2 - 3xy - y^2 + 2y - 6x; \quad R = \{(x, y) \mid |x| \leqslant 3, |y| \leqslant 2\}.$$

EXERCISE 8.7. Find three positive real numbers whose sum is 1000 and whose product is a maximum.

## 9. Lagrange Multipliers

THEOREM 9.1. *Suppose that* $f$ *and* $g$ *are functions of two variables having continuous first partial derivatives and that* $\nabla g \neq \mathbf{0}$ *throughout a region. If* $f$ *has an extremum* $f(x_0, y_0)$ *subject to the constraint* $g(x, y) = 0$*, then there is a real number* $\lambda$ *such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

*By other words they are among solution of the system*

$$\begin{cases} f'_x(x,y) &= \lambda g'_x(x,y) \\ f'_y(x,y) &= \lambda g'_y(x,y) \\ g(x,y) &= 0 \end{cases}.$$

EXERCISE 9.2. Find the extrema of $f$ subject to the stated constrains

$$f(x, y) = 2x^2 + xy - y^2 + y; \quad 2x + 3y = 1.$$

CHAPTER 14

# Multiply Integrals

We consider the next fundamental operation of calculus for functions of several variables.

## 1. Double Integrals

The definite integral of a function of one variable was defined using using *Riemann sum*. We could apply the same idea for definition of definite integral for a function of several variables.

DEFINITION 1.1. Let $f$ be a function of two variables that is defined on a region $R$. The *double integral* of $f$ over $R$, is

$$\iint_R f(x, y)\, dA = \lim_{\|P\| \to 0} \sum_k f(x_k, y_k) \Delta A,$$

provided the limit exists for the *norm of the partition* tensing to $0$.

The following is similar to geometrical meaning of definite integral

DEFINITION 1.2 (Geometrical Meaning of Double Integral). Let $f$ be a continuous function of two variables such that $f(x, y)$ is nonnegative for every $(x, y)$ in a region $R$. The *volume* $V$ of the solid that lies under the graph of $z = f(x, y)$ and over $R$ is

$$V = \iint_R f(x, y)\, dA.$$

Double integral has the following properties (see one variable case).

THEOREM 1.3. (i)

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA.$$

(ii)

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

(iii) *If* $R = R_1 \cup R_2$ *and* $R_1 \cap R_2 = \emptyset$

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

(iv) *If* $(x, y) \geqslant 0$ *throughout* $R$, *then* $\iint_R f(x, y)\, dA \geqslant 0$.

Practically double integrals evaluated by means of *iterated integrals* as follows:

THEOREM 1.4. *Let* $R$ *be a region of* $R_x$ *type. If* $f$ *is continuous on* $R$, *then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

EXERCISE 1.5. Evaluate

$$\int_0^3 \int_{-2}^{-1} (4xy^3 + y)\, dx\, dy \qquad \int_{-1}^{1} \int_{x^3}^{x+1} (3x + 2y)\, dy\, dx.$$

EXERCISE 1.6. Evaluate $\iint_R e^{x/y}\, dA$ if $R$ bounded by $y = 2x$, $y = -x$, $y = 4$.

EXERCISE 1.7. Sketch the region $x = 2\sqrt{y}$, $\sqrt{3}x = \sqrt{y}$, $y = 2x + 5$ and express the double integral as iterated one.

EXERCISE 1.8. Sketch the region of integration for the iterated integral

$$\int_{-1}^{2} \int_{x^2-4}^{x-2} f(x, y)\, dy\, dx.$$

EXERCISE 1.9. Reverse the order of integration and evaluate

$$\int_1^e \int_0^{\ln x} y\, dy\, dx.$$

## 2. Area and Volume

From geometric meaning of double integrals we see that they are usable for finding volumes (and areas).

EXERCISE 2.1. Describe surface and region related to

$$\int_0^1 \int_{3-x}^{1-x^2} (x^2 + y^2)\, dy\, dx.$$

EXERCISE 2.2. Find volume under the graph $z = x^2 + 4y^2$ over triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 2)$.

EXERCISE 2.3. Sketch the solid in the first octant and find its volume $z = y^3$, $y = x^3$, $x = 0$, $z = 0$, $y = 1$.

## 3. Polar Coordinates, Double Integrals in Polar Coordinates

Besides the Cartesian coordinates we could describe a point of the plain by the distance to the preselected point O (*origin* or *pole*) and angle to the ray at origin (*polar axis*). This description is called *polar coordinates*. Here are some interesting curves and their equation in polar coordinates.

(i) circle $(O, R)$: $r = R$.
(ii) circle $(a, a)$: $r = 2a \sin\theta$.
(iii) *cardioid*: $r = a(1 + \cos\theta)$.
(iv) *limaçons*: $r = a + b\cos\theta$.
(v) $n$-*leafed rose*: $r = a \sin n\theta$.
(vi) *spiral of Archimedes*: $r = a\theta$.

EXERCISE* 3.1. Find equation of a straight line in polar coordinates.

Connection between the Cartesian coordinates and polar coordinates is as follows:

THEOREM 3.2. *The rectangular coordinates* $(x, y)$ *and polar coordinates* $(r, \theta)$ *of a point* $P$ *are related as follows:*

(i) $x = r\cos\theta$, $y = r\sin\theta$;
(ii) $r^2 = x^2 + y^2$, $\tan\theta = y/x$ *if* $x \neq 0$.

THEOREM 3.3 (Test for Symmetry). (i) *The graph of* $r = f(\theta)$ *is symmetric with respect to the polar axis if* $f(-\theta) = f(\theta)$.
(ii) *The graph of* $r = f(\theta)$ *is symmetric with respect to the vertical line if* $f(\pi - \theta) = f(\theta)$ *or* $f(-\theta) = -f(\theta)$.
(iii) *The graph of* $r = f(\theta)$ *is symmetric with respect to the pole if* $f(\pi + \theta) = f(\theta)$.

THEOREM 3.4. *The slope* $m$ *of the tangent line to the graph of* $r = f(\theta)$ *at the point* $P(r, \theta)$ *is*

$$m = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

The element of area in polar coordinates equal to $\Delta A = \frac{1}{2}(r_2^2 - r_1^2)\Delta\theta = \bar{r}\Delta r \Delta\theta$, where $\bar{r} = \frac{1}{2}(r_2 - r_1)$. Thus *double integral in polar coordinates* could be presented by iterated integral as follows:

$$\begin{aligned}\iint_R f(r, \theta)\, dA &= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta) r\, dr\, d\theta.\\ &= \int_\alpha^\beta \int_{h_1(r)}^{h_2(r)} f(r, \theta) r\, d\theta\, dr.\end{aligned}$$

EXERCISE 3.5. Use double integral to find the area inside $r = 2 - 2\cos\theta$ and outside $r = 3$.

EXERCISE 3.6. Use polar coordinates to evaluate the integral

$$\iint_R x^2(x^2 + y^2)^3\, dA$$

$R$ is bounded by semicircle $y = \sqrt{1 - x^2}$ and the $x$-axis.

EXERCISE 3.7. Evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\, dy\, dx.$$

EXERCISE 3.8. Find volume bounded by paraboloid $z = 4x^2+4y^2$, the cylinder $x^2+y^2 = 3y$, and plane $z = 0$.

## 4. Surface Area

THEOREM 4.1. *The surface area of the graph* $z = f(x, y)$ *over the region* $R$ *is given by*

$$A = \iint_R \sqrt{[f'_x(x,y)]^2 + [f'_y(x,y)]^2 + 1}\, dA.$$

EXERCISE 4.2. Setup a double integral for the surface area of the graph $x^2 - y^2 + z^2 = 1$ over the square with vertices $(0, 1)$, $(1, 0)$, $(-1, 0)$, $(0, -1)$.

EXERCISE 4.3. Find the area of the surface $z = y^2$ over the triangle with vertices $(0, 0)$, $(0, 2)$, $(2, 2)$.

EXERCISE 4.4. Find the area of the first-octant part of hyperbolic paraboloid $z = x^2 - y^2$ that is inside the cylinder $x^2 + y^2 = 1$.

## 5. Triple Integrals

There is no any principal differences to introduce *triple integral*, it could be done using ideas on definite integrals and double integrals.

DEFINITION 5.1. Triple integral of $f$ over 3d-region $Q$ is defined by Riemann sums:

$$\iiint_Q f(x, y, z)\, dV = \lim_{\|P\|\to 0} \sum_k f(x_k, y_k, z_k)\Delta V_k.$$

To evaluate triple integrals we reduce them by iteration to double integrals:

THEOREM 5.2.

$$\begin{aligned}\iiint_Q f(x, y, z)\, dV &= \iint_R \left[\int_{k_1(x,y)}^{k_2(x,y)} f(x, y, z)\, dz\right] dA\\ &= \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{k_1(x,y)}^{k_2(x,y)} f(x, y, z)\, dz\, dy\, dz.\end{aligned}$$

EXERCISE 5.3. Evaluate the iterated integral

$$\int_0^1 \int_{-1}^2 \int_1^3 (6x^2z + 5xy^2)\, dz\, dx\, dy; \qquad \int_{-1}^2 \int_1^{z^2} \int_{x+z}^{x-z} z\, dy\, dx\, dz.$$

EXERCISE 5.4. Describe region represented by integrals

$$\int_0^1 \int_{z^3}^{\sqrt{z}} \int_0^{4-x} dy\, dx\, dz, \qquad \int_0^1 \int_x^{3x} \int_0^{xy} dz\, dy\, dx.$$

*Physical meaning* of triple integrals is given by

THEOREM 5.5. *Mass of a solid with a mass density $\delta(x, y, z)$ is given by*

$$m = \iiint_Q \delta(x, y, z)\, dV$$

THEOREM 5.6. *Mass of a lamina with an area mass density $\delta(x, y)$ is given by*

$$m = \iiint_R \delta(x, y)\, dA$$

EXERCISE 5.7. Using triple integrals find volume bounded by

(i) $x^2 + z^2 = 4$, $y^2 + z^2 = 4$.
(ii) $z = x^2 + y^2$, $y + z = 2$.

## 7. Cylindrical Coordinates

The *cylindrical coordinates* of a point $P$ is the triple of numbers $(r, \theta, z)$, where $(r, \theta)$ are the polar coordinates of the projection of $P$ on $xy$-plane and $z$ is defined as in rectangular coordinates.

THEOREM 7.1. *The rectangular coordinates $(x, y, z)$ and the cylindrical coordinates $(r, \theta, z)$ of a point are related as follows:*

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{x}{y}.$$

EXERCISE 7.2. Describe the graph in cylindrical ccordinates:

(i) $r = -3\sec\theta$.
(ii) $z = 2r$.

EXERCISE 7.3. Change the equation to cylindrical coordinates:

(i) $x^2 + y^2 = 4z$.
(ii) $x^2 + z^2 = 9$.

THEOREM 7.4. *Evaluation of triple integral in cylindrical coordinates:*

$$\iiint_Q f(r, \theta, z)\, dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r, \theta, z)\, dz\, dr\, d\theta.$$

EXERCISE 7.5. A solid is bounded by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 4$, and the $xy$-plane. Find its volume.

## 8. Spherical Coordinates

The *spherical coordinates* of a point is the triple $(\rho, \phi, \theta)$.

THEOREM 8.1. *The rectangular coordinates* $(x, y, z)$ *and the spherical coordinates* $(\rho, \phi, \theta)$ *of a point related as follows:*

$$\begin{aligned} x &= \rho \sin\phi \cos\theta, x = \rho \sin\phi \sin\theta, \quad z = \rho\cos\theta \\ \rho^2 &= x^2 + y^2 + z^2. \end{aligned}$$

EXERCISE 8.2. Change coordinates

(i) spherical $(1, 3\pi/4, 2\pi/3)$ to rectangular and cylindrical.
(ii) rectangular $(1, \sqrt{3}, 0)$ to spherical and cylindrical.

EXERCISE 8.3. Describe graphs

(i) $\rho = 5$.
(ii) $\phi = 2\pi/3$.
(iii) $\theta = \pi/4$.

EXERCISE 8.4. Change the equation to spherical coordinates.

$$x^2 + y^2 = 4z; \qquad x^2 + (y-2)^2 = 4; \qquad x^2 + z^2 = 9.$$

THEOREM 8.5 (Evaluation theorem).

$$\iiint_Q f(\rho, \phi, \theta)\, dV = \int_m^n \int_c^d \int_a^b f(\rho, \phi, \theta)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

EXERCISE 8.6. Find volume of the solid that lies outside the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

EXERCISE 8.7. Evaluate integral in spherical coordinates:

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy.$$

CHAPTER 15

# Vector Calculus

## 1. Vector Fields

We could make one more step after vector valued functions and function of several variables.

DEFINITION 1.1. A *vector field in three dimensions* is a function $\mathbf{F}$ whose domain $D$ is a subset of $\mathbb{R}^3$ and whose range is is a subset of $\mathbb{V}^3$. If $(x, y, z)$ is in $D$, then

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

where $M$, $N$, and $P$ are scalar functions.

EXERCISE 1.2. Plot the vector field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$.

Example of vector field is as follows:

DEFINITION 1.3. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. A vector field $\mathbf{F}$ is an *inverse square field* if

$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}.$$

Examples of inverse square field are given by *Newton's law of gravitation* and *Coulom's law of charge interaction*.

DEFINITION 1.4. A vector filed $\mathbf{F}$ is *conservative* if

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z)$$

for some scalar function $f$. Then $f$ is *potential function* and its value $f(x, y, z)$ is *potential* in $(x, y, z)$.

EXERCISE 1.5. Find a vector field with potential $f(x, y, z) = \sin(x^2 + y^2 + z^2)$.

THEOREM 1.6. *Every inverse square vector filed is conservative.*

PROOF. The potential is given by $f(r) = \frac{c}{r}$. □

DEFINITION 1.7. Let $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i}+N(x,y,z)\mathbf{j}+P(x,y,z)\mathbf{k}$. The *curl* of $\mathbf{F}$ is given by

$$\begin{aligned} \text{curl}\mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \end{aligned}$$

DEFINITION 1.8. Let $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i}+N(x,y,z)\mathbf{j}+P(x,y,z)\mathbf{k}$. The *divergence* of $\mathbf{F}$ is given by

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

EXERCISE 1.9. Find $\text{curl}\,\mathbf{F}$ and $\text{div}\,\mathbf{F}$ for

$$\mathbf{F}(x,y,z) = (3x+y)\mathbf{i} + xy^2z\mathbf{j} + xz^2\mathbf{k}.$$

EXERCISE 1.10. Prove that for a constant vector $\mathbf{a}$

(i) $\text{curl}\,(\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$;
(ii) $\div(\mathbf{a} \times \mathbf{r}) = 0$.

EXERCISE 1.11. Verify the identities:

$$\begin{aligned} \text{curl}\,(\mathbf{F}+\mathbf{G}) &= \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}; \\ \text{div}\,(\mathbf{F}+\mathbf{G}) &= \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}; \\ \text{curl}\,(f\mathbf{F}) &= f(\text{curl}\,\mathbf{F}) + (\nabla f) \times \mathbf{F}; \end{aligned}$$

## 2. Line Integral

We could introduce a new type of integrals for functions of several variables.

DEFINITION 2.1. The *line integrals along a curve* $C$ with respect to $s, x, y$, respectively are

$$\begin{aligned} \int_C f(x,y)\,ds &= \lim_{\|P\|\to 0} \sum_k f(u_k,u_k)\Delta s_k \\ \int_C f(x,y)\,dx &= \lim_{\|P\|\to 0} \sum_k f(u_k,u_k)\Delta x_k \\ \int_C f(x,y)\,dy &= \lim_{\|P\|\to 0} \sum_k f(u_k,u_k)\Delta y_k \end{aligned}$$

Let a curve $C$ be given parametrically by $x = g(t)$ and $y = h(t)$. Because

$$dx = g'(t)\,dt, \qquad dy = h'(t)\,dt,$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(g'(t))^2 + (h'(t))^2}\,dt.$$

we obtain

THEOREM 2.2 (Evaluation formula for line integrals). *If a smooth curve* C *is given by* $x = g(t)$ *and* $y = h(t)$; $a \leqslant t \leqslant b$ *and* $f(x, y)$ *is continuous in a region containing* C, *then*

$$\begin{aligned} \int_C f(x,y)\,ds &= \int_C f(g(t),h(t))\sqrt{(g'(t))^2+(h'(t))^2}\,dt \\ \int_C f(x,y)\,dx &= \int_C f(g(t),h(t))(g'(t)\,dt \\ \int_C f(x,y)\,dy &= \int_C f(g(t),h(t))h'(t))\,dt \end{aligned}$$

EXERCISE 2.3. Evaluate $\int_C xy^2\,ds$ if C is given by $x = \cos t$, $y = \sin t$; $0 \leqslant t \leqslant \pi/2$.

EXERCISE 2.4. Evaluate $\int_C y\,dy + z\,dy + x\,dz$ if C is the graph of $x = \sin t$, $y = 2\sin t$, $z = \sin^2 t$; $0 \leqslant t \leqslant \pi/2$.

EXERCISE 2.5. Evaluate $\int_C xy\,dx + x^2y^3\,dy$ if C is the graph of $x = y^3$ from $(0,0)$ to $(1,1)$.

EXERCISE 2.6. Evaluate $\int_C (x^2+y^2)\,dx + 2x\,dy$ along three different paths from $(1,2)$ to $(-2,8)$.

EXERCISE 2.7. Evaluate $\int_C (xy+z)\,ds$ if C is the lime segment from $(0,0,0)$ to $(1,2,3)$.

THEOREM 2.8. *The* mass of a wire *is given by*

$$m = \int_C \delta(x,y)\,ds,$$

*where* $\delta(x,y)$ *is the* linear mass density.

THEOREM 2.9. *The* work $W$ done by a force F long a path C *is defined as follows*

$$W = \int_C M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz.$$

*If* $\mathbf{T}$ *is a unit tangent vector to* C *at* $(x,y,z)$ *and* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, *then*

$$W = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\,d\mathbf{r}.$$

## 3. Independence of Path

There is a condition for an integral be independent from the path.

THEOREM 3.1. *If* $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{i}$ *is continuous on an open connected region* D, *then the line integral* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *is independent of path if and only if* $\mathbf{F}$ *is conservative—that is,* $\mathbf{F}(x,y) = \nabla f(x,y)$ *for some scalar function* f.

EXERCISE 3.2. Show that $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path by finding a potential function f for F:

$\mathbf{F}(x,y) = (6xy^2+3y)\mathbf{i}+(6x^2y+2x)\mathbf{j}$; $\quad \mathbf{F}(x,y) = (2xe^{2y}+4y^3)\mathbf{i}+(2x^2e^{2y}+12xy^2)\mathbf{j}$.

In fact we are even able to give a formula for the evaluation:

THEOREM 3.3. *Let* $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{i}$ *be continuous on an open connected region* D, *and* C *be a piecewise-smooth curve in* D *with endpoints* $A(x_1,y_1)$ *and* $B(x_2,y_2)$. *If* $\mathbf{F}(x,y) = \nabla f(x,y)$ *for some scalar function* f, *then*

$$\int_C M(x,y)\,dx + N(x,y)\,dy = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathbf{F}\cdot d\mathbf{r} = [f(x,y)]_{(x_1,y_1)}^{(x_2,y_2)}.$$

Particularly $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ *for every simple closed curve* C.

EXERCISE 3.4. Show that integral is independent of path, and find its value

$$\int_{(0,0)}^{(1,\pi/2)} e^x \sin y\,dx + e^x \cos y\,dy.$$

THEOREM 3.5. *If* F *is a conservative force field in two dimensions, then the work done by* F *along any path* C *from* $A(x_1,y_1)$ *to* $B(x_2,y_2)$ *is equal to the difference in potentials between* A *and* B.

THEOREM 3.6. *If* $M(x,y)$ *and* $N(x,y)$ *have continuous first partial derivatives on a simply connected region* D, *then the line integral*

$$\int_C M(x,y)\,dx + N(x,y)\,dy$$

*is independent of path in* D *if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

EXERCISE 3.7. Use above theorem to show that $\int_C \mathbf{F}\cdot d\mathbf{r}$ is not independent of path:

(i) $\mathbf{F}(x,y) = y^3\cos x\mathbf{i} - 3y^2\sin x\mathbf{j}$.
(ii) $\int_C e^y \cos x\,dx + xe^y\cos z\,dy + xe^y \sin z\,dz$.

## 4. Green's Theorem

THEOREM 4.1 (Green's Theorem). *Let* G *be a piecewise-smooth simple closed curve, and let* R *be the region consisting of* G *and its interior. If* M *and* N *are continuous functions that have continuous first partial derivatives throughout an open region* D *containing* R, *then*

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,da$$

EXERCISE 4.2. Use Green's theorem to evaluate the line integrals

(i) $\oint \sqrt{y}\, dx + \sqrt{x}\, dy$ if C is the tringle with vertices $(1, 1)$, $(3, 1)$, $(2, 2)$.

(ii) $\oint_C y^2\, dx + x^2\, dy$ if C is the boundary of the region bounded by the semicircle $y = \sqrt{4 - x^2}$ and $x$-axis.

As an application we could derive a formula as follows:

THEOREM 4.3. *If a region* $R$ *in the* $xy$*-plane is bounded by a piece-wise-smooth simple closed curve* $C$*, then the area* $A$ *of* $R$ *is*

$$A = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2}\oint_C x\, dy - y\, dx.$$

The region $R$ could contains holes, provided we integrate over the *entire* boundary and always keep the region $R$ to the left of $C$.

EXERCISE 4.4. Use the above theorem to find to fine the area bounded by the graphs $y = x^3$, $y^2 = x$.

THEOREM 4.5 (Vector Form of Green's Theorem).

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$$

## 5. Surface Integral

We could define surface integrals in a way similar to definite integral, double, triple, lines integrals by means of Riemann sums:

$$\iint_S g(x, y, z)\, dS = \lim_{\|P\| \to 0} \sum_k g(x_k, y_k, z_k)\Delta T_k.$$

To calculate surface integrals we use

THEOREM 5.1. *Evaluation formulas for surface integrals are:*

$$\iint_S g(x,y,z)\, dS = \iint_{R_{xy}} g(x, y, f(x, y))\sqrt{[f'_x(x,y)]^2 + [f'_y(x,y)]^2 + 1}\, dA$$

$$\iint_S g(x,y,z)\, dS = \iint_{R_{xz}} g(x, h(x,z)z)\sqrt{[h'_x(x,z)]^2 + [h'_z(x,z)]^2 + 1}\, dA$$

$$\iint_S g(x,y,z)\, dS = \iint_{R_{xy}} g(k(y,z), y, z)\sqrt{[k'_y(y,z)]^2 + [k'_z(y,z)]^2 + 1}\, dA$$

EXERCISE 5.2. Evaluate surface integral of $g(x, y, z) = x^2 + y^2 + z^2$ over the part of plane $z = y + 4$ that is inside the cylinder $x^2 + y^2 = 4$.

EXERCISE 5.3. Express the surface integral $\iint_S (xz + 2y)\, dS$ over the portion of the graph of $y = x^3$ between the plane $y = 0$, $y = 8$, $z = 2$, and $z = 0$ as a double integral over a region in $yz$-plane.

DEFINITION 5.4. The *flux* of vector field $F$ through (or over) a surface $S$ is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dx.$$

EXERCISE 5.5. Find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dx$ for $\mathbf{F} = x\mathbf{i}-y\mathbf{j}$ and $S$ the first octant portion of the sphere $X^2 + y^2 + z^2 = a^2$.

EXERCISE 5.6. Find the flux of $\mathbf{F}(x, y, z) = (x^2 + z)\mathbf{i} + y^2 z\mathbf{j} + (x^2 + y^2 + z)\mathbf{k}$ over $S$ is the first-octant portion of paraboloid $z = x^2 + y^2$ that is cut off by the plane $z = 4$.

## 6. Divergence Theorem

## 7. Stoke's Theorem

# Bibliography

[1] Earl Swokowski, Michael Olinick, and Dennis Pence. *Calculus*. PWS Publishing, Boston, 6-th edition, 1994.

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