Geometry

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CHAPTER 1

General Information

This is an online manual designed for students. The manual is available at the moment in HTML with frames (for easier navigation), HTML without frames and PDF formats. Each from these formats has its own advantages. Please select one better suit your needs.

There is on-line information on the following courses:

- Calculus I.
- Calculus II.
- Geometry.

1. Course description and Schedule

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2. Web page

There is a Web page which contains this course description as well as other information related to this course. Point your Web browser to

http://v-v-kisil.scienceontheweb.net/courses/math255f.html

3. Warnings and Disclaimers

Before proceeding with this interactive manual we stress the following:

- These Web pages are designed in order to help students as a source of *additional information*. They are **NOT** an obligatory part of the course.
- The main material introduced during *lectures* and is contained in *Textbook*. This interactive manual is **NOT** a substitution for any part of those primary sources of information.
- It is **NOT** required to be familiar with these pages in order to pass the examination.
- The entire contents of these pages is continuously improved and updated. Even for material of lectures took place weeks or months ago changes are made.

Contents

CHAPTER 0

Seven Top Reasons to Enjoy Geometry

There are many reasons to enjoy Geometry. No, "I need to pass the exam" is not among them. These reasons are of a much pleasant nature:

- Geometry is *elementary*. To understand even most advanced results you need only to know simple notions like lines, circles, triangles, etc.
- Geometry is *beautiful*. The inner harmony of geometrical constructions is explicit.
- Geometry is *real*. It describes the property of the world around us.
- Geometry is *principal*. It was the first field of mathematical knowledge which set up a model for all other branches of mathematics.
- Geometry is *reach*. You may meet all variety of mathematical tools employed in geometry.
- Geometry is *modern*. Geometric results are used in all contemporary fields of mathematics and are source of inspiration for many new theories.
- Geometry is *surprising*. One is greatly impressed by the unexpected deep and beauty of geometrical results.

EXERCISE 0.1. Illustrate each of the above statements by at least one geometrical construction or theorem.

EXERCISE* 0.2. Give at least one more good reason to enjoy geometry.

CHAPTER 1

Points and Lines Connected with a Triangle

1. The extended Law of Sines

THEOREM 1.1 (Law of Sines). *For a triangle* ABC *with circumradius* R

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

PROOF. The proof is based on the Theorem 0.1 from the Useful Theorem Chapter. □

EXERCISE 1.2. For any triangle ABC, even if B and C is an obtuse angle, $a = b \cos C + c \cos B$. Use the Law of Sines to deduce the additional formula

$$\sin(B + C) = \sin B \cos C + \cos B \sin C.$$

EXERCISE 1.3. In any triangle ABC,

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0.$$

EXERCISE 1.4. In any triangle ABC, $(ABC) = abc/4R$[1].

2. Ceva's theorem

The line segment joining a vertex of a triangle to any given point on the opposite side is called a *cevian*.

THEOREM 2.1 (Ceva's theorem (1678)). *If three cevians* AX, BY, CZ, *one through each vertex of a triangle* ABC, *are concurrent, then*

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = 1$$

PROOF. Three cevians are concurrent so they pass through one point, say P. The proof follows from a consideration of areas of triangles ABP, BPC, CPA. The key point is Lemma on the area of two triangle with a common altitude. □

EXERCISE 2.2. If X, Y, Z are midpoints of the sides, the three cevians are concurrent.

EXERCISE 2.3. Let $XB/XC = p$, $YC/YA = q$, and AX, BY, CZ are concurrent. Find AZ/ZB.

[1] We alway denote *area* of a figure by its name enclosed in parentheses.

EXERCISE 2.4. Cevians perpendicular to the opposite sides are concurrent.

EXERCISE 2.5. Let ABC and $A'B'C'$ be two non-congruent triangles whose sides are respectively parallel. Then the three lines AA', BB', and CC' (extended) are concurrent.

3. Points of intersect

The most important points and lines of intersect in a triangle are:

(i) *Circumcenter*—the center of the circle circumscribed about a triangle, denoted by O. The circle is called *circumcircle*. Its radius (*circumradius*) denoted by R.

(ii) The cevians that joint the vertices of a triangle to the midpoints of the opposite sides are called *medians*. They are concurrent (see Exercise 2.2) and the common point is *centroid* G, which is "center of gravity" of the triangle.

THEOREM 3.1. *A triangle is dissected by its medians into six smaller triangles of equal area.*

(iii) The medians of a triangle divide one another in the ratio $2:1$; in other words, the medians of a triangle "trisect" one another.

(iv) Cevians perpendicular to corresponding lines are *altitudes*. It follows from Exercise 2.4 that altitudes are concurrent, they intersect in *orthocenter* H. *Feets* of altitudes form the *orthic triangle*.

(v) *Bisectors* are cevians which divide angles to two equal parts.

THEOREM 3.2. *Each angle bisector of a triangle divides the opposite side into segments proportional in length to the adjacent sides.*

PROOF. There at least two ways to make a proof:

(a) Applying Law of Sines to two resulting triangles.

(b) Considering the ratio of areas of those two triangles.

□

THEOREM 3.3. *The internal bisectors of the three angles of a triangle are concurrent.*

PROOF. The proof follows from observation that points of bisectors are equidistant from the sides of the triangle. The point of concurrence I is *incenter*, that is the center of inscribed circle, which has all three sides for tangents. Its radius is *inradius*. □

EXERCISE 3.4. The circumcenter and orthocenter of an obtuse-angled triangle lie outside the triangle.

EXERCISE 3.5. Find the ratio of the area of a given triangle to that of triangle whose sides have the same lengths as medians of the original triangle.

EXERCISE 3.6. Any triangle having two equal medians is isosceles.

EXERCISE 3.7. Any triangle having two equal altitudes is isosceles.

EXERCISE 3.8. Use Cevas Theorem to obtain another proof of Theorem 3.3.

EXERCISE 3.9. The product of two sides of a triangle is equal to the product of the circumdiameter and the altitude on the third side.

4. The incircle and excircles

Let the incircle touch sides BC, CA, AB at X, Y, Z. If $x = AZ = AY$, $y = BX = BZ$, and $z = CX = CY$. Let $s = \frac{1}{2}(x + y + z)$ be *semiperimeter*.

THEOREM 4.1. $x = s - a$, $y = s - b$, $z = s - c$.

PROOF. It follows from the Theorem on two tangents. □

THEOREM 4.2. $(ABC) = sr$.

PROOF. It follows from the Theorem on areas. □

THEOREM 4.3. *The external bisectors of any two angles of a triangle are concurrent with the internal bisector of the third angle.*

The circles with with above centers (*excenters*) are *excircles* escribed to the triangle. Their radii (*exradii*) denoted r_a, r_b, r_c. Incircle together with excircles are four *tritangent* circles.

EXERCISE 4.4. The cevians AX, BY, CZ are concurrent. Their common point is the *Gergonne point*.

EXERCISE 4.5. ABC is the orthic triangle of the triangle formed by excenters.

EXERCISE 4.6. $(ABC) = (s - a)r_a = (s - b)r_b = (s - c)r_c$

EXERCISE 4.7.

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}.$$

6. The orthic triangle

THEOREM 6.1. *The orthocenter* H *of an acute-angled triangle is the incenter of its orthic triangle.*

PROOF. Let AD, CF, and BE are altitudes of ABC, O be the circumcenter and $\alpha = 90^\circ - A$. Then the following angles are α: OBC, OCB, ABE, ACF. Because $CDFA$ is inscribe to a circle then $FDA = FCA = \alpha$ and by the same reason $ADE = ABE = \alpha$. Thus DA bisect FDE. □

We also see that $OB \perp FD$, $OC \perp DE$, $OA \perp FE$.

EXERCISE 6.2. $AEF \sim DBF \sim DEC \sim ABC$.

EXERCISE 6.3. $HAO = |B - C|$.

7. The medial triangle and Euler line

The triangle $A'B'C'$ formed by joining the midpoints A', B', C' of the sides of a given triangle ABC will be called the *medial* triangle. We have the following set of conclusions:

(i) The medial triangle $A'B'C'$ is similar to the given ABC with the ration $1:2$. In fact we have four equal triangles!
(ii) ABC and $A'B'C'$ have the same centroid G.
(iii) The circumcenter O of ABC is the orthocenter of $A'B'C'$.
(iv) ABC and $A'B'C'$ are homothetic with a center G and ratio $2:1$.

As a consequence we obtain that

THEOREM 7.1. *The orthocenter, centroid and circumcenter of any triangle are collinear. The centroid divides the distance from the orthocenter to the circumcenter in the ratio* $2:1$.

The line on which these three points lie is called the *Euler line* of the triangle.

THEOREM 7.2. *The circumcenter of the medial triangle lies at the midpoint of segment* HO *of the Euler line of the parent triangle. The circumradius of the medial triangle equals half the circumradius of the parent triangle.*

EXERCISE 7.3. $OH^2 = 9R^2 - a^2 - b^2 - c^2$.

EXERCISE 7.4. $DA' = |b^2 - c^2|/2a$.

EXERCISE 7.5. If ABC has the special property that its Euler line is parallel to its side BC, then $\tan B \tan C = 3$.

CHAPTER 2

Some properties of Circles

1. The power of point with respect to a circle

THEOREM 1.1. *If two lines through a point* P *meet a circle at points* A, A' *(possibly coincident) and* B, B' *(possibly coincident), respectively, then* $PA \times PA' = PB \times PB'$.

PROOF. The proof follows from the similarity of triangles PAB' and PBA' in both cases if P inside or outside of the circle. Notably in the second case $PA \times PA' = PT^2$ where PT is tangent to circle and T belong to it. □

Let R is the radius of the circle and d is the distance to its center. If P is inside then $PA \times PA' = R^2 - d^2$ and if it is outside then $PA \times PA' = d^2 - R^2$.

THEOREM 1.2. *Let* O *and* I *be the circumcenter and incenter, respectively, of a triangle with circumradius* R *and* r*; let* d *be the distance* OI. *Then*

$$d^2 = R^2 - 2rR.$$

PROOF. Let bisector AL meet circumcircle at L, ML be diameter of circumcircle, then $LM \perp BC$. BLI is isosceles, thus $BL = IL$. Then

$$\begin{aligned} R^2 - d^2 &= LI \times IA = BL \times IA \\ &= LM\frac{LB/LM}{IY/IA}IY = LM\frac{\sin A/2}{\sin A/2}IY \\ &= LM \times IY = 2rR. \end{aligned}$$

This is an example of *synthetic proof*, compare with proof of Radial Axis Theorem. □

If we adopt the Newton convention:

$$PA = -AP \tag{1.1}$$

then identity

$$d^2 - R^2 = PA \times PA'$$

became universally true for any secant or chord. Its value is *power of* P *with respect to the circle*.

EXERCISE 1.3. What is smallest possible value of the power of a point with respect to a circle of radius R? Which point has this critical power?

EXERCISE 1.4. What is the locus of points of constant power?

EXERCISE 1.5. If PT and PU are tangents from P to two concentric circles, with T on the smaller, and if the segment PT meets the larger circle at Q, then $PT^2 - PU^2 = QT^2$.

2. The radical axis of two circles

THEOREM 2.1. *The locus of all points whose powers with respect to two nonconcentric circles are equal is a line perpendicular to the line of centers of the two circles.*

PROOF. The proof could be done by means of *analytic geometry*, or *analytic proof*. Namely we express the problem by means of equations and solve them afterwards. The key ingredient is the equation of circle known from the **Calculus I** course. □

The locus of points of equal power with respect to two non-concentric circles is called their *radical axis*.

EXERCISE 2.2. Give a simple indication of radical axis when two circles intersect or are tangent.

EXERCISE 2.3. Let PAB, AQB, ABR, $P'AB$, $AQ'B$, ABR' be six *similar* triangles all on the same side of their common side AB. Then points P, Q, R, P', Q', R' all lie on one circle.

3. Coaxal circles

A *pencil of coaxial circle* is infinite family of circles, represented by the equation

$$x^2 + y^2 - 2ax + c = 0$$

for a fixed c and

(i) a any if $c < 0$.
(ii) $a \in [-\sqrt{c}, \sqrt{c}]$ if $c > 0$.

Any two circles from the pencil have the same radical axis—the y-axis.

THEOREM 3.1. *If the centers of three circles form a triangle, there is just one point whose powers with respect to the three circles are equal. Its name is* radical center.

EXERCISE 3.2. Two circles are in contact internally at a point T. Let the chord AB of the largest circle be tangent to the smaller circle at point P. Then the line TP bisect ATB.

5. Simpson line

THEOREM 5.1. *The feet of the perpendiculars from a point to the sides of a triangle are collinear iff the point lies on the circumcircle.*

PROOF. Let A', B', C' be feets of perpediculars from point P on the circumcircle. Observations:

(i) Quadragles $PB'A'C$ and $C'PB'A$ are inscribed to a circles.
(ii) From the above: $A'PC = A'B'C$ and $AB'C' = APC'$.
(iii) $C'PA' = APC$;
(iv) From the above $C'PA = A'PC$.
(v) From 5.1(ii) and 5.1(iv): $AB'C' = CB'A'$, thus A', B', and C' are collinear.

□

EXERCISE 5.2. What point on the circumcirle has CA as its Simpson line?

EXERCISE 5.3. The tangent at two points B and C on a circle meet at A. Let $A_1B_1C_1$ be the pedal triangle of the isosceles triangle ABC for an arbitrary point P on the circle. Then $PA_1^2 = PB_1 \times PC_1$.

6. Ptolemy's theorem and its extension

The orthic triangle and medial triangle are two instances of a more general type of associated triangle. Let P be any point inside a given tringle ABC, and PA_1, PB_1, PC_1 are three perpendicular to its sides. Then $A_1B_1C_1$ is *pedal triangle* for *pedal point* P.

It is easy to see that AB_1C_1 inscribed to a circle with the diameter AP. Then from the Theorem of sines follows that:

THEOREM 6.1. *If the pedal point is distant x, y, z from the vertices of ABC, the pedal triangle has sides*

$$\frac{ax}{2R}, \qquad \frac{by}{2R}, \qquad \frac{cz}{2R}. \tag{6.1}$$

The Simpson line is degenerate case of the pedal triangle, nevertheless the above formulas are true and moreover $A_1B_1 + B_1C_1 = A_1C_1$ we deduce $c\,CP + a\,AP = b\,BP$, thus

$$AB \times CP + BC \times AP = AC \times BP.$$

THEOREM 6.2. *If a quadralaterial is inscribed in a circle, the sum of the product of the two pairs of opposite sides is equal to the product of the diagonals.*

The inverse theorem could be modified accordingly to the triangle inequality $A_1B_1 + B_1C_1 > A_1C_1$.

THEOREM 6.3. *If* ABC *is a triangle and* P *is not on the arc* CA *of the circumcircle, then*

$$AB \times CP + BC \times AP > AC \times BP.$$

EXERCISE 6.4. If a point P lies on the arc CD of the circumcircle of a square $ABCD$, then $PA(PA + PC) = PB(PB + PD)$.

CHAPTER 3

Collinearity and Concurrence

1. Quadrangles; Varignon's theorem

A *polygon* is a cyclically ordered set of points in a plane, with no three successive points collinear, together with the line segments joining consecutive pairs of the points. First few names are *triangle*, *quadrangle*, *pentagon*, *hexagon*, and so on.

Two sides of a quadrangle are said to be *adjacent* or *opposite* according as they do or do not have a vertex in common. The lines joining pairs of opposite vertices are called *diagonals*.

There three different types of the quadrangles:

(i) *convex*—both diagonals are inside;
(ii) *re-entrant*—one diagonal is in, another is out;
(iii) *crossed*—both diagonals are outside.

We agree to count the area of triangle *positive* or *negative* if its vertices are named in counterclockwise or clockwise order. For example

$$(ABC) = -(BAC). \tag{1.1}$$

For all convex and re-entrant quadrangles *area* is:

$$(ABCD) = (ABC) + (CDA).$$

REMARK 1.1. Combined the idea of signed area with directed segments could extend the proof of the Ceva's theorem to the case, then points divides sides externally.

THEOREM 1.2 (Varignon 1731). *The figure formed when the midpoints of the sides of a quadrangle are joined in order is a parallelogram, and its area is half that of the quadrangle.*

THEOREM 1.3. *The segments joining the midpoints of pairs of the opposite sides of the a quadrangle and segment joining the midpoints of the diagonals are concurrent and bisect one another.*

THEOREM 1.4. *If one diagonal divides a quadrangle into two triangles of equal area, it bisect the other diagonal. Conversely, if one diagonal bisect the other, it bisect the area of the quadrangle.*

THEOREM 1.5. *If a quadrangle* $ABCD$ *has its opposite sides* AD *and* BC *(extended) meeting at* W*, while* X *and* Y *are the midpoints of the diagonals* AC *and* BD*, then* $(WXY) = 1/4(ABCD)$.

2. Cyclic quadrangles; Brahmagupta's formula

THEOREM 2.1 (Brahmagupta). *If a cyclic quadrangle has sides* a, b, c, d *and semiperimeter* s, *its* area K *is given by*

$$K^2 = (s-a)(s-b)(s-c)(s-d).$$

COROLLARY 2.2 (Heron). Area *of a triangle is given by*

$$(ABC)^2 = s(s-a)(s-b)(s-c).$$

4. Menelaus's theorem

THEOREM 4.1 (Menelaus). *If points* X, Y, Z *on sides* BC, CA, AB *(suitable extended) of* ΔABC *are collinear, then*

$$\frac{BX}{CX}\frac{CY}{AY}\frac{AZ}{BZ} = 1.$$

Conversely, if this equation holds for points X, Y, Z *on the three sides, then these three points are collinear.*

For the directed segments it could be rewritten as follows:

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = -1.$$

5. Pappus's theorem

The following theorem is the first belonging to *projective geometry*. It is formulated entirely in terms of collinearity.

THEOREM 5.1 (Pappus, 300 A.D.). *If* A, C, E *are three points on one line,* B, D, F *on another, and if the three lines* AB, CD, EF *meet* DE, FA, BC, *respectively, then three points of intersection* L, M, N *are collinear.*

PROOF. Let lines AB, CD, EF form triangle UVW. Apply the Menelaus's Theorem to the five triads of points

$$LDE, \quad AMF, \quad BCN, \quad ACE, \quad BDF$$

on the sides of this triangle UVW. Then the product of first three identities divided by the last two ones gives

$$\frac{VL}{LM}\frac{WM}{MU}\frac{UN}{NV} = -1.$$

Thus by Menelaus's Theorem L, M, N are collinear. □

6. Perspective triangles; Desargues's theorem

If two specimens of a figure, composed of points and lines, can be put into correspondence in such a way that pairs of corresponding points are joined by concurrent lines, we say that two specimens are *perspective from a point*. If the correspondence is such that pairs of corresponding lines meet at collinear points, we say that two specimens are *perspective from a line*.

THEOREM 6.1 (Desargues, 1650). *If two triangles are perspective from a point then they are perspective from a line.*

In other words, If two triangles are perspective from a point, and if their pairs of corresponding sides meet, then three points of intersection are collinear.

PROOF. Let PQR and $P'Q'R'$ are the triangles perspective from point O, and let $D = RQ \cdot R'Q'$, $E = PR \cdot P'R'$, $F = PQ \cdot P'Q'$. Apply Menelaus's Theorem to triads $DR'Q'$, $EP'R'$, $FQ'P'$ and triangles OQR, ORP, OPQ. □

The converse theorem is also true.

THEOREM 6.2. *If two triangles are perspective from a line, they are perspective from a point.*

If two triangles are perspective from a line, and if two pairs of corresponding vertices are joined by intersecting lines, the triangles are perspective from the point of intersection of these lines.

7. Hexagons

Two vertices of a hexagon are said to be *adjacent*, *alternate*, *opposite* according as they are separated by one sides, two sides, or three sides. The join of two opposite vertices is called a *diagonal*.

EXERCISE 7.1. Count the number of ways a given hexagon could be labelled as $ABCDEF$ (*Answer:* 12).

EXERCISE 7.2. Count number of different hexagons defined by given 6 point, no three collinear. (*Answer:* 60).

In term of hexagon we could reformulate Pappus's Theorem as follows:

If each set of three alternate vertices of a hexagon is a set of three collinear points, and the three pairs of opposite sides intersect, then the three points of intersection are collinear.

8. Pascal's theorem

THEOREM 8.1 (Pascal's Theorem). *If all six vertices of a hexagon lie on a circle and the three pairs of opposite sides intersect, then the three points of intersection are collinear.*

PROOF. The proof consists of application Menelaus's Theorem four times. □

This theorem of a projective nature and hexagon could be in fact inscribed in any conic. Under such a formulation it has an inverse:

THEOREM 8.2. *If the three pairs of opposite sides of a hexagon meet at three collinear points, then the six vertices lie on a conic.*

Some degenerated cases of the Pascal's Theorem are of interest

COROLLARY 8.3. *Let* $ABDE$ *be a cyclic crossed quadrangle. Tangents to the circle in points* B *and* E *meet in a point* N *which is collinear with points* $L = AB \cdot DE$ *and* $M = BD \cdot EA$.

CHAPTER 4

Transformations

The *groups of transformations* are very important in geometry. In fact they are could characterize different geometries as was stated by Felix Klein in his famous Erlangen program. We will consider most fundamental groups of *transformations*.

For Eucleadean geometry the important transformations are *isometries*. There are several of them: translations, rotations (particularly half-turn), reflections.

1. Translations

We refer for properties of *translations* or vectors in the Calculus course. As geometrical application of vectors we could consider the deducing formula of parallelogramm area. Another example is

EXERCISE 1.1. Inscribe in a given circle a rectangle with two opposite sides equal and parallel to a given line segment a.

The *characteristic property of translation* among isometries is: *each ray come to a parllel ray* (prove it!).

2. Rotations

Other important isometries are *rotations* around a point O by an angle α.

The *characteristic property of rotations* among isometries is: *each ray come to ray rotated by the* α.

3. Half-turn

The *half-turn* is rotation by the angle $180°$ and is completely defined by its center. The *characteristic property of half-turn* among isometries is: *each ray come to the opposite ray*. Thus

THEOREM 3.1. *Composition of two half-turn is a traslation by the vector* $2\overrightarrow{O_1O_2}$.

Using half-turns we could easily prove that *if digonals of a quadrangle bisect each other then it is a parallelogramm*.

4. Reflections

The third type of isometries is *reflections* in a *mirror*. It interesting that they give a geometrical solution for the following extremal problem: find the shortes path (which is physically the path of a light ray) between two points via a point of the mirror.

7. Dilation

Isometries transfor a figure into a *congruent* figure. Another important class is transformations which change each figure to into a *similar* figure, i.e. all distances increased in the same ratio, *ration of magnification*.

EXERCISE* 7.1. Prove that suchtransformations preserve collinearity and angles.

A simplest kind is *dilation*, which *transforms each line into a parallel line*. If a dilation is not a translation then its *central dilation*. Translations and half-turn are partucular cases of dilations with ratio 1 and -1 correspondingly.

8. Spiral symmetry

It is possible to see that the composition of a translation and a dilation or composition of two tarnslations are again a dilation (sinse parallel lines come to parallel lines). But composition of a dilation and rotation around the same point is something different—*spiral similarity*, which is a kind of *direct similarity* (preserves angles in magnitude and sign). They are completely determined by their center O, ratio k, and angle θ, we will denote it by $O(k, \theta)$.

THEOREM 8.1. *If squares, with centers* O_1, O_2, O_3, *are erected externally on the sides of* ΔABC, *then line segments* O_1O_2 *and* CO_3. *are equal and perpendicular.*

PROOF. It is follows from consideration of $A(\sqrt{2}, 45°)$ and $C(\sqrt{2}, -45°)$. □

It is interesting that there are no other direct similarities besides spiral ones:

THEOREM 8.2. *Any two directly similar figures are related either by a translation or by a spiral similarity.*

COROLLARY 8.3. *If* ABC *and* $A'B'C'$ *are two directly similar triangles, while* $AA'A''$, $BB'B''$, $CC'C''$ *are three directly similar triangles, then* $\Delta A''B''C''$ *and* ΔABC *are directly similar.*

9. A genealogy of transformations

We could put the following transformation in a genealogical tree:

Transformation			
Continuous transormation			
Linear transformation			
Similarity			Procrustean stretch
Isometry	Dilation		Spiral similarity
Reflection	Translation	Rotation	Central dilation
Half-turn			

CHAPTER 5

An Introduction to Inversive Geometry

1. Separation

THEOREM 1.1. *If four points* A, B, C, D *do not all lie on the circle or line, there exist two non-intercecting circles, one through* A *and* C*, the other through* B *and* D.

Two distinct point pairs, AC and BD are said to *sepatrate* each other if A, B, C, D lie on a circle (or a line) in such an order that either of the arcs AC contains one but not both of the remaining points B and D. It is denoted by AC//BD. Another characterizations are

THEOREM 1.2. *Two distinct point pairs,* AC *and* BD *are said to* sepatrate *each other if every circle through* A *and* C *intersects (or coinsides with) every circle through* B *and* D.

Alternatively

THEOREM 1.3. *The mutual distances of four distinct points* A, B, C, D *satisfy*

$$AB \times CD + BC \times AD \geqslant AC \times BD,$$

with the equals sign only then AC//BD.

PROOF. It is follows directly from consideration of directed line segments if the points are collinear and is a consequence of the Ptolemy's theorem if points lie on a circle or are not collinear. □

2. Cross Ratio

We introduce *cross ratio* as follows

$$\{AB, CD\} = \frac{AC \times BD}{AD \times BC}.$$

Then we obtainfrom the Separation Theorem

THEOREM 2.1. *The cross ratios of four distinct points* A, B, C, D *satisfy*

$$\{AD, BC\} + \{AB, DC\} = 1$$

iff AC//BD.

Now instead of defining separation in the term of circles we could define circles in the term of separation:

DEFINITION 2.2. The circle determined by three points A, B, C is set of points consisiting of the three points themselves along with all the points X such that

$$BC//AX \text{ or } CA//BX \text{ or } AB//CX.$$

3. Inversion

For a given circle ω with the center O and radius k we define a point $P' = i(P)$ being *inverse* to P if $P' \in OP$ and

$$OP \times OP' = k^2.$$

It is obvious from this conditions that $P = i(i(P))$ for any point P (different from O). The inverse for O is not defined. There is a simple geometrical constraction.

THEOREM 3.1. *The inverse of any line a, not through 0, is a circle through O, and the diametre through O of the circle is perpendicular to a.*

The inverse of any circle through O is a perpendicular to the diametr through O.

We could construct inverse points using *Peaucellier's cell*.

Considering images under inversion of three points we could observe

THEOREM 3.2. *For a suitable circel inversion, any three distinct points A, B, C can be inverted into the vertices of a triangle $A'B'C'$ congruent to a given triangle.*

4. The inversive plane

THEOREM 4.1. *If a circle with center O and radius k invert point pair AB into $A'B'$, the distance are related by the equation*

$$A'B' = \frac{k^2 AB}{OA \times OB}.$$

THEOREM 4.2. *If A, B, C, D invert into A', B', C', D', then*

$$\{A'B', C'D'\} = \{AB, CD\}.$$

THEOREM 4.3. *If A, B, C, D invert into A', B', C', D' and $Ac//BD$ then $A'C'//B'D'$.*

If we will think on lines as circles with infinite radius then we could state

THEOREM 4.4. *The inverse of any circle is a circle.*

To define inversion for all points we may add to a plane a one special point: *point at infinity* p_∞. Then inverse of O is p_∞ and vise verse. A plane together with $p_\infty q$ form the *inversive plane*.

CHAPTER 6

An Introduction to Projective Geometry

1. Reciprocation

Let ω be a circle with center O and radius k. Each point P (different from O) determine a corresponding line p, called the *polar* of P; it is the line perpendicular to OP through the inverse of P. Conversely, each line p determine a point P, the *pole* of p; it is the inverse of the foot of the perpendicular from O to p.

THEOREM 1.1. *If* B *lies on* a, *then* b *passes through* A.

We say that A and B are *conjugate points*; a and b are *conjugate line*. Any point on a tangent a is conjugate to the point of contact A, which is *self-conjugate point*, and any line through A (on ω) is conjugate to the tangent a, which is a *self-conjugate line*.

Reciprocation allows us to introduce a vocabulary for projective duality

point	line
lie on	pass through
line joining two points	intersection of two lines
concurrent	collinear
quadrangle	quadrilateral
pole	polar
locus	envelope
tangent	point of contact

THEOREM 1.2. *The pole of any secant* AB *(except a diameter) is the common point of the tangents at* A *and* B. *The polar of any exterior point is the line joining the points of contact of two tangents from this point. The pole of any line* p *(except a diameter) is the common point of the polars of two exterior points on* p. *The polar of any point* P *(except the center) is the line joining the poles of two secants through* P.

3. Conics

We meet already conics (or conic sections) in the course of Calculus I. Their they was defined by means of equations in Cartesian coordinates or as sections of cones. Now we could give a projective definition. Let ω be a circle with center O.

DEFINITION 3.1. A *conic* is the reciprocal of a circle with a center A and radius r. Let $\epsilon = OA/r$ be the *eccentricity* of the conic.

(i) If $\epsilon < 1$ then it is *ellipce*, particularly $\epsilon = 0$ is the circle.
(ii) If $\epsilon = 1$ then it is *parabola*.
(iii) If $\epsilon > 1$ then it is *hyperbola*.

5. The projective plane

Similarly for definition of the inversive plane we could make an extension of Euclidean plane for the projective case. To define reciprocation for all points we need to introduce a one additional line: *line at infinity* l_∞. This line is polar for O and its points (*points at infinity*) are poles for lines through O. Those points are common points for *pencil of parallel lines*. Thus *any two distinct lines* a *and* b *determine a unique point* $a \cdot b$.

THEOREM 5.1. *If* P *is not on the conic, its polar joins the points of intersection* $AB \cdot DE$ *and* $AE \cdot BD$*, where* AD *and* BE *are any two secant through* P.

THEOREM 5.2. *With respect to any conic except a circle, a directrix is the polar of the corresponding focus.*

7. Stereographic and gnomonic projection

In the same way as we introduce the inversion we could introduce in $\mathbb{R}^3$ with respect to a sphere Σ with a center O and radius K by relation $OA \times OA' = k^2$. As a corollary from the plane we see that the image of any sphere (including a plane as a limit case) is a sphere again.

If a plane is tangent to the sphere of inversion at A then its image is the sphere σ with a diameter OA. And the image of any point P in the plane is just another point of intersection of line OP with σ. Sphere σ is a *model* for inversive plane. The mapping between plane and sphere is *stereographic projection*. Its preserve angles between directions in any points.

If we take a sphere Σ and construct the map from a tangent plane to pairs of antipodal points as intersections of line OP and Σ then we obtain *gnomonic map*. It maps big circles (shortest distances on the sphere) to straight lines (shortest distances on the plane). Identifying pairs antipodal points on the sphere we obtain a *model* of projective plane.

APPENDIX A

Some Useful Theorems

THEOREM 0.1. *An angle inscribed in an arc of a circle has a measure which is a half of angular measure of the complementary arc.*

COROLLARY 0.2. *An angle inscribed in semicircle is is a right angle.*

THEOREM 0.3. *Two tangemts to a circle from any external point are equal.*

THEOREM 0.4. *The following geometric objects have indicated areas* S*:*

(i) *Rectangle with sides* a *and* b*:* $S = ab$.
(ii) *Parallelogram with a base* a *and altitude* h*:* $S = ah$.
(iii) *Triangle with a side* a *and corresponding altitude* h_a*:* $S = ah_a/2$.

COROLLARY 0.5. (i) *Two triangles with a common altitude* h *have areas proportional to their sides:* $\frac{S_1}{S_2} = \frac{a_1}{a_2}$.
(ii) *Two triangles with a common side* a *have areas proportional to their altitudes:* $\frac{S_1}{S_2} = \frac{h_1}{h_2}$.

APPENDIX B

Some Useful Tricks

1. Look for a triangle—the golden rule of geometry

If the unknown element is a line segment or an angle—*try to find a triangle*, which contains this element and such that other parameters of the triangle are given or could be found from the given conditions.

If you are questioned about two elements (like a ratio of two line segments, for example) *try to find two triangles* with some common elements and each containing one of the unknown line segments.

2. Investigate a particular case

If you meet a problem—*try to investigate a particular case*. If you study an angle inscribed in a circle—consider first a case when the angle goes through the center of circle. If this particular investigation was successful try to use this particular case for solution the general one.

Bibliography

[1] H.S.M. Coxeter and S.L. Greitzer. *Geometry Revisited*. This famous book was reprinted many times and translated to many foreign languages.

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