INTRODUCTION TO FUNCTIONAL ANALYSIS

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ABSTRACT. This is lecture notes for several courses **on Functional Analysis** at School of Mathematics of University of Leeds. They are based on the notes of Dr. Matt Daws, Prof. Jonathan R. Partington, Dr. David Salinger, and Prof. Alex Strohmaier used in the previous years. Some sections are borrowed from the textbooks, which I used since being a student myself. However all misprints, omissions, and errors are only my responsibility. I am very grateful to Filipa Soares de Almeida, Eric Borgnet, Pasc Gavruta for pointing out some of them. Please let me know if you find more.

The notes are available also for download in PDF.

The suggested textbooks are [1,9,12,13]. The other nice books with many interesting problems are [3,11].

Exercises with stars **are not** a part of mandatory material but are nevertheless worth to hear about. And they are not necessarily difficult, try to solve them!

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Notations and Assumptions

$\mathbb{Z}_+$, $\mathbb{R}_+$ denotes non-negative integers and reals.
$x, y, z, \ldots$ denotes vectors.
$\lambda, \mu, \nu, \ldots$ denotes scalars.
$\Re z$, $\Im z$ stand for real and imaginary parts of a complex number z.

Integrability conditions. In this course, the functions we consider will be real or complex valued functions defined on the real line which are *locally Riemann integrable*. This means that they are Riemann integrable on any finite closed interval $[a, b]$. (A complex valued function is Riemann integrable iff its real and imaginary parts are Riemann-integrable.) In practice, we shall be dealing mainly with bounded functions that have only a finite number of points of discontinuity in any finite interval. We can relax the boundedness condition to allow improper Riemann integrals, but we then require the integral of the absolute value of the function to converge.

We mention this right at the start to get it out of the way. There are many fascinating subtleties connected with Fourier analysis, but those connected with technical aspects of integration theory are beyond the scope of the course. It turns out that one needs a "better" integral than the Riemann integral: the Lebesgue integral, and I commend the module, Linear Analysis 1, which includes an introduction to that topic which is available to MM students (or you could look it up in *Real and Complex Analysis* by Walter Rudin). Once one has the Lebesgue integral, one can start thinking about the different classes of functions to which Fourier analysis applies: the modern theory (not available to Fourier himself) can even go beyond functions and deal with generalized functions (distributions) such as the Dirac delta function which may be familiar to some of you from quantum theory.

From now on, when we say "function", we shall assume the conditions of the first paragraph, unless anything is stated to the contrary.

0. Motivating Example: Fourier Series

0.1. **Fourier series: basic notions.** Before proceed with an abstract theory we consider a motivating example: Fourier series.

0.1.1. *2π-periodic functions.* In this part of the course we deal with functions (as above) that are periodic.

We say a function $f : \mathbb{R} \to \mathbb{C}$ is *periodic* with *period* $T > 0$ if $f(x + T) = f(x)$ for all $x \in \mathbb{R}$. For example, $\sin x$, $\cos x$, $e^{ix} (= \cos x + i \sin x)$ are periodic with period 2π. For $k \in R \setminus \{0\}$, $\sin kx$, $\cos kx$, and e^{ikx} are periodic with period $2\pi/|k|$. Constant functions are periodic with period T, for any $T > 0$. We shall specialize to periodic functions with period 2π: we call them 2π-periodic functions, for short. Note that $\cos nx$, $\sin nx$ and e^{inx} are 2π-periodic for $n \in \mathbb{Z}$. (Of course these are also $2\pi/|n|$-periodic.)

Any half-open interval of length T is a *fundamental domain* of a periodic function f of period T. Once you know the values of f on the fundamental domain, you know them everywhere, because any point x in $\mathbb{R}$ can be written uniquely as $x = w + nT$ where $n \in \mathbb{Z}$ and w is in the fundamental domain. Thus $f(x) = f(w+(n-1)T+T) = \cdots = f(w + T) = f(w)$.

For 2π-periodic functions, we shall usually take the fundamental domain to be $]-\pi, \pi]$. By abuse of language, we shall sometimes refer to $[-\pi, \pi]$ as the fundamental domain. We then have to be aware that $f(\pi) = f(-\pi)$.

0.1.2. *Integrating the complex exponential function.* We shall need to calculate $\int_a^b e^{ikx}\, dx$, for $k \in \mathbb{R}$. Note first that when $k = 0$, the integrand is the constant function 1, so the result is $b - a$. For non-zero k, $\int_a^b e^{ikx}\, dx = \int_a^b (\cos kx + i \sin kx)\, dx = (1/k)[(\sin kx - i \cos kx)]_a^b = (1/ik)[(\cos kx + i \sin kx)]_a^b = (1/ik)[e^{ikx}]_a^b = (1/ik)(e^{ikb} - e^{ika})$. Note that this is exactly the result you would have got by treating i as a real constant and using the usual formula for integrating e^{ax}. Note also that the cases $k = 0$ and $k \neq 0$ have to be treated separately: this is typical.

Definition 0.1. Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function which is Riemann integrable on $[-\pi, \pi]$. For each $n \in \mathbb{Z}$ we define the *Fourier coefficient* $\hat{f}(n)$ by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx\,.$$

Remark 0.2. (i) $\hat{f}(n)$ is a complex number whose modulus is the amplitude and whose argument is the phase (of that component of the original function).

(ii) If f and g are Riemann integrable on an interval, then so is their product, so the integral is well-defined.

(iii) The constant before the integral is to divide by the length of the interval.

(iv) We could replace the range of integration by any interval of length 2π, without altering the result, since the integrand is 2π-periodic.

(v) Note the minus sign in the exponent of the exponential. The reason for this will soon become clear.

Example 0.3. (i) $f(x) = c$ then $\hat{f}(0) = c$ and $\hat{f}(n) = 0$ when $n \neq 0$.

(ii) $f(x) = e^{ikx}$, where k is an integer. $\hat{f}(n) = \delta_{nk}$.

(iii) f is 2π periodic and $f(x) = x$ on $]-\pi, \pi]$. (Diagram) Then $\hat{f}(0) = 0$ and, for $n \neq 0$,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx = \left[\frac{-xe^{-inx}}{2\pi in}\right]_{-\pi}^{\pi} + \frac{1}{in}\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx}\, dx = \frac{(-1)^n i}{n}.$$

Proposition 0.4 (Linearity). *If f and g are 2π-periodic functions and c and d are complex constants, then, for all $n \in \mathbb{Z}$,*

$$(cf + dg)\widehat{\ }(n) = c\hat{f}(n) + d\hat{g}(n)\,.$$

Corollary 0.5. *If* $p(x)$ *is a* trigonometric polynomial, $p(x) = \sum_{-k}^{k} c_n e^{inx}$, *then* $\hat{p}(n) = c_n$ *for* $|n| \leqslant k$ *and* $= 0$, *for* $|n| \geqslant k$.

$$p(x) = \sum_{n \in \mathbb{Z}} \hat{p}(n) e^{inx}\,.$$

This follows immediately from Ex. 0.3(ii) and Prop.0.4.

Remark 0.6. (i) This corollary explains why the minus sign is natural in the definition of the Fourier coefficients.

(ii) The first part of the course will be devoted to the question of how far this result can be extended to other 2π-periodic functions, that is, for which functions, and for which interpretations of infinite sums is it true that

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}. \tag{0.1}$$

Definition 0.7. $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$ is called the *Fourier series* of the 2π-periodic function f.

For real-valued functions, the introduction of complex exponentials seems artificial: indeed they can be avoided as follows. We work with (0.1) in the case of a finite sum: then we can rearrange the sum as

$$\begin{aligned} &\hat{f}(0) + \sum_{n>0} (\hat{f}(n)e^{inx} + \hat{f}(-n)e^{-inx}) \\ = \;& \hat{f}(0) + \sum_{n>0} [(\hat{f}(n) + \hat{f}(-n)) \cos nx + i(\hat{f}(n) - \hat{f}(-n)) \sin nx] \\ = \;& \frac{a_0}{2} + \sum_{n>0} (a_n \cos nx + b_n \sin nx) \end{aligned}$$

Here

$$\begin{aligned} a_n &= (\hat{f}(n) + \hat{f}(-n)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(e^{-inx} + e^{inx})\, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \end{aligned}$$

for $n > 0$ and

$$b_n = i((\hat{f}(n) - \hat{f}(-n)) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

for $n > 0$. $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$, the constant chosen for consistency.

The a_n and b_n are also called Fourier coefficients: if it is necessary to distinguish them, we may call them *Fourier cosine* and *sine coefficients*, respectively.

We note that if f is real-valued, then the a_n and b_n are real numbers and so $\Re\hat{f}(n) = \Re\hat{f}(-n)$, $\Im\hat{f}(-n) = -\Im\hat{f}(n)$: thus $\hat{f}(-n)$ is the complex conjugate of $\hat{f}(n)$. Further, if f is an even function then all the sine coefficients are 0 and if f is an odd function, all the cosine coefficients are zero. We note further that the sine and cosine coefficients of the functions $\cos kx$ and $\sin kx$ themselves have a particularly simple form: $a_k = 1$ in the first case and $b_k = 1$ in the second. All the rest are zero.

For example, we should expect the 2π-periodic function whose value on $]-\pi, \pi]$ is x to have just sine coefficients: indeed this is the case: $a_n = 0$ and $b_n = i(\hat{f}(n) - \hat{f}(-n)) = (-1)^{n+1} 2/n$ for $n > 0$.

The above question can then be reformulated as "to what extent is $f(x)$ represented by the Fourier series $a_0/2 + \sum_{n>0}(a_n \cos x + b_n \sin x)$?" For instance how well does $\sum(-1)^{n+1}(2/n)\sin nx$ represent the 2π-periodic sawtooth function f whose value on $]-\pi, \pi]$ is given by $f(x) = x$. The easy points are $x = 0$, $x = \pi$, where the terms are identically zero. This gives the 'wrong' value for $x = \pi$, but, if we look at the periodic function near π, we see that it jumps from π to $-\pi$, so perhaps the mean of those values isn't a bad value for the series to converge to. We could conclude that we had defined the function incorrectly to begin with and that its value at the points $(2n + 1)\pi$ should have been zero anyway. In fact one can show (ref.) that the Fourier series converges at all other points to the given values of f, but I shan't include the proof in this course. The convergence is not at all uniform (it can't be, because the partial sums are continuous functions, but the limit is discontinuous.) In particular we get the expansion

$$\frac{\pi}{2} = 2(1 - 1/3 + 1/5 - \cdots)$$

which can also be deduced from the Taylor series for $\tan^{-1}$.

0.2. **The vibrating string.** In this subsection we shall discuss the formal solutions of the wave equation in a special case which Fourier dealt with in his work.

We discuss the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{K^2}\frac{\partial^2 y}{\partial t^2}, \tag{0.2}$$

subject to the boundary conditions

$$y(0, t) = y(\pi, t) = 0, \tag{0.3}$$

for all $t \geqslant 0$, and the initial conditions

$$\begin{aligned} y(x, 0) &= F(x), \\ y_t(x, 0) &= 0. \end{aligned}$$

This is a mathematical model of a string on a musical instrument (guitar, harp, violin) which is of length π and is plucked, i.e. held in the shape $F(x)$ and released at time $t = 0$. The constant K depends on the length, density and tension of the string. We shall derive the formal solution (that is, a solution which assumes existence and ignores questions of convergence or of domain of definition).

0.2.1. *Separation of variables.* We first look (as Fourier and others before him did) for solutions of the form $y(x,t) = f(x)g(t)$. Feeding this into the wave equation (0.2) we get

$$f''(x)g(t) = \frac{1}{K^2} f(x)g''(t)$$

and so, dividing by $f(x)g(t)$, we have

$$\frac{f''(x)}{f(x)} = \frac{1}{K^2}\frac{g''(t)}{g(t)}. \tag{0.4}$$

The left-hand side is an expression in x alone, the right-hand side in t alone. The conclusion must be that they are both identically equal to the same constant C, say.

We have $f''(x) - Cf(x) = 0$ subject to the condition $f(0) = f(\pi) = 0$. Working through the method of solving linear second order differential equations tells you that the only solutions occur when $C = -n^2$ for some positive integer n and the corresponding solutions, up to constant multiples, are $f(x) = \sin nx$.

Returning to equation (0.4) gives the equation $g''(t)+K^2n^2g(t) = 0$ which has the general solution $g(t) = a_n \cos Knt + b_n \sin Knt$. Thus the solution we get through separation of variables, using the boundary conditions but ignoring the initial conditions, are

$$y_n(x,t) = \sin nx(a_n \cos Knt + b_n \sin Knt)\,,$$

for $n \geqslant 1$.

0.2.2. *Principle of Superposition.* To get the general solution we just add together all the solutions we have got so far, thus

$$y(x,t) = \sum_{n=1}^{\infty} \sin nx(a_n \cos Knt + b_n \sin Knt) \tag{0.5}$$

ignoring questions of convergence. (We can do this for a finite sum without difficulty because we are dealing with a linear differential equation: the iffy bit is to extend to an infinite sum.)

We now apply the initial condition $y(x,0) = F(x)$ (note F has $F(0) = F(\pi) = 0$). This gives

$$F(x) = \sum_{n=1}^{\infty} a_n \sin nx\,.$$

We apply the reflection trick: the right-hand side is a series of odd functions so if we extend F to a function G by reflection in the origin, giving

$$G(x) := \begin{cases} F(x) & , \text{ if } 0 \leqslant x \leqslant \pi; \\ -F(-x) & , \text{ if } -\pi < x < 0. \end{cases}$$

we have

$$G(x) = \sum_{n=1}^{\infty} a_n \sin nx\,,$$

for $-\pi \leqslant x \leqslant \pi$.

If we multiply through by $\sin rx$ and integrate term by term, we get

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} G(x) \sin rx \, dx$$

so, assuming that this operation is valid, we find that the a_n are precisely the sine coefficients of G. (Those of you who took Real Analysis 2 last year may remember that a sufficient condition for integrating term-by -term is that the series which is integrated is itself uniformly convergent.)

If we now assume, further, that the right-hand side of (0.5) is differentiable (term by term) we differentiate with respect to t, and set $t = 0$, to get

$$0 = y_t(x, 0) = \sum_{n=1}^{\infty} b_n Kn \sin nx. \tag{0.6}$$

This equation is solved by the choice $b_n = 0$ for all n, so we have the following result

Proposition 0.8 (Formal). *Assuming that the formal manipulations are valid, a solution of the differential equation* (0.2) *with the given boundary and initial conditions is*

$$y(x, t) = \sum_{1}^{\infty} a_n \sin nx \cos Knt\,, \tag{2.11}$$

where the coefficients a_n *are the Fourier sine coefficients*

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} G(x) \sin nx \, dx$$

of the 2π *periodic function* G*, defined on* $]-\pi, \pi]$ *by reflecting the graph of* F *in the origin.*

Remark 0.9. This leaves us with the questions

(i) For which F are the manipulations valid?
(ii) Is this the only solution of the differential equation? (which I'm not going to try to answer.)
(iii) Is $b_n = 0$ all n the only solution of (0.6)? This is a special case of the **uniqueness** problem for trigonometric series.

0.3. **Historic: Joseph Fourier.** *Joseph Fourier*, Civil Servant, Egyptologist, and mathematician, was born in 1768 in Auxerre, France, son of a tailor. Debarred by birth from a career in the artillery, he was preparing to become a Benedictine monk (in order to be a teacher) when the French Revolution violently altered the course of history and Fourier's life. He became president of the local revolutionary committee, was arrested during the Terror, but released at the fall of Robespierre.

Fourier then became a pupil at the Ecole Normale (the teachers' academy) in Paris, studying under such great French mathematicians as Laplace and Lagrange. He became a teacher at the Ecole Polytechnique (the military academy).

He was ordered to serve as a scientist under Napoleon in Egypt. In 1801, Fourier returned to France to become Prefect of the Grenoble region. Among his most notable achievements in that office were the draining of some 20 thousand acres of swamps and the building of a new road across the alps.

During that time he wrote an important survey of Egyptian history ("a masterpiece and a turning point in the subject").

In 1804 Fourier started the study of the theory of heat conduction, in the course of which he systematically used the sine-and-cosine series which are named after him. At the end of 1807, he submitted a memoir on this work to the Academy of Science. The memoir proved controversial both in terms of his use of Fourier series and of his derivation of the heat equation and was not accepted at that stage. He was able to resubmit a revised version in 1811: this had several important new features, including the introduction of the Fourier transform. With this version of his memoir, he won the Academy's prize in mathematics. In 1817, Fourier was finally elected to the Academy of Sciences and in 1822 his 1811 memoir was published as "Théorie de la Chaleur".

For more details see *Fourier Analysis* by T.W. Körner, 475-480 and for even more, see the biography by J. Herivel *Joseph Fourier: the man and the physicist*.

What is Fourier analysis. The idea is to analyse functions (into sine and cosines or, equivalently, complex exponentials) to find the underlying frequencies, their strengths (and phases) and, where possible, to see if they can be recombined (synthesis) into the original function. The answers will depend on the original properties of the functions, which often come from physics (heat, electronic or sound waves). This course will give basically a mathematical treatment and so will be interested in mathematical classes of functions (continuity, differentiability properties).

1. Basics of Metric Spaces

1.1. **Metric Spaces.**

1.1.1. *Metric spaces: definition and examples.* In Analysis and Calculus the definition of convergence was based on the notion of a distance between points, namely the standard distance between two real numbers is given by

$$d(x, y) = |x - y|.$$

Similarly, the distance between two points in the plane, given by

$$d(\mathbf{x},\mathbf{y}) = d((x_1,x_2),(y_1,y_2)) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}.$$

A metric space formalises this notion. This will give us the flexibility to talk about distances on function spaces, for example, or introduce other notions of distance on spaces.

Definition 1.1 (Metric Space)**.** A *metric space* (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}$ that satisfies the following properties

(i) $d(x, y) \geqslant 0$; and $d(x, y) = 0 \iff x = y$ (positive definite);
(ii) $d(x, y) = d(y, x)$ (symmetric);
(iii) $d(x, z) \leqslant d(x, y) + d(y, z)$ (triangle inequality).

The function d is called the *metric*. The word *distance* will be used interchangeably with the same meaning.

Example 1.2. (i) $X = \mathbb{R}$. The *standard* metric is given by $d_1(x, y) = |x - y|$. There are many other metrics on $\mathbb{R}$, for example

$$d(x, y) = |e^x - e^y|;$$

$$d(x,y) = \begin{cases} |x-y| & \text{if } |x-y| \leqslant 1, \\ 1 & \text{if } |x-y| \geqslant 1. \end{cases}$$

(ii) Let X be any set whatsoever, then we can define the *discrete metric*

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

(iii) $X = \mathbb{R}^m$. The standard metric is the Euclidean metric: if $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ then

$$d_2(\mathbf{x},\mathbf{y}) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_m-y_m)^2}.$$

This is linked to the inner-product (scalar product), $\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + \ldots + x_my_m$, since it is just $\sqrt{(x-y).(x-y)}$. We will study inner products more carefully later, so for the moment we won't prove the (well-known) fact that it is indeed a metric.

Other possible metrics include

$$d_\infty(\mathbf{x},\mathbf{y}) = \max\{|x_1-y_1|, |x_2-y_2|, \ldots, |x_m-y_m|\}.$$

Another metric on $\mathbb{R}^m$ comes from the generalisation of our first example:

$$d_1(\mathbf{x},\mathbf{y}) = |x_1-y_1| + |x_2-y_2| + \ldots + |x_m-y_m|.$$

These metrics d_1, d_2, d_∞ are all *translation-invariant* (i.e., $d(\mathbf{x}+\mathbf{z},\mathbf{y}+\mathbf{z}) = d(\mathbf{x},\mathbf{y})$), and *positively homogeneous* (i.e., $d(k\mathbf{x},k\mathbf{y}) = |k|d(\mathbf{x},\mathbf{y})$), see Ex. 1.8 for further discussion.

(iv) Take $X = C[a,b]$. Here are three metrics similar to above ones:

$$d_2(f,g) = \sqrt{\int_a^b |f(x)-g(x)|^2\,dx}.$$

Again, this is linked to the idea of an inner product, so we will delay proving that it is a metric.

$$d_1(f,g) = \int_a^b |f(x)-g(x)|\,dx,$$

the area between two graphs

$$d_\infty(f,g) = \max\{|f(x)-g(x)| : a \leqslant x \leqslant b\},$$

the maximum vertical separation between two graphs.

Example 1.3. On $C[0,1]$ take $f(x) = x$ and $g(x) = x^2$ and calculate

$$\begin{aligned} d_2(f,g) &= \left(\int_0^1 (x-x^2)^2\,dx\right)^{1/2} = \sqrt{1/30}, \\ d_1(f,g) &= \int_0^1 |x-x^2|\,dx = 1/6, \text{ and} \\ d_\infty(f,g) &= \max_{x\in[0,1]} |x-x^2| = 1/4. \end{aligned}$$

Remark 1.4. Any subset of a metric space is again a metric space its own right, by restricting the distance function to the subset.

Example 1.5. (i) The interval $[a,b]$ with $d(x,y) = |x-y|$ is a subspace of $\mathbb{R}$.

(ii) The unit circle $\{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ with $d_2(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ is a subspace of $\mathbb{R}^2$.

(iii) The space of polynomials $\mathcal{P}$ is a metric space with any of the metrics inherited from $C[a,b]$ above.

Definition 1.6. A *normed space* $(V, \|\cdot\|)$ is a real vector space V with a map $\|\cdot\| : V \to \mathbb{R}$ (called *norm*) satisfying

(i) $\|\mathbf{v}\| \geqslant \mathbf{0}$, and $(\|\mathbf{v}\| = \mathbf{0} \Leftrightarrow \mathbf{v} = \mathbf{0})$,

(ii) $\|\lambda \mathbf{v}\| = |\breve{\ }|\|\mathbf{v}\|$,

(iii) $\|\mathbf{v} + \mathbf{w}\| \leqslant \|\mathbf{v}\| + \|\mathbf{w}\|$.

Exercise 1.7. Prove that V is a metric space with metric $d(\mathbf{v}, \mathbf{w}) := \|\mathbf{v} - \mathbf{w}\|$.

Exercise 1.8. (i) Write norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ on $\mathbb{R}^m$ which produces metrics d_1, d_2, d_∞ from Ex. 1.2.1.2(iii).
Hint: see (2.4) and (2.2) below.

(ii) Show, that the following are norms on the vector space $V = C[a, b]$:

$$\|f\|_1 = \int_a^b |f(x)|\, dx,$$

$$\|f\|_2 = \int_a^b |f(x)|^2\, dx,$$

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|.$$

Furthermore, these norms generate the respective metrics d_1, d_2 and d_∞ from Ex. 1.2(1.2(iv)) as indicated in the previous exercise.

Definition 1.9. An *inner product space*$(V, \langle\cdot,\cdot\rangle)$ is a real vector space V with a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ (called *inner product*) satisfying

(i) $\langle \lambda \mathbf{v}, \mathbf{w}\rangle = \breve{\ }\langle \mathbf{v}, \mathbf{w}\rangle$,

(ii) $\langle \mathbf{v_1} + \mathbf{v_2}, \mathbf{w}\rangle = \langle \mathbf{v_1}, \mathbf{w}\rangle + \langle \mathbf{v_2}, \mathbf{w}\rangle$,

(iii) $\langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{v}\rangle$,

(iv) $\langle \mathbf{v}, \mathbf{v}\rangle \geqslant \mathbf{0}$, and $(\langle \mathbf{v}, \mathbf{v}\rangle = \mathbf{0} \Leftrightarrow \mathbf{v} = \mathbf{0})$.

Exercise 1.10. (i) Prove that the *Cauchy–Schwarz inequality* $|\langle \mathbf{v}, \mathbf{w}\rangle|^2 \leqslant \langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle$ holds.
Hint: start by considering the expression $\langle \mathbf{v} + \breve{\ }\mathbf{w}, \mathbf{v} + \breve{\ }\mathbf{w}\rangle \geqslant \mathbf{0}$ and analyse the discriminant of the quadratic expression for λ.

(ii) Then prove that V is a normed space with norm $\|\mathbf{v}\| := \langle \mathbf{v}, \mathbf{v}\rangle^{\frac{1}{2}}$.

(iii) Which of the above norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ from Ex. 1.8 can be obtained from an inner product as described in the previous item?

There is a natural name for a class of maps, which preserve metrics:

Definition 1.11 (Isometry). Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $\phi : X \to Y$ is an *isometry* if

$$d_Y(\phi(x_1), \phi(x_2)) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

A metric space (X, d_X) is *isometric* to a metric space (Y, d_Y) if there is an isometry bijection between X and Y.

1.1.2. *Open and closed sets.*

Definition 1.12 (Open and closed balls). Let (X, d) be a metric space, let $x \in X$ and let $r > 0$. The *open ball* centred at x, with radius r, is the set

$$B_r(x) = \{y \in X : d(x, y) < r\},$$

and the *closed ball* is the set

$$\overline{B_r(x)} = \{y \in X : d(x, y) \leqslant r\}.$$

A trivial but useful observation is: $x \in B_r(x) \subset \overline{B_r(x)}$ for all $x \in X$ and $r > 0$.

Note that in $\mathbb{R}$ with the usual metric the open ball is $B_r(x) = (x - r, x + r)$, an open interval, and the closed ball is $\overline{B_r(x)} = [x - r, x + r]$, a closed interval.

For the d_2 metric on $\mathbb{R}^2$, the *unit ball*, $B_1(\mathbf{0})$, is disc centred at the origin, excluding the boundary. You may like to think about what you get for other metrics on $\mathbb{R}^2$.

Definition 1.13 (Open sets). A subset U of a metric space (X, d) is said to be *open*, if for each point $x \in U$ there is an $r > 0$ such that the open ball $B_r(x)$ is contained in U ("room to swing a cat").

Clearly X itself is an open set, that is the whole metric space is open in itself. Also *by convention* the empty set $\emptyset$ is also considered to be *open*.

Remark 1.14. Note that the property "be open" of a set depends on the metric space. For example if we consider the set $[0, 1]$ it is open in the metric space $[0, 1]$ with the standard metric, but not open in the set $\mathbb{R}$ with standard metric.

Proposition 1.15. *Every "open ball" $B_r(x)$ is an open set.*

Proof. For if $y \in B_r(x)$, choose $\delta = r - d(x, y)$. We claim that $B_\delta(y) \subset B_r(x)$. If $z \in B_\delta(y)$, i.e., $d(z, y) < \delta$, then by the triangle inequality

$$d(z, x) \leqslant d(z, y) + d(y, x) < \delta + d(x, y) = r.$$

So $z \in B_r(x)$. □

Definition 1.16 (Closed set)**.** A subset F of (X, d) is said to be *closed*, if its complement $X \setminus F$ is open.

Note that closed does not mean "not open". In a metric space the sets $\emptyset$ and X are both open and closed. In $\mathbb{R}$ we have:

(a, b) is open.
$[a, b]$ is closed, since its complement $(-\infty, a) \cup (b, \infty)$ is open.
$[a, b)$ is not open, since there is no open ball $B(a, r)$ contained in the set. Nor it is closed, since its complement $(-\infty, a) \cup [b, \infty)$ isn't open (no ball centred at b can be contained in the set).

Remark 1.17. As it can be seen from the definitions the property of a subset F to be open or closed depends from the surrounding space X. For example:

- The interval $[0, 1)$ is open as a subset of the space $[0, 2]$ and is not open as a subset of $\mathbb{R}$ (both are taken with the usual metric).
- The same interval $[0, 1)$ is closed as a subset of the space $(-1, 1)$ and is not open as subset $\mathbb{R}$ (again, both are taken with the usual metric).

Example 1.18. If we take the discrete metric,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

then each point $\{x\} = B_{1/2}(x)$ so is an open set. Hence every set U is open, since for $x \in U$ we have $B_{1/2}(x) \subseteq U$. Hence, by taking complements, every set is also closed.

Theorem 1.19. *In a metric space, every one-point set $\{x_0\}$ is closed.*

Proof. We need to show that the set $U = \{x \in X : x \neq x_0\}$ is open, so take a point $x \in U$. Now $d(x, x_0) > 0$, and the ball $B_r(x)$ is contained in U for every $0 < r < d(x, x_0)$. □

Theorem 1.20. *Let $(U_\alpha)_{\alpha \in A}$ be any collection of open subsets of a metric space (X, d) (not necessarily finite!). Then $\bigcup_{\alpha \in A} U_\alpha$ is open. Let U and V be open subsets of a metric space (X, d). Then $U \cap V$ is open. Hence (by induction) any finite intersection of open subsets is open.*

Proof. If $x \in \bigcup_{\alpha\in A} U_\alpha$ then there is an α with $x \in U_\alpha$. Now U_α is open, so $B_r(x) \subset U_\alpha$ for some $r > 0$. Then $B_r(x) \subset \bigcup_{\alpha\in A} U_\alpha$ so the union is open.
If now U and V are open and $x \in U \cap V$, then $\exists r > 0$ and $s > 0$ such that $B_r(x) \subset U$ and $B(x,s) \subset V$, since U and V are open. Then $B(x,t) \subset U \cap V$ if $t \leqslant \min(r,s)$. □

Remark 1.21. Here we used a common property, which is helpful to remember: the minimum of a finite set of positive numbers is always positive. However, the infimum of an infinite set of positive numbers can be zero, e.g. $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$. Therefore, a transition from a given infinite set to a suitable finite set will be a reacquiring theme in our course, cf. compact set later in the course.

So the collection of open sets is preserved by arbitrary unions and finite intersections.

However, an *arbitrary* intersection of open sets is not always open; for example $(-\frac{1}{n}, \frac{1}{n})$ is open for each $n = 1, 2, 3, \ldots$, but $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$, which is not an open set.

For closed sets we swap union and intersection.

Theorem 1.22. *Let* $(F_\alpha)_{\alpha\in A}$ *be any collection of closed subsets of a metric space* (X,d) *(not necessarily finite!). Then* $\bigcap_{\alpha\in A} F_\alpha$ *is closed. Let* F *and* G *be closed subsets of a metric space* (X,d). *Then* $F \cup G$ *is closed. Hence (by induction) any finite union of closed subsets is closed.*

Proof. To prove this we recall *de Morgan's laws*. We use the notation S^c for the complement $X \setminus S$ of a set $S \subset X$.

$$x \notin \bigcup_\alpha A_\alpha \iff x \notin A_\alpha \text{ for all } \alpha, \text{ so } (\bigcup A_\alpha)^c = \bigcap A_\alpha^c.$$

$$x \notin \bigcap_\alpha A_\alpha \iff x \notin A_\alpha \text{ for some } \alpha, \text{ so } (\bigcap A_\alpha)^c = \bigcup A_\alpha^c.$$

Write $U_\alpha = F_\alpha^c = X \setminus F_\alpha$ which is open. So $\bigcup_{\alpha\in A} U_\alpha$ is open by Theorem 1.20. Now, by de Morgan's laws, $(\bigcap_{\alpha\in A} F_\alpha)^c = \bigcup_{\alpha\in A} F_\alpha^c$. This is just $\bigcup_{\alpha\in A} U_\alpha$. Since its complement is open, $\bigcap_{\alpha\in A} F_\alpha$ is closed.
Similarly, the complement of $F \cup G$ is $F^c \cap G^c$, which is the intersection of two open sets and hence open by Theorem 1.20. Hence $F \cup G$ is closed. □

Infinite unions of closed sets do not need to be closed. An example is

$$\bigcup_{n=1}^{\infty} [\frac{1}{n}, \infty) = (0, \infty),$$

which is open but not closed.

Definition 1.23 (Closure of a set)**.** The *closure of* S, written $\overline{S}$, is the smallest closed set containing S, and is contained in all other closed sets containing S.

The above smallest closed set containing S does exist, because we can define

$$\overline{S} = \bigcap \{F : F \supset S, F \text{ closed}\},$$

the intersection of all closed sets containing S. There is at least one, namely X itself.

Example 1.24. In the metric space $\mathbb{R}$ the closure of $S = [0, 1)$ is $[0, 1]$. This is closed, and there is nothing smaller that is closed and contains S.

Definition 1.25 (Dense subset)**.** A subset $S \subset X$ is *dense* in X if $\overline{S} = X$.

Theorem 1.26. *The set $\mathbb{Q}$ of rationals is dense in $\mathbb{R}$, with the usual metric.*

Proof. Suppose that F is a closed subset of $\mathbb{R}$ which contains $\mathbb{Q}$: we claim that it $F = \mathbb{R}$.
For $U = \mathbb{R} \setminus F$ is open and contains no points of $\mathbb{Q}$. But an open set U (unless it is empty) must contain an interval $B_r(x)$ for some $x \in U$, and hence a rational number.
Our only conclusion is that $U = \emptyset$ and $F = \mathbb{R}$, so that $\overline{\mathbb{Q}} = \mathbb{R}$. □

Definition 1.27 (Neighbourhood)**.** We say that V is a *neighbourhood* (nbh) of x if there is an open set U such that $x \in U \subseteq V$; this means that $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq V$. Thus a set is open precisely when it is a neighbourhood of each of its points.

Example 1.28. The half-open interval $[0, 1)$ is a neighbourhood of every point in it except for 0.

Theorem 1.29. *For a subset S of a metric space X, we have $x \in \overline{S}$ iff $V \cap S \neq \emptyset$ for all nhds V of x (i.e., all neighbourhoods of x meet S).*

Proof. If there is a neighbourhood of x that doesn't meet S, then there is an open subset U with $x \in U$ and $U \cap S = \emptyset$.
But then $X \setminus U$ is a closed set containing S and so $\overline{S} \subset X \setminus U$, and then $x \notin \overline{S}$ because $x \in U$.
Conversely, if every neighbourhood of x does meet S, then $x \in \overline{S}$, as otherwise $X \setminus \overline{S}$ is as open neighbourhood of x that doesn't meet S. □

Definition 1.30 (Interior)**.** The *interior of* S, $\operatorname{int} S$, is the largest open set contained in S, and can be written as

$$\operatorname{int} S = \bigcup \{U : U \subset S, U \text{ open}\}.$$

the union of all open sets contained in S. There is at least one, namely $\emptyset$.

We see that S is open exactly when $S = \operatorname{int} S$, otherwise $\operatorname{int} S$ is smaller.

Example 1.31.
(i) In the metric space $\mathbb{R}$ we have $\operatorname{int}[0, 1) = (0, 1)$; clearly this is open and there is no larger open set contained in $[0, 1)$.
(ii) $\operatorname{int} \mathbb{Q} = \emptyset$. For any non-empty open set must contain an interval $B_r(x)$ and then it contains an irrational number, so isn't contained in $\mathbb{Q}$.

Proposition 1.32. $\operatorname{int} S = X \setminus (\overline{X \setminus S})$.

Proof. By De Morgan's laws,

$$\begin{aligned} \operatorname{int} S &= \bigcup \{U : U \subset S, U \text{ open}\} \\ &= X \setminus \bigcap \{U^c : U \subset S, U \text{ open}\} \\ &= X \setminus \bigcap \{F : F \supset X \setminus S, F \text{ closed}\} = X \setminus (\overline{X \setminus S}). \end{aligned}$$

This is because $U \subset S$ if and only if $U^c = X \setminus U \supset X \setminus S$. Also $F = U^c$ is closed precisely when U is open. That is, there is a correspondence between open sets contained in S and closed sets containing its complement. □

1.1.3. *Convergence and continuity.* Let (x_n) be a sequence in a metric space (X, d), i.e., $x_1, x_2, \ldots$. (Sometimes we may start counting at x_0.)

Definition 1.33 (Convergence)**.** We say $x_n \to x$ (i.e., x_n *converges* to x) if $d(x_n, x) \to 0$ as $n \to \infty$.
In other words: $x_n \to x$ if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $d(x, x_n) < \varepsilon$.

This is the usual notion of convergence if we think of points in $\mathbb{R}^d$ with the Euclidean metric.

Theorem 1.34. *Let (x_n) be a sequence in a metric space (X, d). Then the following are equivalent:*

(i) $x_n \to x$;
(ii) *for every open* U *with* $x \in U$, *there exists an* $N > 0$ *such that* $(n > N) \implies x_n \in U$;
(iii) *for every* $\varepsilon > 0$ *there exists an* $N > 0$ *such that* $(n > N) \implies x_n \in B_\varepsilon(x)$.

Proof. 1.34(i) ⇒ 1.34(ii) If $x_n \to x$ and $x \in U$, then there is a ball $B_\varepsilon(x) \subset U$, since U is open. But $x_n \to x$ so $d(x_n, x) < \varepsilon$ for n sufficiently large, i.e., $x_n \in U$ for n sufficiently large.
1.34(ii) ⇒ 1.34(iii) is obvious.
Finally, 1.34(iii) ⇒ 1.34(i). If the 1.34(iii) condition works for a given $\varepsilon > 0$ and large n the inclusion $x_n \in B_\varepsilon(x)$ implies $d(x_n, x) < \varepsilon$. □

Theorem 1.35. *Let S be a subset of the metric space X. Then $x \in \overline{S}$ if and only if there is a sequence (x_n) of points of S with $x_n \to x$.*

Proof. If $x \in \overline{S}$, then for each n we have $B_{\frac{1}{n}}(x) \cap S \neq \emptyset$ by Theorem 1.29. So choose $x_n \in B_{\frac{1}{n}}(x) \cap S$. Clearly $d(x_n, x) \to 0$, i.e., $x_n \to x$.
Conversely, if $x \notin \overline{S}$, then there is a neighbourhood U of x with $U \cap S = \emptyset$. Now no sequence in S can get into U so it cannot converge to x. □

This can also be phrased as follows, characterising closed set in terms of sequences.

Corollary 1.36 (Closedness under taking limits)**.** *A subset $Y \subset X$ of a metric space (X, d) is closed if and only if for every sequence (x_n) in Y that is convergent in X its limit is also in Y.*

Hence, the closure $\overline{S}$ is obtained from S by adding all possible limit points of sequences in S.

Example 1.37. (i) Take $(\mathbb{R}^2, d_1)$, where $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, and consider the sequence $(\frac{1}{n}, \frac{2n+1}{n+1})$. We *guess* its limit is $(0, 2)$. To see if this is right, look at

$$d_1\left(\left(\frac{1}{n}, \frac{2n+1}{n+1}\right), (0,2)\right) = \left|\frac{1}{n}\right| + \left|\frac{2n+1}{n+1} - 2\right| = \frac{1}{n} + \frac{1}{n+1} \to 0$$

as $n \to \infty$. So the limit is $(0, 2)$.

(ii) In $C[0, 1]$ let $f_n(t) = t^n$ and $f(t) = 0$ for $0 \leqslant t \leqslant 1$. Does $f_n \to f$, (a) in d_1, and (b) in d_∞?

(a)

$$d_1(f_n, f) = \int_0^1 t^n \, dt = \frac{1}{n+1} \to 0$$

as $n \to \infty$. So $f_n \to f$ in d_1.

(b)

$$d_\infty(f_n, f) = \max\{t^n : 0 \leqslant t \leqslant 1\} = 1 \not\to 0$$

as $n \to \infty$. So $f_n \not\to f$ in d_∞.

Note: Say $g_n \to g$ *pointwise* on $[a, b]$ as $n \to \infty$ if $g_n(x) \to g(x)$ for all $x \in [a, b]$. If we define $g(x) = \begin{cases} 0 & \text{for } 0 \leqslant x < 1, \\ 1 & \text{for } x = 1, \end{cases}$ then $f_n \to g$ pointwise on $[0, 1]$. But $g \notin C[0, 1]$, as it is not continuous at 1.

(iii) Take the discrete metric

$$d_0(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then $x_n \to x \iff d_0(x_n, x) \to 0$. But since $d_0(x_n, x) = 0$ or 1, this happens if and only if $d_0(x_n, x) = 0$ for n sufficiently large. That is, there is an n_0 such that $x_n = x$ for all $n \geqslant n_0$.

All convergent sequences in this metric are eventually constant. So, for example $d_0(1/n, 0) \not\to 0$.

A result on convergence in $\mathbb{R}^m$.

Proposition 1.38. *Take* $\mathbb{R}^2$ *with any of the metrics* d_1, d_2 *and* d_∞. *Then a sequence* $\mathbf{x}_n = (a_n, b_n)$ *converges to* $\mathbf{x} = (a, b)$ *if and only if* $a_n \to a$ *and* $b_n \to b$.

Proof. A useful observation is that for any $\mathbf{x}_n$ and $\mathbf{x}$:

$$d_1(\mathbf{x}_n, \mathbf{x}) \geqslant d_2(\mathbf{x}_n, \mathbf{x}) \geqslant d_\infty(\mathbf{x}_n, \mathbf{x}).$$

If $a_n \to a$ and $b_n \to b$, then for any $\varepsilon > 0$ there are N_a and N_b such that for $N > N_a$ we have $|a_n - a| < \varepsilon/2$ and for $n > N_b$ $|b_n - b| < \varepsilon/2$. Thus for any $n > N = \max(N_a, N_b)$:

$$\varepsilon > |a_n - a| + |b_n - b| = d_1(\mathbf{x}_n, \mathbf{x}) \geqslant d_2(\mathbf{x}_n, \mathbf{x}) \geqslant d_\infty(\mathbf{x}_n, \mathbf{x}),$$

which shows the convergence in all three metrics.
To show the opposite, WLOG assume towards a contradiction that $a_n \not\to a$, that is, there exits $\varepsilon > 0$ such that for any N there exists $n > N$ such that $|a_n - a| > \varepsilon$. Then:

$$d_1(\mathbf{x}_n, \mathbf{x}) \geqslant d_2(\mathbf{x}_n, \mathbf{x}) \geqslant d_\infty(\mathbf{x}_n, \mathbf{x}) = \max\{|a_n - a|, |b_n - b|\} > |a_n - a| > \varepsilon$$

showing the divergence in all three norms.

□

A similar result holds for $\mathbb{R}^m$ in general.
Now let's look at continuous functions again.

Theorem 1.39. *If* $f_n \to f$ *in* $(C[a, b], d_\infty)$, *then* $f_n \to f$ *in* $(C[a, b], d_1)$. *Informally speaking,* d_∞ *convergence is stronger than* d_1 *convergence.*

Proof. $d_\infty(f_n, f) = \max\{|f_n(x) - f(x)| : a \leqslant x \leqslant b\} \to 0$ as $n \to \infty$, so, given $\varepsilon > 0$ there is an N so that $d_\infty(f_n, f) < \varepsilon$ for $n \geqslant N$. It follows that if $n \geqslant N$ then

$$d_1(f_n, f) = \int_a^b |f_n(x) - f(x)|\, dx \leqslant \int_a^b \varepsilon\, dx = \varepsilon(b - a),$$

so $d_1(f_n, f) \to 0$ as $n \to \infty$. □

Remark 1.40. It is also true that if $d_\infty(f_n, f) \to 0$ then $f_n \to f$ point-wise on $[a, b]$. The converse is *false*, cf. 1.37(1.37(ii)).

Now we look at continuous functions between general metric spaces.

Definition 1.41 (Continuity). Let $f : (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. We say that f is *continuous at* $x \in X$ if for each $\varepsilon > 0$ there is a $\delta_{\varepsilon,x} > 0$ such that $d_Y(f(x'), f(x)) < \varepsilon$ for all $x' \in X$ whenever $d_X(x', x) < \delta_{\varepsilon,x}$.

Another way of saying the same is that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(B_\delta(x)) \subset B_\varepsilon(f(x)).$$

The map f is *continuous*, if it is continuous at all points of X.

Theorem 1.42 (Sequential continuity). *For f as above, f is continuous at a if and only if, whenever a sequence $x_n \to a$, then $f(x_n) \to f(a)$.*
In short, f is continuous at a if and only if f permutes with the limit*:*

$$f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) \tag{1.1}$$

for any sequence $x_n \to a$.

Proof. Same proof as in real analysis, more or less. If f is continuous at a and $x_n \to a$, then for each $\varepsilon > 0$ we have a $\delta > 0$ such that $d_Y(f(x), f(a)) < \varepsilon$ whenever $d_X(x, a) < \delta$.
Then there's an n_0 with $d(x_n, a) < \delta$ for all $n \geqslant n_0$, and so $d(f(x_n), f(a)) < \varepsilon$ for all $n \geqslant n_0$. Thus $f(x_n) \to f(x)$.
Conversely, if f is not continuous at a, then there is an ε for which no δ will do, so we can find x_n with $d(x_n, a) < \frac{1}{n}$, but $d(f(x_n), f(a)) \geqslant \varepsilon$. Then $x_n \to a$ but $f(x_n) \not\to f(a)$. □

But there is a nicer way to define continuity. For a mapping $f : X \to Y$ and a set $U \subset Y$, let $f^{-1}(U)$ be the set

$$f^{-1}(U) = \{x \in X : f(x) \in U\}.$$

This makes sense even if f^{-1} is not defined as a function.

Theorem 1.43 (Continuity and open sets). *A function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X for every open subset $U \subset Y$. In short: the inverse image of an open set is open.*

Proof. Suppose that f is continuous, that $U \subset Y$ is open, and that $x_0 \in f^{-1}(U)$, so $f(x_0) \in U$. Now there is a ball $B_\varepsilon(f(x_0)) \subset U$, since U is open, and then by continuity there is a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. This means that for $d(x, x_0) < \delta$, $f(x) \in U$ and so $x \in f^{-1}(U)$. That is, $f^{-1}(U)$ is open.
Conversely, if the inverse image of an open set is open, and $x_0 \in X$, let $\varepsilon > 0$ be given. We know that $B_\varepsilon(f(x_0))$ is open, so $f^{-1}(B(f(x_0), \varepsilon))$ is open, and contains x_0. So it contains some $B_\delta(x_0)$ with $\delta > 0$.
But now if $d(x, x_0) < \delta$, we have $x \in B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$ so $f(x) \in B_\varepsilon(f(x_0))$ and we have $d(f(x), f(x_0)) < \varepsilon$. □

Remark 1.44. Note that for f continuous we do not expect $f(U)$ to be open for all open subsets of X, for example $f : \mathbb{R} \to \mathbb{R}$, $f \equiv 0$, then $f(\mathbb{R}) = \{0\}$, not open.

Example 1.45. Let $X = \mathbb{R}$ with the discrete metric, and Y any metric space. Then all functions $f : X \to Y$ are continuous! Indeed, in either way:

- Because the inverse image of an open set is an open set, since all sets are open.
- Because whenever $x_n \to x_0$ we have $x_n = x_0$ for n large, so obviously $f(x_n) \to f(x_0)$.

Exercise 1.46. Which functions from a metric space X to the discrete metric space $Y = \mathbb{R}$ are continuous?

Proposition 1.47. *Let X and Y be metric spaces.*

(i) *A function $f : X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed whenever F is a closed subset of Y.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is the composition $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$.*

Proof. (i) We can do this by complements, as if F is closed, then $U = F^c$ is open, and $f^{-1}(F) = f^{-1}(U)^c$ (a point is mapped into F if and only if it isn't mapped into U).

Then $f^{-1}(F)$ is always closed when F is closed $\iff$ $f^{-1}(U)$ is always open when U is open.

(ii) Take $U \subset Z$ open; then $(g'f)^{-1}(U) = f^{-1}(g^{-1}(U))$; for these are the points which map under f into $g^{-1}(U)$ so that they map under $g'f$ into U.

Now $g^{-1}(U)$ is open in Y, as g is continuous, and then $f^{-1}(g^{-1}(U))$ is open in X since f is continuous.

□

In many cases we may need a stronger notion.

Definition 1.48 (Uniform continuity)**.** A function $f : (X, d_X) \to (Y, d_Y)$ is called *uniformly continuous* if for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that whenever $x, x' \in X$ satisfy $d_X(x, x') \leqslant \delta_\varepsilon$, we have that $d_Y(f(x), f(x')) \leqslant \varepsilon$.

Note, that here the same δ_ε shall work for all $x \in X$. Thus any uniformly continuous function is continuous at every point. On the other hand the function $f(x) = \frac{1}{x}$ on $(0, 1)$ is continuous but *not* uniformly continuous.

1.2. **Useful properties of metric spaces.** Metric spaces may or may not have some useful properties which we are discussing in the following subsections: *completeness* and *compactness*.

1.2.1. *Cauchy sequences and completeness.* Recall that if (X, d) is a metric space, then a sequence (x_n) of elements of X converges to $x \in X$ if $d(x_n, x) \to 0$, i.e., if given $\varepsilon > 0$ there exists N such that $d(x_n, x) < \varepsilon$ whenever $n \geqslant N$. Thus, to show that a sequence is convergent from the definition we need to present its limit x which may not belong to the sequence (x_n). It would be convenient to deduce convergence of (x_n) just through its own properties without a reference to extraneous x. This is possible for complete metric spaces studied in this subsection.

Often we think of convergent sequences as ones where x_n and x_m are close together when n and m are large. This is almost, but not quite, the same thing in a general metric space.

Definition 1.49 (Cauchy Sequence)**.** A sequence (x_n) in a metric space (X, d) is a *Cauchy sequence* if for any $\varepsilon > 0$ there is an N such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geqslant N$.

Example 1.50. Take $x_n = 1/n$ in $\mathbb{R}$ with the usual metric. Now $d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right|$. Suppose that n and m are both at least as big as N; then $d(x_n, x_m) \leqslant 1/N$. Hence if $\varepsilon > 0$ and we take $N > 1/\varepsilon$, we have $d(x_n, x_m) \leqslant 1/N < \varepsilon$ whenever n and m are both $\geqslant N$.

In fact all convergent sequences are Cauchy sequences, by the following result.

Theorem 1.51. *Suppose that (x_n) is a convergent sequence in a metric space (X, d), i.e., there is a limit point x such that $d(x_n, x) \to 0$. Then (x_n) is a Cauchy sequence.*

Proof. Take $\varepsilon > 0$. Then there is an N such that $d(x_n, x) < \varepsilon/2$ whenever $n \geqslant N$. Now suppose both $n \geqslant N$ and $m \geqslant N$. Then

$$d(x_n, x_m) \leqslant d(x_n, x) + d(x, x_m) = d(x_n, x) + d(x_m, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and we are done. □

Proposition 1.52. *Every subsequence of a Cauchy sequence is a Cauchy sequence.*

Proof. If (x_n) is Cauchy and (x_{n_k}) is a subsequence, then given $\varepsilon > 0$ there is an N such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \geqslant N$. Now there is a K such that $n_k \geqslant N$ whenever $k \geqslant K$. So $d(x_{n_k}, x_{n_l}) < \varepsilon$ whenever $k, l \geqslant K$. □

Does *every* Cauchy sequence converge?

Example 1.53. (i) $(X, d) = \mathbb{Q}$, as a subspace of $\mathbb{R}$ with the usual metric. Take $x_0 = 2$ and define $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$. The sequence continues $3/2$, $17/12$, $577/408, \ldots$ and indeed the sequence converges in $\mathbb{R}$ as $x_n \to x$ where $x = \frac{x}{2} + \frac{1}{x}$, i.e., $x^2 = 2$. But this isn't in $\mathbb{Q}$.

Thus (x_n) is Cauchy in $\mathbb{R}$, since it converges to $\sqrt{2}$ when we think of it as a sequence in $\mathbb{R}$. So it is Cauchy in $\mathbb{Q}$, but doesn't converge to a point of $\mathbb{Q}$.

(ii) Easier. Take $(X, d) = (0, 1)$. Then $\left(\frac{1}{n}\right)$ is a Cauchy sequence in X (since it is Cauchy in $\mathbb{R}$, as seen above), and has no limit in X.

In each case there are "points missing from X".

Definition 1.54 (Completeness)**.** A metric space (X, d) is *complete* if every Cauchy sequence in X converges to a limit in X.

Theorem 1.55. *The metric space $\mathbb{R}$ is complete.*

Remark 1.56. In parts of the literature $\mathbb{R}$ is simply defined as the completion of $\mathbb{Q}$. In this case one does not have to prove that $\mathbb{R}$ is complete, but it is complete by construction. One then has to work a bit to show that it is also a field.

This is a result from the first year. Since its proof depends on the definition of $\mathbb{R}$ we will not demonstrate it here.

Example 1.57. (i) Open intervals in $\mathbb{R}$ are not complete; closed intervals are complete.

(ii) What about $C[a, b]$ with d_1, d_2 or d_∞?

Following our consideration in Ex. 1.37.1.37(ii), define f_n in $C[0, 2]$ by

$$f_n(x) = \begin{cases} x^n & \text{for } 0 \leqslant x \leqslant 1, \\ 1 & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

[DIAGRAM]

Then

$$\begin{aligned} d_1(f_n, f_m) &= \int_0^2 |f_n(x) - f_m(x)|\, dx \\ &= \int_0^1 |x^n - x^m|\, dx \\ &= \int_0^1 (x^m - x^n)\, dx \qquad \text{if} \quad n \geqslant m \\ &= \frac{1}{m+1} - \frac{1}{n+1} \leqslant \frac{1}{m+1} \to 0, \end{aligned}$$

and hence (f_n) is Cauchy in $(C[0,2], d_1)$. Does the sequence converge?

If there is an $f \in C[0,2]$ with $f_n \to f$ as $n \to \infty$, then $\int_0^2 |f_n(x) - f(x)|\, dx \to 0$, so $\int_0^1$ and $\int_1^2$ both tend to zero. So $f_n \to f$ in $(C[0,1], d_1)$, which means that $f(x) = 0$ on $[0,1]$ (from an example we did earlier). Likewise, $f = 1$ on $[1,2]$, which doesn't give a continuous limit.

(iii) Similarly, $(C[a,b], d_1)$ is incomplete in general. Also it is incomplete in the d_2 metric, as the same example shows (a similar calculation with squares of functions). We will see later that it is complete in the d_∞ metric.

Remark 1.58. Note that $\mathbb{R}^2$ is also complete with any of the metrics d_1, d_2 and d_∞; since a Cauchy/ convergent sequence $(\mathbf{v}_n) = (x_n, y_n)$ in $\mathbb{R}^2$ is just one in which both (x_n) and (y_n) are Cauchy/ convergent sequences in $\mathbb{R}$ (cf. Prop. 1.38).
Similar arguments show that $\mathbb{R}^k$ is also complete for $k = 1, 2, 3, \ldots$, and (with the same proof as for Corollary) all closed subsets of $\mathbb{R}^k$ are complete.

If a metric space (X, d) is not complete one can always pass to its abstract completion in the following sense.

Proposition 1.59 (Abstract completion)**.** *Any metric space* (X, d) *is isometric to a dense subspace of a complete metric space, which is called its* abstract completion *if* (X, d).

Sketch of proof. We describe a metric space (X', d') in which X is isometric to a dense subset. Consider the space $\tilde{X}$ of Cauchy sequences of X. We define an equivalence relation $\sim$ on $\tilde{X}$ by

$$(x_n) \sim (y_n) \Leftrightarrow d(x_n, y_n) \to 0.$$

The set X' is defined to be the set of equivalence classes $[(x_n)]$. It has a well defined metric given by

$$d'([(x_n)], [(y_n)]) := \lim_{n\to\infty} d(x_n, y_n).$$

One checks easily that this is metric and is well defined (does not depend on the chosen representative x_n of $[(x_n)]$). Now there is an injective map $X \to X'$ defined by sending x to the constant sequence $(x, x, x, \ldots)$. This map is an isometry. We can therefore think of (X, d) as a subset of (X', d'). This subset is dense because every Cauchy sequence can be approximated by a sequence of constant sequences. So the only difficult bit in this construction is to show that (X', d') is complete. We will sketch the construction of a limit here. It turns out that it verifies completeness on a dense set.

Lemma 1.60. *Suppose that (X, d) is a metric space and let $Y \subset X$ be a dense set with the property that every Cauchy sequence in Y has a limit in X. Then (X, d) is complete.*

Proof. Let (x_n) be a Cauchy sequence in X. Now replace x_n with another sequence y_n in Y such that $d(x_n, y_n) < \frac{1}{n}$. Then, by the triangle inequality, y_n is again a Cauchy sequence and converges, by assumption, to some $x \in X$. Then also x_n converges to x. □

Let us turn to the proof of completeness of X'. Suppose that (x_n) is a Cauchy sequence in X. Then, in X' this sequence has the form $((x_1, x_1, \ldots), (x_2, x_2, \ldots), (x_3, x_3, \ldots), \ldots)$. This sequence has a limit, namely, (x_n) itself. □

Exercise 1.61 (Extension by continuity)**.** Let (X, d) be a metric space and X_1 be a dense subset of X. Let $f : X_1 \to Y$ be a uniformly continuous function to a complete metric space (Y, d'). Show that there is a *unique* function $\tilde{f} : X \to Y$ which satisfies two properties:

(i) restriction of $\tilde{f}$ to X_1 coincides with f, that is $\tilde{f}(x) = f(x)$ for all $x \in X_1$;
(ii) $\tilde{f}$ is continuous on X.

Furthermore, it can be shown that $\tilde{f}$ is uniformly continuous on X. We will call $\tilde{f}$ the *extension of f by continuity* and will often keep the same letter f to denote $\tilde{f}$.

There are many important consequences of Ex. 1.61, in particular the following.

Corollary 1.62. *All abstract completions of a metric space* (X, d) *are isometric, in other words, the abstract completions is unique up to isometry.*

1.2.2. *Compactness.* Accordingly to a dictionary: *compact—closely and firmly united or packed together*. For a metric space a meaning of "closely and firmly united" can be defined in several different forms—through open coverings or convergent subsequences—and we will see that these interpretations are equivalent.

An *open cover* of a metric space (X, d) is a family of open sets $(U_\alpha)_{\alpha \in I}$ such that

$$\bigcup_{\alpha \in I} U_\alpha = X.$$

A *subcover* of a cover is a subset $I' \subset I$ of the index set such that $(U_\alpha)_{\alpha \in I'}$ is still a cover.

Definition 1.63 (Compactness)**.** A metric space (X, d) is called *compact* if every open cover has a finite subcover.

Informally: a space is compact if any infinite open covering is excessive and can be reduced just to a finite one. An example of a compact set is $[0, 1]$ and example of non-compact—all reals or the open interval $(0, 1)$. An importance of this concept is clarified by Rem. 1.21.

Definition 1.64 (Sequential Compactness)**.** A metric space (X, d) is called *sequentially compact* if every sequence $(x_n)_{n \in \mathbb{N}}$ in X has a convergent subsequence.

Informally: a space is sequentially compact if there is no room to place infinite number of points sufficiently apart from each other to avoid their condensation to a limit. Taking the sequence $x_n = n$ shows that the set of all reals is not sequentially compact. On the other hand, we know from previous years that bounded closed set in $\mathbb{R}^n$ every sequence has a convergent subsequence. Therefore, bounded closed sets in $\mathbb{R}^n$ are sequentially compact.

Exercise 1.65. What are compact sets in a discrete metric space? What are sequentially compact sets in a discrete metric space?

Lemma 1.66. *Let* (X, d) *be a sequentially compact metric space. Then for every* $\varepsilon > 0$ *there exist finitely many points* $x_1, \ldots, x_n$ *such that* $\{B_\varepsilon(x_i) \mid i = 1, \ldots, n\}$ *is a cover.*

Proof. Suppose this were not the case. Then there would exist an $\varepsilon > 0$ such that for any finite number of points $x_1, \ldots, x_n$ the collection of balls $B_\varepsilon(x_i)$ does not cover, i.e.

$$\bigcup_{i=1}^{n} B_\varepsilon(x_i) \neq X.$$

Starting with $n = 1$ and then inductively adding points that are in the complement of $\cup_{i=1}^{n} B_\varepsilon(x_i)$ we end up with an infinite sequence of points x_i such that $d(x_i, x_k) \geqslant \varepsilon$. This sequence cannot have a Cauchy subsequence (required for convergence) in contradiction with the sequential compactness of X. □

Theorem 1.67. *A metric space* (X, d) *is compact if and only if it is sequentially compact.*

Proof. We show the two directions separately.

Compactness implies sequential compactness: Suppose that X is compact and let $(x_i)_{i\in\mathbb{N}}$ be a sequence. We want to show that it has a convergent subsequence. Suppose (x_i) did not have a convergent subsequence. Then no point x is an *accumulation point*, i.e. a limit of a subsequence. Therefore, for each $x \in X$ there exists an $\varepsilon(x) > 0$ such that only finitely many $i \in \mathbb{N}$ for which $x_i \in B_{\varepsilon(x)}$. Since $(B_{\varepsilon(x)})_{x\in X}$ is an open cover it has a finite subcover, that is a finite number of balls with a finite number of x_i in each. This contradicts to the infinite number of elements in the sequence (x_i).

Sequential compactness implies compactness: This implication is quite tricky. The proof is again by contradiction. Let us assume our space is sequentially compact and there exists a cover U_α that does not have a finite subcover. By the above lemma there are finitely many points $x_1, \ldots, x_{N_1}$ such that $B_1(x_i)$ is a cover. Each of the balls $B_1(x_i)$ is covered by U_α as well. Since our cover does not have a finite subcover one of the balls $B_1(x_i)$ does not have a finite subcover. Denote the relevant point x_i by z_1.

Again there are finitely many points $x'_1, \ldots, x'_{N_2}$ such that $B_{\frac{1}{2}}(x_i)$ is a cover of X. The collection of sets $B_1(z_1) \cap B_{\frac{1}{2}}(x_i)$, with $i = 1, \ldots, N_2$ is also a covering of $B_1(z_1)$. In the same way as before there is at least one of the x_i, such that $B_1(z_1) \cap B_{\frac{1}{2}}(z_2)$ can not be covered by a finite subcover of U_α. Call that point z_2. Continuing like this we construct a sequence of points z_i such that none of the sets

$$B_1(z_1) \cap B_{\frac{1}{2}}(z_2) \cap \ldots \cap B_{\frac{1}{N}}(z_N)$$

can be covered by a finite subcover of U_α.

By assumption the sequence (z_i) has a convergent subsequence. Say z is a limit point of that subsequence. Since U_α is an open cover the point z is contained

in one of the U_α and of course that means that an open ball $B_\varepsilon(z)$ around z is contained in U_α for some $\varepsilon > 0$.
Now we show that there exits an $N \in \mathbb{N}$ such that $B_{\frac{1}{N}}(z_N)$ is a subset of U_α (this will be the desired contradiction!). Indeed, choose N large enough so that $d(z_N, z) + \frac{1}{N} < \varepsilon$. Then $x \in B_{\frac{1}{N}}(z_N)$ implies that $d(x, z) \leqslant d(z_N, z) + d(x, z_N) < d(z_N, z) + \frac{1}{N} < \varepsilon$. This means in particular that

$$B_1(z_1) \cap B_{\frac{1}{2}}(z_2) \cap \ldots \cap B_{\frac{1}{N}}(z_N)$$

is a subset of U_α. Thus, there is a subcover of the set $B_1(z_1) \cap \ldots \cap B_{\frac{1}{N}}(z_N)$ consisting of one element U_α. This is a contradiction as we constructed the sequence of balls in such a way that these sets cannot be covered by a finite number of the U_α. □

Definition 1.68 (Boundedness)**.** A subset $A \subset X$ of a metric space is called *bounded* if there exists $x_0 \in X$ and $C > 0$ such that for all $x \in A$ we have $d(x_0, x) \leqslant C$.

Remark 1.69. One can easily see, using the triangle inequality, that the reference point x_0 can be chosen as any point in X. This means if $A \subset X$ is bounded and $x_0 \in X$, then there exist a $C > 0$ such that $d(x_0, x) \leqslant C$ for any $x \in A$.

Theorem 1.70. *Suppose that $A \subset X$ is a compact subset of a metric space. Then A is closed and bounded.*

Proof. First we show A is bounded. Choose any $x_0 \in X$ and note that the set $B_n(x_0)$ indexed by $n \in \mathbb{N}$ is an open cover of A. Hence, there exists a finite subcover¡ $B_{n_1}(x_0), \ldots, B_{n_N}(x_0)$. Hence, $A \subset B_C(x_0)$, where $C = \max\{n_1, \ldots, c_N\}$. Hence, A is bounded.
Next assume that (x_k) is a sequence in A that converges in X. Since A is compact there exists a subsequence that converges in A. Hence, the limit of x_k must also be in A. Therefore, A is closed. □

The converse of this statement is not correct in general. It is however famously correct in $\mathbb{R}^m$.

Theorem 1.71 (Heine–Borel)**.** *A subset $K \subset \mathbb{R}^m$ is compact if and only if it is closed and bounded.*

Proof. We just need to combine the above statements. We have already shown that compactness implies closedness and boundedness. If K is closed and bounded we know from Analysis that it is sequentially compact. Therefore it is compact. □

As an illustration of further nice properties of compact spaces we mention the following result:

Exercise 1.72. (i) Any continuous function on a compact set is bounded.
(ii) Any continuous function $f : K \to X$ from a compact space K to a metric space X is uniformly continuous.

Remark 1.73. Note that there are two different sorts of properties of metric spaces:

- the first sort of *absolute* properties can be verified on a metric space itself;
- the second sort of *relative* properties is meaningful only for subsets of another metric spaces. Such a property may be true for X as a subspace of X but false if X is considered as a subspace of a different space Z.

Completeness and compactness are of the first sort, closedness is of the second, cf. Rem 1.17.

2. Basics of Linear Spaces

> A person is solely the concentration of an infinite set of interrelations with another and others, and to separate a person from these relations means to take away any real meaning of the life.
>
> Vl. Soloviev

A space around us could be described as a three dimensional Euclidean space. To single out a point of that space we need a fixed *frame of references* and three real numbers, which are *coordinates* of the point. Similarly to describe a pair of points from our space we could use six coordinates; for three points—nine, end so on. This makes it reasonable to consider Euclidean (linear) spaces of an arbitrary finite dimension, which are studied in the courses of linear algebra.

The basic properties of Euclidean spaces are determined by its *linear* and *metric* structures. The *linear space* (or *vector space*) structure allows to add and subtract vectors associated to points as well as to multiply vectors by real or complex numbers (scalars).

The *metric space* structure assign a *distance*—non-negative real number—to a pair of points or, equivalently, defines a *length of a vector* defined by that pair. A metric (or, more generally a topology) is essential for definition of the core analytical notions like limit or continuity. The importance of linear and metric (topological) structure in analysis sometime encoded in the formula:

$$\textbf{Analysis} = \textbf{Algebra} + \textbf{Geometry}\,. \tag{2.1}$$

On the other hand we could observe that many sets admit a sort of linear *and* metric structures which are linked each other. Just few among many other examples are:

- The set of convergent sequences;
- The set of continuous functions on $[0, 1]$.

It is a very *mathematical way of thinking* to declare such sets to be *spaces* and call their elements *points*.

But shall we lose all information on a particular element (e.g. a sequence $\{1/n\}$) if we represent it by a shapeless and size-less "point" without any inner configuration? Surprisingly not: all properties of an element could be now retrieved not from its *inner configuration* but from interactions with other elements through linear and metric structures. Such a "sociological" approach to all kind of mathematical objects was codified in the abstract *category theory*.

Another surprise is that starting from our three dimensional Euclidean space and walking far away by a road of abstraction to infinite dimensional Hilbert spaces we are arriving just to yet another picture of the surrounding space—that time on the language of *quantum mechanics*.

> The distance from Manchester to Liverpool is 35 miles—just about the mileage in the opposite direction!
>
> *A tourist guide to England*

2.1. **Banach spaces (basic definitions only).** The following definition generalises the notion of *distance* known from the everyday life.

Definition 2.1. A *metric* (or *distance function*) d on a set M is a function $d : M \times M \to \mathbb{R}_+$ from the set of pairs to non-negative real numbers such that:

(i) $d(x, y) \geqslant 0$ for all $x, y \in M$, $d(x, y) = 0$ implies $x = y$.
(ii) $d(x, y) = d(y, x)$ for all x and y in M.
(iii) $d(x, y) + d(y, z) \geqslant d(x, z)$ for all x, y, and z in M (*triangle inequality*).

Exercise 2.2. Let M be the set of UK's cities are the following function are metrics on M:

(i) $d(A, B)$ is the price of 2nd class railway ticket from A to B.
(ii) $d(A, B)$ is the off-peak driving time from A to B.

The following notion is a useful specialisation of metric adopted to the linear structure.

Definition 2.3. Let V be a (real or complex) vector space. A *norm* on V is a real-valued function, written $\|x\|$, such that

(i) $\|x\| \geqslant 0$ for all $x \in V$, and $\|x\| = 0$ implies $x = 0$.
(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all scalar λ and vector x.
(iii) $\|x + y\| \leqslant \|x\| + \|y\|$ (*triangle inequality*).

A vector space with a norm is called a *normed space*.

The connection between norm and metric is as follows:

Proposition 2.4. *If* $\|\cdot\|$ *is a norm on* V, *then it gives a metric on* V *by* $d(x, y) = \|x - y\|$.

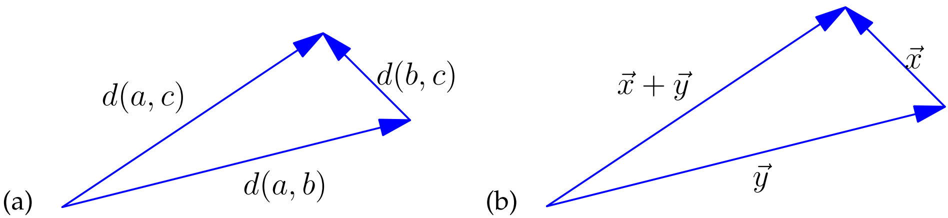


FIGURE 1. Triangle inequality in metric (a) and normed (b) spaces.

Proof. This is a simple exercise to derive items 2.1(i)–2.1(iii) of Definition 2.1 from corresponding items of Definition 2.3. For example, see the Figure 1 to derive the triangle inequality. □

An important notions known from real analysis are limit and convergence. Particularly we usually wish to have enough limiting points for all "reasonable" sequences.

Definition 2.5. A sequence $\{x_k\}$ in a metric space (M, d) is a *Cauchy sequence*, if for every $\epsilon > 0$, there exists an integer n such that $k, l > n$ implies that $d(x_k, x_l) < \epsilon$.

Definition 2.6. (M, d) is a *complete metric space* if every Cauchy sequence in M converges to a limit in M.

For example, the set of integers $\mathbb{Z}$ and reals $\mathbb{R}$ with the natural distance functions are complete spaces, but the set of rationals $\mathbb{Q}$ is not. The complete normed spaces deserve a special name.

Definition 2.7. A *Banach space* is a complete normed space.

Exercise* 2.8. A convenient way to define a norm in a Banach space is as follows. The *unit ball* U in a normed space B is the set of x such that $\|x\| \leqslant 1$. Prove that:

(i) U is a *convex set*, i.e. $x, y \in U$ and $\lambda \in [0,1]$ the point $\lambda x + (1-\lambda)y$ is also in U.
(ii) $\|x\| = \inf\{\lambda \in \mathbb{R}_+ \mid \lambda^{-1}x \in U\}$.
(iii) U is closed if and only if the space is Banach.

Example 2.9. Here is some examples of normed spaces.

(i) ℓ_2^n is either $\mathbb{R}^n$ or $\mathbb{C}^n$ with norm defined by

$$\|(x_1,\ldots,x_n)\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}. \tag{2.2}$$

(ii) ℓ_1^n is either $\mathbb{R}^n$ or $\mathbb{C}^n$ with norm defined by

$$\|(x_1,\ldots,x_n)\|_1 = |x_1| + |x_2| + \cdots + |x_n|. \tag{2.3}$$

(iii) ℓ_∞^n is either $\mathbb{R}^n$ or $\mathbb{C}^n$ with norm defined by

$$\|(x_1,\ldots,x_n)\|_\infty = \max(|x_1|, |x_2|, \cdots, |x_n|). \tag{2.4}$$

(iv) Let X be a topological space, then $C_b(X)$ is the space of continuous bounded functions $f : X \to \mathbb{C}$ with norm $\|f\|_\infty = \sup_X |f(x)|$.
(v) Let X be any set, then $\ell_\infty(X)$ is the space of all bounded (not necessarily continuous) functions $f : X \to \mathbb{C}$ with norm $\|f\|_\infty = \sup_X |f(x)|$.

All these normed spaces are also complete and thus are Banach spaces. Some more examples of both complete and incomplete spaces shall appear later.

—We need an extra space to accommodate this product!
A manager to a shop assistant

2.2. **Hilbert spaces.** Although metric and norm capture important geometric information about linear spaces they are not sensitive enough to represent such geometric characterisation as angles (particularly *orthogonality*). To this end we need a further refinements.

From courses of linear algebra known that the scalar product $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ is important in a space $\mathbb{R}^n$ and defines a norm $\|x\|^2 = \langle x, x\rangle$. Here is a suitable generalisation:

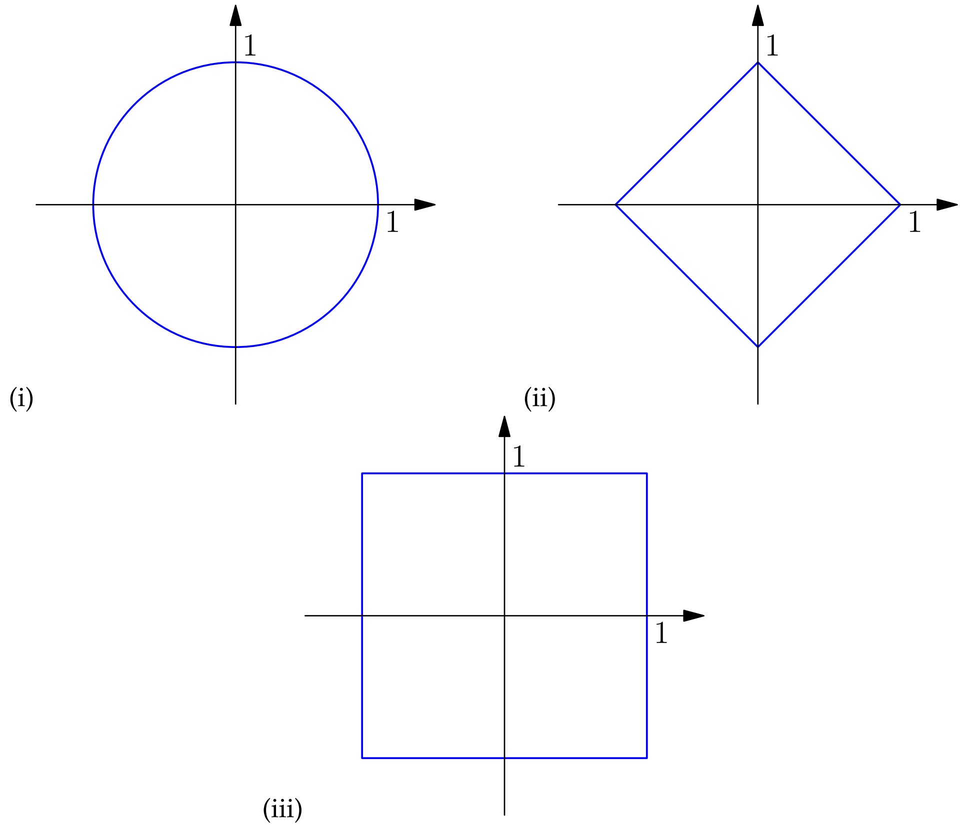


FIGURE 2. Different unit balls defining norms in $\mathbb{R}^2$ from Example 2.9.

Definition 2.10. A *scalar product* (or *inner product*) on a real or complex vector space V is a mapping $V \times V \to \mathbb{C}$, written $\langle x, y \rangle$, that satisfies:

(i) $\langle x, x \rangle \geqslant 0$ and $\langle x, x \rangle = 0$ implies $x = 0$.
(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ in complex spaces and $\langle x, y \rangle = \langle y, x \rangle$ in real ones for all $x, y \in V$.
(iii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$, for all $x, y \in V$ and scalar λ. (What is $\langle x, \lambda y \rangle$?).
(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, for all x, y, and $z \in V$. (What is $\langle x, y + z \rangle$?).

Last two properties of the scalar product is oftenly encoded in the phrase: "it is linear in the first variable if we fix the second and anti-linear in the second if we fix the first".

Definition 2.11. An *inner product space* V is a real or complex vector space with a scalar product on it.

Example 2.12. Here is some examples of inner product spaces which demonstrate that expression $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm.

(i) The inner product for $\mathbb{R}^n$ was defined in the beginning of this section. The inner product for $\mathbb{C}^n$ is given by $\langle x, y\rangle = \sum_1^n x_j \bar{y}_j$. The norm $\|x\| = \sqrt{\sum_1^n |x_j|^2}$ makes it ℓ_2^n from Example 2.9(i).

(ii) The extension for infinite vectors: let ℓ_2 be

$$\ell_2 = \{\text{sequences } \{x_j\}_1^\infty \mid \sum_1^\infty |x_j|^2 < \infty\}. \tag{2.5}$$

Let us equip this set with operations of term-wise addition and multiplication by scalars, then ℓ_2 is closed under them. Indeed it follows from the triangle inequality and properties of absolutely convergent series. From the standard Cauchy–Bunyakovskii–Schwarz inequality follows that the series $\sum_1^\infty x_j \bar{y}_j$ absolutely converges and its sum defined to be $\langle x, y\rangle$.

(iii) Let $C_b[a, b]$ be a space of continuous functions on the interval $[a, b] \in \mathbb{R}$. As we learn from Example 2.9(iv) a normed space it is a normed space with the norm $\|f\|_\infty = \sup_{[a,b]} |f(x)|$. We could also define an inner product:

$$\langle f, g\rangle = \int_a^b f(x)\bar{g}(x)\, dx \quad \text{and} \quad \|f\|_2 = \left(\int_a^b |f(x)|^2\, dx\right)^{\frac{1}{2}}. \tag{2.6}$$

Now we state, probably, the most important inequality in analysis.

Theorem 2.13 (Cauchy–Schwarz–Bunyakovskii inequality). *For vectors x and y in an inner product space V let us define $\|x\| = \sqrt{\langle x, x\rangle}$ and $\|y\| = \sqrt{\langle y, y\rangle}$ then we have*

$$|\langle x, y\rangle| \leqslant \|x\|\,\|y\|, \tag{2.7}$$

with equality if and only if x and y are scalar multiple each other.

Proof. For simplicity we start from a real vector space. Let we have two vectors u and v and want to define an inner product on the two-dimensional vector space spanned by them. That is we need to know a value of $\langle au + bv, cu + dv\rangle$ for all possible scalars a, b, c, d.

By the linearity $\langle au+bv, cu+dv\rangle = ac\langle u,u\rangle + (bc+ad)\langle u,v\rangle + db\langle v,v\rangle$, thus everything is defined as soon as we know three inner products $\langle u,u\rangle$, $\langle u,v\rangle$ and $\langle v,v\rangle$. First of all we need to demand $\langle u,u\rangle \geqslant 0$ and $\langle v,v\rangle \geqslant 0$.
Furthermore, they shall be such that $\langle au+bv, au+bv\rangle \geqslant 0$ for all scalar a and b. If $a=0$, that is reduced to the previous case $\langle v,v\rangle \geqslant 0$. If a is non-zero we note $\langle au+bv, au+bv\rangle = a^2\langle u+(b/a)v, u+(b/a)v\rangle$ and letting $\lambda = b/a$ we reduce our consideration to the quadratic expression

$$\langle u+\lambda v, u+\lambda v\rangle = \lambda^2\langle v,v\rangle + 2\lambda\langle u,v\rangle + \langle u,u\rangle .$$

The graph of this function of λ is an upward parabolabecause $\langle v,v\rangle \geqslant 0$. Thus, it will be non-negative for all λ if its lowest value is non-negative. From the theory of quadratic expressions, the latter is achieved at $\lambda = -\langle u,v\rangle / \langle v,v\rangle$ and is equal to

$$\frac{\langle u,v\rangle^2}{\langle v,v\rangle^2}\langle v,v\rangle - 2\frac{\langle u,v\rangle}{\langle v,v\rangle}\langle u,v\rangle + \langle u,u\rangle = -\frac{\langle u,v\rangle^2}{\langle v,v\rangle} + \langle u,u\rangle$$

If $-\frac{\langle u,v\rangle^2}{\langle v,v\rangle} + \langle u,u\rangle \geqslant 0$ then $\langle v,v\rangle\langle u,u\rangle \geqslant \langle u,v\rangle^2$.
Therefore, the Cauchy-Schwarz inequality is *necessary and sufficient condition* for the non-negativity of the inner product defined by the three values $\langle u,u\rangle$, $\langle u,v\rangle$ and $\langle v,v\rangle$.
After the previous discussion it is easy to get the result for complex vector space as well. For any $x, y \in V$ and any $t \in \mathbb{R}$ we have:

$$0 < \langle x+ty, x+ty\rangle = \langle x,x\rangle + 2t\Re\langle y,x\rangle + t^2\langle y,y\rangle),$$

Thus, the discriminant of this quadratic expression in t is non-positive: $(\Re\langle y,x\rangle)^2 - \|x\|^2\|y\|^2 \leqslant 0$, that is $|\Re\langle x,y\rangle| \leqslant \|x\|\,\|y\|$. Replacing y by $e^{i\alpha}y$ for an arbitrary $\alpha \in [-\pi,\pi]$ we get $\left|\Re(e^{i\alpha}\langle x,y\rangle)\right| \leqslant \|x\|\,\|y\|$, this implies the desired inequality.

□

Corollary 2.14. *Any inner product space is a normed space with norm* $\|x\| = \sqrt{\langle x,x\rangle}$ *(hence also a metric space, Prop. 2.4).*

Proof. Just to check items 2.3(i)–2.3(iii) from Definition 2.3. □

Again complete inner product spaces deserve a special name

Definition 2.15. A complete inner product space is *Hilbert space*.

The relations between spaces introduced so far are as follows:

$$\begin{array}{ccccc} \text{Hilbert spaces} & \Rightarrow & \text{Banach spaces} & \Rightarrow & \text{Complete metric spaces} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{inner product spaces} & \Rightarrow & \text{normed spaces} & \Rightarrow & \text{metric spaces.} \end{array}$$

How can we tell if a given norm comes from an inner product?

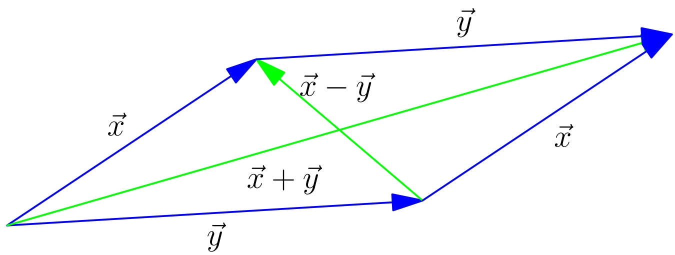


FIGURE 3. To the parallelogram identity.

Theorem 2.16 (Parallelogram identity). *In an inner product space* H *we have for all* x *and* $y \in H$ *(see Figure 3):*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 . \tag{2.8}$$

Proof. Just by linearity of inner product:

$$\langle x+y, x+y\rangle + \langle x-y, x-y\rangle = 2\langle x,x\rangle + 2\langle y,y\rangle,$$

because the cross terms cancel out. □

Exercise 2.17. Show that (2.8) is also a sufficient condition for a norm to arise from an inner product. Namely, for a norm on a complex Banach space satisfying to (2.8) the formula

$$\begin{aligned} \langle x,y\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) \\ &= \frac{1}{4}\sum_0^3 i^k \left\|x+i^k y\right\|^2 \end{aligned} \tag{2.9}$$

defines an inner product. What is a suitable formula for a real Banach space?

Divide and rule!

Old but still much used recipe

2.3. **Subspaces.** To study Hilbert spaces we may use the traditional mathematical technique of *analysis* and *synthesis*: we split the initial Hilbert spaces into smaller and probably simpler subsets, investigate them separately, and then reconstruct the entire picture from these parts.

As known from the linear algebra, a *linear subspace* is a subset of a linear space is its subset, which inherits the linear structure, i.e. possibility to add vectors and multiply them by scalars. In this course we need also that subspaces inherit topological structure (coming either from a norm or an inner product) as well.

Definition 2.18. By a *subspace* of a normed space (or inner product space) we mean a linear subspace with the same norm (inner product respectively). We write $X \subset Y$ or $X \subseteq Y$.

Example 2.19.
(i) $C_b(X) \subset \ell_\infty(X)$ where X is a metric space.
(ii) Any linear subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$ with any norm given in Example 2.9(i)–2.9(iii).
(iii) Let c_{00} be the *space of finite sequences*, i.e. all sequences (x_n) such that exist N with $x_n = 0$ for $n > N$. This is a subspace of ℓ_2 since $\sum_1^\infty |x_j|^2$ is a finite sum, so finite.

We also wish that the both inhered structures (linear and topological) should be in agreement, i.e. the subspace should be complete. Such inheritance is linked to the property be closed.

A subspace need *not* be closed—for example the sequence

$$x = (1, 1/2, 1/3, 1/4, \ldots) \in \ell_2 \qquad \text{because} \qquad \sum 1/k^2 < \infty$$

and $x_n = (1, 1/2, \ldots, 1/n, 0, 0, \ldots) \in c_{00}$ converges to x thus $x \in \overline{c_{00}} \subset \ell_2$.

Proposition 2.20.
(i) *Any closed subspace of a Banach/Hilbert space is complete, hence also a Banach/Hilbert space.*
(ii) *Any complete subspace is closed.*
(iii) *The closure of subspace is again a subspace.*

Proof.
(i) This is true in any metric space X: any Cauchy sequence from Y has a limit $x \in X$ belonging to $\bar{Y}$, but if Y is closed then $x \in Y$.
(ii) Let Y is complete and $x \in \bar{Y}$, then there is sequence $x_n \to x$ in Y and it is a Cauchy sequence. Then completeness of Y implies $x \in Y$.
(iii) If $x, y \in \bar{Y}$ then there are x_n and y_n in Y such that $x_n \to x$ and $y_n \to y$. From the triangle inequality:

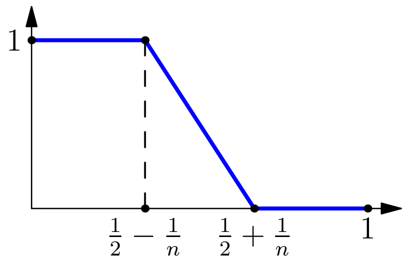
$$\|(x_n + y_n) - (x + y)\| \leqslant \|x_n - x\| + \|y_n - y\| \to 0,$$

so $x_n + y_n \to x + y$ and $x + y \in \bar{Y}$. Similarly $x \in \bar{Y}$ implies $\lambda x \in \bar{Y}$ for any λ.

□

Hence c_{00} is an *incomplete* inner product space, with inner product $\langle x, y\rangle = \sum_1^\infty x_k \bar{y}_k$ (this is a finite sum!) as it is not closed in ℓ_2.

(a)



(b)

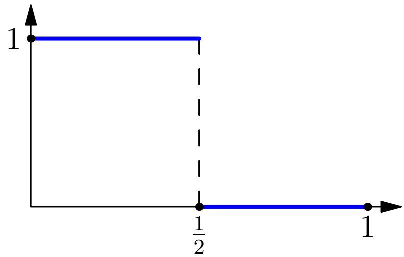


FIGURE 4. Jump function on (b) as a L_2 limit of continuous functions from (a).

Similarly $C[0,1]$ with inner product norm $\|f\| = \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}$ is incomplete—take the large space X of functions continuous on $[0,1]$ except for a possible jump at $\frac{1}{2}$ (i.e. left and right limits exists but may be unequal and $f(\frac{1}{2}) = \lim_{t\to\frac{1}{2}+} f(t)$. Then the sequence of functions defined on Figure 4(a) has the limit shown on Figure 4(b) since:

$$\|f - f_n\| = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} |f - f_n|^2\,dt < \frac{2}{n} \to 0.$$

Obviously $f \in \overline{C[0,1]} \setminus C[0,1]$.

Exercise 2.21. Show alternatively that the sequence of function f_n from Figure 4(a) is a Cauchy sequence in $C[0,1]$ but has no continuous limit.

Similarly the space $C[a,b]$ is *incomplete* for any $a < b$ if equipped by the inner product and the corresponding norm:

$$\langle f, g\rangle = \int_a^b f(t)\bar{g}(t)\,dt \tag{2.10}$$

$$\|f\|_2 = \left(\int_a^b |f(t)|^2\,dt\right)^{1/2}. \tag{2.11}$$

Definition 2.22. Define a Hilbert space $L_2[a,b]$ to be the smallest complete inner product space containing space $C[a,b]$ with the restriction of inner product given by (2.10).

It is practical to realise $L_2[a,b]$ as a certain space of "functions" with the inner product defined via an integral. There are several ways to do that and we mention just two:

(i) Elements of $L_2[a,b]$ are equivalent classes of Cauchy sequences $f^{(n)}$ of functions from $C[a,b]$.

(ii) Let integration be extended from the Riemann definition to the wider *Lebesgue integration* (see Section 13). Let L be a set of square integrable in Lebesgue sense functions on $[a,b]$ with a finite norm (2.11). Then $L_2[a,b]$ is a quotient space of L with respect to the equivalence relation $f \sim g \Leftrightarrow \|f-g\|_2 = 0$.

Example 2.23. Let the *Cantor function* on $[0,1]$ be defined as follows:

$$f(t) = \begin{cases} 1, & t \in \mathbb{Q}; \\ 0, & t \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

This function is *not* integrable in the Riemann sense but *does* have the Lebesgue integral. The later however is equal to 0 and as an L_2-function the Cantor function equivalent to the function identically equal to 0.

(iii) The third possibility is to map $L_2(\mathbb{R})$ onto a space of "true" functions but with an additional structure. For example, in *quantum mechanics* it is useful to work with the *Segal–Bargmann space* of analytic functions on $\mathbb{C}$ with the inner product [4–6]:

$$\langle f_1, f_2\rangle = \int_{\mathbb{C}} f_1(z)\bar{f}_2(z)e^{-|z|^2}\,dz.$$

Theorem 2.24. *The sequence space ℓ_2 is complete, hence a Hilbert space.*

Proof. Take a Cauchy sequence $x^{(n)} \in \ell_2$, where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots)$. Our proof will have three steps: identify the limit x; show it is in ℓ_2; show $x^{(n)} \to x$.

(i) If $x^{(n)}$ is a Cauchy sequence in ℓ_2 then $x_k^{(n)}$ is also a Cauchy sequence of numbers for any fixed k:

$$\left|x_k^{(n)} - x_k^{(m)}\right| \leqslant \left(\sum_{k=1}^{\infty} \left|x_k^{(n)} - x_k^{(m)}\right|^2\right)^{1/2} = \left\|x^{(n)} - x^{(m)}\right\| \to 0.$$

Let x_k be the limit of $x_k^{(n)}$.

(ii) For a given $\epsilon > 0$ find n_0 such that $\left\|x^{(n)} - x^{(m)}\right\| < \epsilon$ for all $n, m > n_0$. For any K and m:

$$\sum_{k=1}^{K} \left|x_k^{(n)} - x_k^{(m)}\right|^2 \leqslant \left\|x^{(n)} - x^{(m)}\right\|^2 < \epsilon^2.$$

Let $m \to \infty$ then $\sum_{k=1}^{K} \left|x_k^{(n)} - x_k\right|^2 \leqslant \epsilon^2$.

Let $K \to \infty$ then $\sum_{k=1}^{\infty} \left|x_k^{(n)} - x_k\right|^2 \leqslant \epsilon^2$. Thus $x^{(n)} - x \in \ell_2$ and because ℓ_2 is a linear space then $x = x^{(n)} - (x^{(n)} - x)$ is also in ℓ_2.

(iii) We saw above that for any $\epsilon > 0$ there is n_0 such that $\left\|x^{(n)} - x\right\| < \epsilon$ for all $n > n_0$. Thus $x^{(n)} \to x$.

Consequently ℓ_2 is complete. □

All good things are covered by a thick layer of chocolate (well, if something is not yet–it certainly will)

2.4. **Linear spans.** As was explained into introduction 2, we describe "internal" properties of a vector through its relations to other vectors. For a detailed description we need sufficiently many external reference points.

Let A be a subset (finite or infinite) of a normed space V. We may wish to upgrade it to a linear subspace in order to make it subject to our theory.

Definition 2.25. The *linear span* of A, write $\mathrm{Lin}(A)$, is the intersection of all linear subspaces of V containing A, i.e. the smallest subspace containing A, equivalently the set of all finite linear combination of elements of A. The *closed linear span* of A write $\mathrm{CLin}(A)$ is the intersection of all *closed* linear subspaces of V containing A, i.e. the smallest *closed* subspace containing A.

Exercise* 2.26. (i) Show that if A is a subset of finite dimension space then $\mathrm{Lin}(A) = \mathrm{CLin}(A)$.
(ii) Show that for an infinite A spaces $\mathrm{Lin}(A)$ and $\mathrm{CLin}(A)$could be different. (*Hint*: use Example 2.19(iii).)

Proposition 2.27. $\overline{\mathrm{Lin}(A)} = \mathrm{CLin}(A)$.

Proof. Clearly $\overline{\mathrm{Lin}(A)}$ is a closed subspace containing A thus it should contain $\mathrm{CLin}(A)$. Also $\mathrm{Lin}(A) \subset \mathrm{CLin}(A)$ thus $\overline{\mathrm{Lin}(A)} \subset \overline{\mathrm{CLin}(A)} = \mathrm{CLin}(A)$. Therefore $\overline{\mathrm{Lin}(A)} = \mathrm{CLin}(A)$. □

Consequently $\mathrm{CLin}(A)$ is the set of all limiting points of finite linear combination of elements of A.

Example 2.28. Let $V = C[a, b]$ with the sup norm $\|\cdot\|_\infty$. Then:
$\mathrm{Lin}\{1, x, x^2, \ldots\} = \{\text{all polynomials}\}$
$\mathrm{CLin}\{1, x, x^2, \ldots\} = C[a, b]$ by the Weierstrass approximation theorem proved later.

Remark 2.29. Note, that the relation $P \subset \mathrm{CLin}(Q)$ between two sets P and Q is transitive: if $P \subset \mathrm{CLin}(Q)$ and $Q \subset \mathrm{CLin}(R)$ then $P \subset \mathrm{CLin}(R)$. This observation is often used in the following way. To show that $P \subset \mathrm{CLin}(R)$ we introduce some intermediate sets $Q_1, \ldots, Q_n$ such that $P \subset \mathrm{CLin}(Q_1)$, $Q_j \subset \mathrm{CLin}(Q_{j+1})$ and $Q_n \subset \mathrm{CLin}(R)$, see the proof of Weierstrass Approximation Thm. 5.17 or § 14.2 for an illustration.

The following simple result will be used later many times without comments.

Lemma 2.30 (about Inner Product Limit)**.** *Suppose H is an inner product space and sequences x_n and y_n have limits x and y correspondingly. Then $\langle x_n, y_n\rangle \to \langle x, y\rangle$ or equivalently:*

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \left\langle \lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n \right\rangle.$$

Proof. Obviously by the Cauchy–Schwarz inequality:

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \\ &\leqslant |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle| \\ &\leqslant \|x_n - x\| \, \|y_n\| + \|x\| \, \|y_n - y\| \to 0, \end{aligned}$$

since $\|x_n - x\| \to 0$, $\|y_n - y\| \to 0$, and $\|y_n\|$ is bounded. □

3. Orthogonality

Pythagoras is forever!
The catchphrase from TV commercial of Hilbert Spaces course

As was mentioned in the introduction the Hilbert spaces is an analog of our 3D Euclidean space and theory of Hilbert spaces similar to plane or space geometry. One of the primary result of Euclidean geometry which still survives in high school curriculum despite its continuous nasty de-geometrisation is Pythagoras' theorem based on the notion of *orthogonality*[1].

So far we was concerned only with distances between points. Now we would like to study angles between vectors and notably *right angles*. Pythagoras' theorem states that if the angle C in a triangle is right then $c^2 = a^2 + b^2$, see Figure 5 .

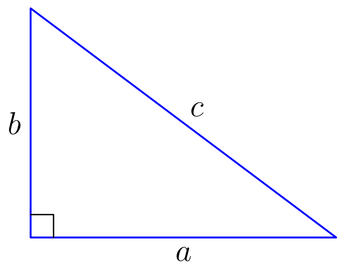


Figure 5. The Pythagoras' theorem $c^2 = a^2 + b^2$

It is a very *mathematical way of thinking* to turn this *property* of right angles into their *definition*, which will work even in infinite dimensional Hilbert spaces.

Look for a triangle, or even for a right triangle
A universal advice in solving problems from elementary geometry.

3.1. Orthogonal System in Hilbert Space.

In inner product spaces it is even more convenient to give a definition of orthogonality not from Pythagoras' theorem but from an equivalent property of inner product.

[1] Some more "strange" types of orthogonality can be seen in the paper *Elliptic, Parabolic and Hyperbolic Analytic Function Theory–1: Geometry of Invariants*.

Definition 3.1. Two vectors x and y in an inner product space are *orthogonal* if $\langle x, y\rangle = 0$, written $x \perp y$.
An *orthogonal sequence* (or *orthogonal system*) e_n (finite or infinite) is one in which $e_n \perp e_m$ whenever $n \neq m$.
An *orthonormal sequence* (or *orthonormal system*) e_n is an orthogonal sequence with $\|e_n\| = 1$ for all n.

Exercise 3.2. (i) Show that if $x \perp x$ then $x = 0$ and consequently $x \perp y$ for any $y \in H$.
(ii) Show that if all vectors of an orthogonal system are non-zero then they are linearly independent.

Example 3.3. These are orthonormal sequences:
(i) Basis vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ in $\mathbb{R}^3$ or $\mathbb{C}^3$.
(ii) Vectors $e_n = (0, \ldots, 0, 1, 0, \ldots)$ (with the only 1 on the nth place) in ℓ_2. (Could you see a similarity with the previous example?)
(iii) Functions $e_n(t) = 1/(\sqrt{2}\pi)e^{int}$, $n \in \mathbb{Z}$ in $C[0, 2\pi]$:

$$\langle e_n, e_m\rangle = \int_0^{2\pi} \frac{1}{2\pi} e^{int} e^{-imt} dt = \begin{cases} 1, & n = m; \\ 0, & n \neq m. \end{cases} \tag{3.1}$$

Exercise 3.4. Let A be a subset of an inner product space V and $x \perp y$ for any $y \in A$. Prove that $x \perp z$ for all $z \in \mathrm{CLin}(A)$.

Theorem 3.5 (Pythagoras'). *If* $x \perp y$ *then* $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. *Also if* e_1, $\ldots$, e_n *is orthonormal then*

$$\left\|\sum_1^n a_k e_k\right\|^2 = \left\langle \sum_1^n a_k e_k, \sum_1^n a_k e_k \right\rangle = \sum_1^n |a_k|^2 .$$

Proof. A one-line calculation. □

The following theorem provides an important property of Hilbert spaces which will be used many times. Recall, that a subset K of a linear space V is *convex* if for all $x, y \in K$ and $\lambda \in [0, 1]$ the point $\lambda x + (1 - \lambda)y$ is also in K. Particularly any subspace is convex and any unit ball as well (see Exercise 2.8(i)).

Theorem 3.6 (about the Nearest Point). *Let* K *be a non-empty convex closed subset of a Hilbert space* H. *For any point* $x \in H$ *there is the unique point* $y \in K$ *nearest to* x.

Proof. Let $d = \inf_{y \in K} d(x, y)$, where $d(x, y)$—the distance coming from the norm $\|x\| = \sqrt{\langle x, x \rangle}$ and let y_n a sequence points in K such that $\lim_{n\to\infty} d(x, y_n) = d$. Then y_n is a Cauchy sequence. Indeed from the parallelogram identity for the parallelogram generated by vectors $x - y_n$ and $x - y_m$ we have:

$$\|y_n - y_m\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - \|2x - y_n - y_m\|^2 .$$

Note that $\|2x - y_n - y_m\|^2 = 4\left\|x - \frac{y_n + y_m}{2}\right\|^2 \geqslant 4d^2$ since $\frac{y_n + y_m}{2} \in K$ by its convexity. For sufficiently large m and n we get $\|x - y_m\|^2 \leqslant d + \epsilon$ and $\|x - y_n\|^2 \leqslant d + \epsilon$, thus $\|y_n - y_m\| \leqslant 4(d^2 + \epsilon) - 4d^2 = 4\epsilon$, i.e. y_n is a Cauchy sequence.

Let y be the limit of y_n, which exists by the completeness of H, then $y \in K$ since K is closed. Then $d(x, y) = \lim_{n\to\infty} d(x, y_n) = d$. This show the existence of the nearest point. Let y' be another point in K such that $d(x, y') = d$, then the parallelogram identity implies:

$$\|y - y'\|^2 = 2\|x - y\|^2 + 2\|x - y'\|^2 - \|2x - y - y'\|^2 \leqslant 4d^2 - 4d^2 = 0.$$

This shows the uniqueness of the nearest point. □

Exercise* 3.7. The essential rôle of the parallelogram identity in the above proof indicates that the theorem does not hold in a general Banach space.

(i) Show that in $\mathbb{R}^2$ with either norm $\|\cdot\|_1$ or $\|\cdot\|_\infty$ form Example 2.9 the nearest point could be non-unique;

(ii) Could you construct an example (in Banach space) when the nearest point does not exists?

Liberte, Egalite, Fraternite!
A longstanding ideal approximated in the real life by something completely different

3.2. **Bessel's inequality.** For the case then a convex subset is a subspace we could characterise the nearest point in the term of orthogonality.

Theorem 3.8 (on Perpendicular). *Let* M *be a subspace of a Hilbert space* H *and a point* $x \in H$ *be fixed. Then* $z \in M$ *is the nearest point to* x *if and only if* $x - z$ *is orthogonal to any vector in* M.

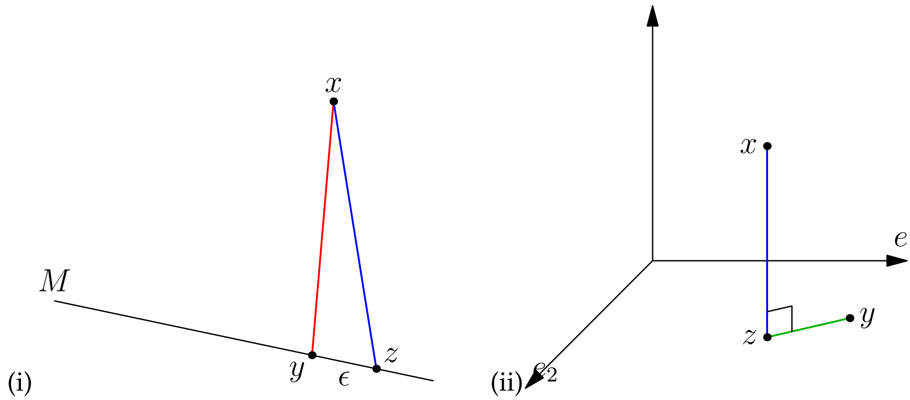


FIGURE 6. (i) A smaller distance for a non-perpendicular direction; and
(ii) Best approximation from a subspace

Proof. Let z is the nearest point to x existing by the previous Theorem. We claim that $x - z$ orthogonal to any vector in M, otherwise there exists $y \in M$ such that $\langle x - z, y \rangle \neq 0$. Then

$$\begin{aligned} \|x - z - \epsilon y\|^2 &= \|x - z\|^2 - 2\epsilon\Re \langle x - z, y \rangle + \epsilon^2 \|y\|^2 \\ &< \|x - z\|^2, \end{aligned}$$

if ϵ is chosen to be small enough and such that $\epsilon\Re \langle x - z, y \rangle$ is positive, see Figure 6(i). Therefore we get a contradiction with the statement that z is closest point to x.
On the other hand if $x - z$ is orthogonal to all vectors in H_1 then particularly $(x - z) \perp (z - y)$ for all $y \in H_1$, see Figure 6(ii). Since $x - y = (x - z) + (z - y)$ we got by the Pythagoras' theorem:

$$\|x - y\|^2 = \|x - z\|^2 + \|z - y\|^2 .$$

So $\|x - y\|^2 \geqslant \|x - z\|^2$ and the are equal if and only if $z = y$. □

Exercise 3.9. The above proof does not work if $\langle x - z, y \rangle$ is an imaginary number, what to do in this case?

Consider now a basic case of *approximation*: let $x \in H$ be fixed and $e_1, \ldots, e_n$ be orthonormal and denote $H_1 = \mathrm{Lin}\{e_1, \ldots, e_n\}$. We could try to approximate x by a vector $y = \lambda_1 e_1 + \cdots + \lambda_n e_n \in H_1$.

Corollary 3.10. *The minimal value of* $\|x - y\|$ *for* $y \in H_1$ *is achieved when* $y = \sum_1^n \langle x, e_i \rangle e_i$.

Proof. Let $z = \sum_1^n \langle x, e_i \rangle e_i$, then $\langle x - z, e_i \rangle = \langle x, e_i \rangle - \langle z, e_i \rangle = 0$. By the previous Theorem z is the nearest point to x. □

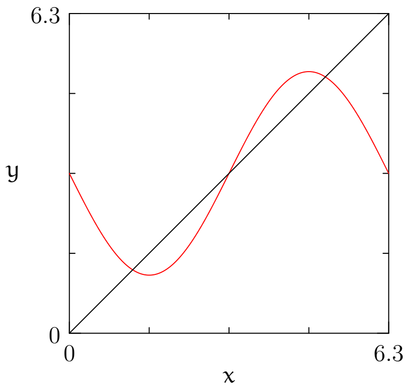


FIGURE 7. Best approximation by three trigonometric polynomials

Example 3.11. (i) In $\mathbb{R}^3$ find the best approximation to $(1, 0, 0)$ from the plane $V : \{x_1 + x_2 + x_3 = 0\}$. We take an orthonormal basis $e_1 = (2^{-1/2}, -2^{-1/2}, 0)$, $e_2 = (6^{-1/2}, 6^{-1/2}, -2 \cdot 6^{-1/2})$ of V (Check this!). Then:

$$z = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 = \left(\frac{1}{2}, -\frac{1}{2}, 0\right) + \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right).$$

(ii) In $C[0, 2\pi]$ what is the best approximation to $f(t) = t$ by functions $a + be^{it} + ce^{-it}$? Let

$$e_0 = \frac{1}{\sqrt{2\pi}}, \qquad e_1 = \frac{1}{\sqrt{2\pi}} e^{it}, \qquad e_{-1} = \frac{1}{\sqrt{2\pi}} e^{-it}.$$

We find:

$$\begin{aligned}
\langle f, e_0\rangle &= \int_0^{2\pi} \frac{t}{\sqrt{2\pi}}\,dt = \left[\frac{t^2}{2}\frac{1}{\sqrt{2\pi}}\right]_0^{2\pi} = \sqrt{2}\pi^{3/2};\\
\langle f, e_1\rangle &= \int_0^{2\pi} \frac{te^{-it}}{\sqrt{2\pi}}\,dt = i\sqrt{2\pi} \qquad \text{(Check this!)}\\
\langle f, e_{-1}\rangle &= \int_0^{2\pi} \frac{te^{it}}{\sqrt{2\pi}}\,dt = -i\sqrt{2\pi} \qquad \text{(Why we may not check this one?)}
\end{aligned}$$

Then the best approximation is (see Figure 7):

$$\begin{aligned}
f_0(t) &= \langle f, e_0\rangle\, e_0 + \langle f, e_1\rangle\, e_1 + \langle f, e_{-1}\rangle\, e_{-1}\\
&= \frac{\sqrt{2}\pi^{3/2}}{\sqrt{2\pi}} + ie^{it} - ie^{-it} = \pi - 2\sin t.
\end{aligned}$$

Corollary 3.12 (Bessel's inequality). *If* (e_i) *is orthonormal then*

$$\|x\|^2 \geqslant \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 .$$

Proof. Let $z = \sum_1^n \langle x, e_i\rangle e_i$ then $x - z \perp e_i$ for all i therefore by Exercise 3.4 $x - z \perp z$. Hence:

$$\begin{aligned}
\|x\|^2 &= \|z\|^2 + \|x - z\|^2\\
&\geqslant \|z\|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 .
\end{aligned}$$

□

—Did you say "rice and fish for them"?
A student question

3.3. **The Riesz–Fischer theorem.** When (e_i) is orthonormal we call $\langle x, e_n\rangle$ the nth *Fourier coefficient* of x (with respect to (e_i), naturally).

Theorem 3.13 (Riesz–Fisher). *Let* $(e_n)_1^\infty$ *be an orthonormal sequence in a Hilbert space* H. *Then* $\sum_1^\infty \lambda_n e_n$ *converges in* H *if and only if* $\sum_1^\infty |\lambda_n|^2 < \infty$. *In this case* $\|\sum_1^\infty \lambda_n e_n\|^2 = \sum_1^\infty |\lambda_n|^2$.

Proof. Necessity: Let $x_k = \sum_1^k \lambda_n e_n$ and $x = \lim_{k\to\infty} x_k$. So $\langle x, e_n\rangle = \lim_{k\to\infty} \langle x_k, e_n\rangle = \lambda_n$ for all n. By the Bessel's inequality for all k

$$\|x\|^2 \geqslant \sum_1^k |\langle x, e_n\rangle|^2 = \sum_1^k |\lambda_n|^2 ,$$

hence $\sum_1^k |\lambda_n|^2$ converges and the sum is at most $\|x\|^2$.

Sufficiency: Consider $\|x_k - x_m\| = \left\|\sum_m^k \lambda_n e_n\right\| = \left(\sum_m^k |\lambda_n|^2\right)^{1/2}$ for $k > m$. Since $\sum_m^k |\lambda_n|^2$ converges x_k is a Cauchy sequence in H and thus has a limit x. By the Pythagoras' theorem $\|x_k\|^2 = \sum_1^k |\lambda_n|^2$ thus for $k \to \infty$ $\|x\|^2 = \sum_1^\infty |\lambda_n|^2$ by the Lemma about inner product limit. □

Observation: the closed linear span of an orthonormal sequence in any Hilbert space looks like ℓ_2, i.e. ℓ_2 is a universal model for a Hilbert space.

By Bessel's inequality and the Riesz–Fisher theorem we know that the series $\sum_1^\infty \langle x, e_i\rangle e_i$ converges for any $x \in H$. What is its limit?

Let $y = x - \sum_1^\infty \langle x, e_i\rangle e_i$, then

$$\langle y, e_k\rangle = \langle x, e_k\rangle - \sum_1^\infty \langle x, e_i\rangle \langle e_i, e_k\rangle = \langle x, e_k\rangle - \langle x, e_k\rangle = 0 \qquad \text{for all } k. \tag{3.2}$$

Definition 3.14. An orthonormal sequence (e_i) in a Hilbert space H is *complete* if the identities $\langle y, e_k\rangle = 0$ for all k imply $y = 0$.
A complete orthonormal sequence is also called *orthonormal basis* in H.

Theorem 3.15 (on Orthonormal Basis). *Let e_i be an orthonormal basis in a Hilber space H. Then for any $x \in H$ we have*

$$x = \sum_{n=1}^\infty \langle x, e_n\rangle e_n \qquad \textit{and} \qquad \|x\|^2 = \sum_{n=1}^\infty |\langle x, e_n\rangle|^2 .$$

Proof. By the Riesz–Fisher theorem, equation (3.2) and definition of orthonormal basis. □

There are constructive existence theorems in mathematics.
An example of pure existence statement

3.4. **Construction of Orthonormal Sequences.** Natural questions are: Do orthonormal sequences always exist? Could we construct them?

Theorem 3.16 (Gram–Schmidt). *Let (x_i) be a sequence of linearly independent vectors in an inner product space V. Then there exists orthonormal sequence (e_i) such that*

$$\mathrm{Lin}\{x_1, x_2, \ldots, x_n\} = \mathrm{Lin}\{e_1, e_2, \ldots, e_n\}, \qquad \textit{for all } n.$$

Proof. We give an explicit algorithm working by induction. The *base* of induction: the first vector is $e_1 = x_1 / \|x_1\|$. The *step* of induction: let $e_1, e_2, \ldots, e_n$ are already constructed as required. Let $y_{n+1} = x_{n+1} - \sum_{i=1}^{n} \langle x_{n+1}, e_i \rangle e_i$. Then by (3.2) $y_{n+1} \perp e_i$ for $i = 1, \ldots, n$. We may put $e_{n+1} = y_{n+1} / \|y_{n+1}\|$ because $y_{n+1} \neq 0$ due to linear independence of x_k's. Also

$$\begin{aligned}
\mathrm{Lin}\{e_1, e_2, \ldots, e_{n+1}\} &= \mathrm{Lin}\{e_1, e_2, \ldots, y_{n+1}\} \\
&= \mathrm{Lin}\{e_1, e_2, \ldots, x_{n+1}\} \\
&= \mathrm{Lin}\{x_1, x_2, \ldots, x_{n+1}\}.
\end{aligned}$$

So (e_i) are orthonormal sequence. □

Example 3.17. Consider $C[0,1]$ with the usual inner product (2.10) and apply orthogonalisation to the sequence $1, x, x^2, \ldots$. Because $\|1\| = 1$ then $e_1(x) = 1$. The continuation could be presented by the table:

$$e_1(x) = 1$$

$$y_2(x) = x - \langle x, 1 \rangle 1 = x - \frac{1}{2}, \quad \|y_2\|^2 = \int_0^1 (x - \frac{1}{2})^2 \, dx = \frac{1}{12}, \quad e_2(x) = \sqrt{12}(x - \frac{1}{2})$$

$$y_3(x) = x^2 - \langle x^2, 1 \rangle 1 - \left\langle x^2, x - \frac{1}{2} \right\rangle (x - \frac{1}{2}) \cdot 12, \quad \ldots, \quad e_3 = \frac{y_3}{\|y_3\|}$$

....

Example 3.18. Many famous sequences of orthogonal polynomials, e.g. Chebyshev, Legendre, Laguerre, Hermite, can be obtained by orthogonalisation of $1, x, x^2, \ldots$ with various inner products.

(i) *Legendre polynomials* in $C[-1,1]$ with inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)} \, dt. \tag{3.3}$$

(ii) *Chebyshev polynomials* in $C[-1,1]$ with inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)} \frac{dt}{\sqrt{1-t^2}} \tag{3.4}$$

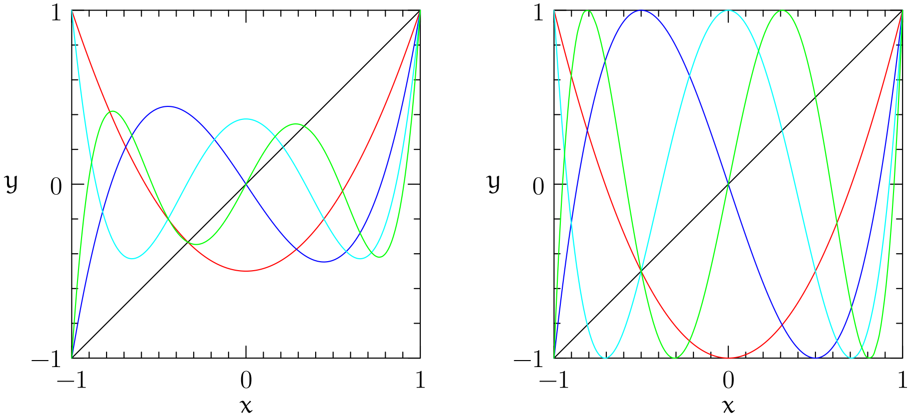


FIGURE 8. Five first Legendre P_i and Chebyshev T_i polynomials

(iii) *Laguerre polynomials* in the space of polynomials $P[0,\infty)$ with inner product

$$\langle f, g \rangle = \int_0^\infty f(t)\overline{g(t)}e^{-t}\,dt.$$

See Figure 8 for the five first Legendre and Chebyshev polynomials. Observe the difference caused by the different inner products (3.3) and (3.4). On the other hand note the similarity in oscillating behaviour with different "frequencies".

Another natural question is: When is an orthonormal sequence complete?

Proposition 3.19. *Let (e_n) be an orthonormal sequence in a Hilbert space H. The following are equivalent:*

(i) *(e_n) is an orthonormal basis.*
(ii) $\mathrm{CLin}((e_n)) = H$.
(iii) $\|x\|^2 = \sum_1^\infty |\langle x, e_n\rangle|^2$ *for all* $x \in H$.

Proof. Clearly 3.19(i) implies 3.19(ii) because $x = \sum_1^\infty \langle x, e_n\rangle e_n$ in $\mathrm{CLin}((e_n))$ and $\|x\|^2 = \sum_1^\infty \langle x, e_n\rangle e_n$ by Theorem 3.15. The same theorem tells that 3.19(i) implies 3.19(iii).

If (e_n) is *not* complete then there exists $x \in H$ such that $x \neq 0$ and $\langle x, e_k\rangle = 0$ for all k, so 3.19(iii) fails, consequently 3.19(iii) implies 3.19(i).

Finally if $\langle x, e_k\rangle = 0$ for all k then $\langle x, y\rangle = 0$ for all $y \in \mathrm{Lin}((e_n))$ and moreover for all $y \in \mathrm{CLin}((e_n))$, by the Lemma on continuity of the inner product. But then $x \notin \mathrm{CLin}((e_n))$ and 3.19(ii) also fails because $\langle x, x \rangle = 0$ is not possible. Thus 3.19(ii) implies 3.19(i). □

Corollary 3.20. *A separable Hilbert space (i.e. one with a countable dense set) can be identified with either ℓ_2^n or ℓ_2, in other words it has an orthonormal basis (e_n) (finite or infinite) such that*

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \qquad \text{and} \qquad \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 .$$

Proof. Take a countable dense set (x_k), then $H = \mathrm{CLin}((x_k))$, delete all vectors which are a linear combinations of preceding vectors, make orthonormalisation by Gram–Schmidt the remaining set and apply the previous proposition. □

Most pleasant compliments are usually orthogonal to our real qualities.

An advise based on observations

3.5. **Orthogonal complements.** Orthogonality allow us split a Hilbert space into subspaces which will be "independent from each other" as much as possible.

Definition 3.21. Let M be a subspace of an inner product space V. The *orthogonal complement*, written $M^\perp$, of M is

$$M^\perp = \{x \in V : \langle x, m \rangle = 0 \ \forall m \in M\}.$$

Theorem 3.22. *If M is a closed subspace of a Hilbert space H then $M^\perp$ is a closed subspace too (hence a Hilbert space too).*

Proof. Clearly $M^\perp$ is a subspace of H because $x, y \in M^\perp$ implies $ax + by \in M^\perp$:

$$\langle ax + by, m \rangle = a \langle x, m \rangle + b \langle y, m \rangle = 0.$$

Also if all $x_n \in M^\perp$ and $x_n \to x$ then $x \in M^\perp$ due to inner product limit Lemma. □

Theorem 3.23. *Let M be a closed subspace of a Hilber space H. Then for any $x \in H$ there exists the unique decomposition $x = m + n$ with $m \in M$, $n \in M^\perp$ and $\|x\|^2 = \|m\|^2 + \|n\|^2$. Thus $H = M \oplus M^\perp$ and $(M^\perp)^\perp = M$.*

Proof. For a given x there exists the unique closest point m in M by the Theorem on nearest point and by the Theorem on perpendicular $(x-m) \perp y$ for all $y \in M$. So $x = m + (x - m) = m + n$ with $m \in M$ and $n \in M^\perp$. The identity $\|x\|^2 = \|m\|^2 + \|n\|^2$ is just Pythagoras' theorem and $M \cap M^\perp = \{0\}$ because null vector is the only vector orthogonal to itself.
Finally $(M^\perp)^\perp = M$. We have $H = M \oplus M^\perp = (M^\perp)^\perp \oplus M^\perp$, for any $x \in (M^\perp)^\perp$ there is a decomposition $x = m + n$ with $m \in M$ and $n \in M^\perp$, but then n is orthogonal to itself and therefore is zero. □

4. Duality of Linear Spaces

Everything has another side

Orthonormal basis allows to reduce any question on Hilbert space to a question on sequence of numbers. This is powerful but sometimes heavy technique. Sometime we need a smaller and faster tool to study questions which are represented by a single number, for example to demonstrate that two vectors are different it is enough to show that there is a unequal values of a single coordinate. In such cases *linear functionals* are just what we needed.

–Is it functional?
–Yes, it works!

4.1. Dual space of a normed space.

Definition 4.1. A *linear functional* on a vector space V is a linear mapping $\alpha : V \rightarrow \mathbb{C}$ (or $\alpha : V \rightarrow \mathbb{R}$ in the real case), i.e.

$$\alpha(ax + by) = a\alpha(x) + b\alpha(y), \qquad \text{for all } x, y \in V \text{ and } a, b \in \mathbb{C}.$$

Exercise 4.2. Show that $\alpha(0)$ is necessarily 0.

We will not consider any functionals but linear, thus below *functional* always means *linear functional*.

Example 4.3. (i) Let $V = \mathbb{C}^n$ and c_k, $k = 1, \ldots, n$ be complex numbers. Then $\alpha((x_1, \ldots, x_n)) = c_1 x_1 + \cdots + c_2 x_2$ is a linear functional.

(ii) On $C[0, 1]$ a functional is given by $\alpha(f) = \int_0^1 f(t)\, dt$.

(iii) On a Hilbert space H for any $x \in H$ a functional α_x is given by $\alpha_x(y) = \langle y, x \rangle$.

Theorem 4.4. *Let V be a normed space and α is a linear functional. The following are equivalent:*

(i) *α is continuous (at any point of V).*
(ii) *α is continuous at point 0.*
(iii) $\sup\{|\alpha(x)| : \|x\| \leqslant 1\} < \infty$, *i.e.* α *is a* bounded linear functional.

Proof. Implication 4.4(i) $\Rightarrow$ 4.4(ii) is trivial.
Show 4.4(ii) $\Rightarrow$ 4.4(iii). By the definition of continuity: for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|v\| < \delta$ implies $|\alpha(v) - \alpha(0)| < \epsilon$. Take $\epsilon = 1$ then $|\alpha(\delta x)| < 1$ for all x with norm less than 1 because $\|\delta x\| < \delta$. But from linearity of α the inequality $|\alpha(\delta x)| < 1$ implies $|\alpha(x)| < 1/\delta < \infty$ for all $\|x\| \leqslant 1$.
4.4(iii) $\Rightarrow$ 4.4(i). Let mentioned supremum be M. For any $x, y \in V$ such that $x \neq y$ vector $(x-y)/\|x-y\|$ has norm 1. Thus $|\alpha((x-y)/\|x-y\|)| < M$. By the linearity of α this implies that $|\alpha(x) - \alpha(y)| < M\|x-y\|$. Thus α is continuous. □

Definition 4.5. The *dual space* X^* of a normed space X is the set of continuous linear functionals on X. Define a norm on it by

$$\|\alpha\| = \sup_{\|x\|=1} |\alpha(x)|. \tag{4.1}$$

Exercise 4.6. (i) Show that the chain of inequalities:

$$\|\alpha\| \leqslant \sup_{\|x\|\leqslant 1} |\alpha(x)| \leqslant \sup_{x\neq 0} \frac{|\alpha(x)|}{\|x\|} \leqslant \|\alpha\|.$$

Deduce that any of the mentioned supremums deliver the norm of α. Which of them you will prefer if you need to show boundedness of α? Which of them is better to use if boundedness of α is given?
(ii) Show that $|\alpha(x)| \leqslant \|\alpha\| \cdot \|x\|$ for all $x \in X$, $\alpha \in X^*$.

The important observations is that linear functionals form a normed space as follows:

Exercise 4.7. (i) Show that X^* is a linear space with natural (point-wise) operations.
(ii) Show that (4.1) defines a norm on X^*.

Furthermeore, X^* is always complete, regardless of properties of X!

Theorem 4.8. X^* *is a Banach space with the defined norm (even if* X *was incomplete).*

Proof. Due to Exercise 4.7 we only need to show that X^* is complete. Let (α_n) be a Cauchy sequence in X^*, then for any $x \in X$ scalars $\alpha_n(x)$ form a Cauchy sequence, since $|\alpha_m(x) - \alpha_n(x)| \leqslant \|\alpha_m - \alpha_n\| \cdot \|x\|$. Thus the sequence has a limit and we define α by $\alpha(x) = \lim_{n\to\infty} \alpha_n(x)$. Clearly α is a linear functional on X. We should show that it is bounded and $\alpha_n \to \alpha$. Given $\epsilon > 0$ there exists N such that $\|\alpha_n - \alpha_m\| < \epsilon$ for all $n, m \geqslant N$. If $\|x\| \leqslant 1$ then $|\alpha_n(x) - \alpha_m(x)| \leqslant \epsilon$, let $m \to \infty$ then $|\alpha_n(x) - \alpha(x)| \leqslant \epsilon$, so

$$|\alpha(x)| \leqslant |\alpha_n(x)| + \epsilon \leqslant \|\alpha_n\| + \epsilon,$$

i.e. $\|\alpha\|$ is finite and $\|\alpha_n - \alpha\| \leqslant \epsilon$, thus $\alpha_n \to \alpha$. □

Definition 4.9. The *kernel of linear functional* α, write $\ker \alpha$, is the set all vectors $x \in X$ such that $\alpha(x) = 0$.

Exercise 4.10. Show that

(i) $\ker \alpha$ is a subspace of X.
(ii) If $\alpha \not\equiv 0$ then obviously $\ker \alpha \neq X$. Furthermore, if X has at least two linearly independent vectors then $\ker \alpha \neq \{0\}$, thus $\ker \alpha$ is a *proper* subspace of X.
(iii) If α is continuous then $\ker \alpha$ is closed.

Study one and get any other for free!
Hilbert spaces sale

4.2. Self-duality of Hilbert space.

Lemma 4.11 (Riesz–Fréchet). *Let* H *be a Hilbert space and* α *a continuous linear functional on* H*, then there exists the unique* $y \in H$ *such that* $\alpha(x) = \langle x, y \rangle$ *for all* $x \in H$. *Also* $\|\alpha\|_{H^*} = \|y\|_H$.

Proof. Uniqueness: if $\langle x, y \rangle = \langle x, y' \rangle \Leftrightarrow \langle x, y - y' \rangle = 0$ for all $x \in H$ then $y - y'$ is self-orthogonal and thus is zero (Exercise 3.2(i)).
Existence: we may assume that $\alpha \not\equiv 0$ (otherwise take $y = 0$), then $M = \ker \alpha$ is a closed *proper* subspace of H. Since $H = M \oplus M^\perp$, there exists a non-zero $z \in M^\perp$, by scaling we could get $\alpha(z) = 1$. Then for any $x \in H$:

$$x = (x - \alpha(x)z) + \alpha(x)z, \qquad \text{with } x - \alpha(x)z \in M,\ \alpha(x)z \in M^\perp.$$

Because $\langle x, z\rangle = \alpha(x)\langle z, z\rangle = \alpha(x)\|z\|^2$ for any $x \in H$ we set $y = z/\|z\|^2$. Equality of the norms $\|\alpha\|_{H^*} = \|y\|_H$ follows from the Cauchy–Bunyakovskii–Schwarz inequality in the form $\alpha(x) \leqslant \|x\| \cdot \|y\|$ and the identity $\alpha(y/\|y\|) = \|y\|$. □

Example 4.12. On $L_2[0,1]$ let $\alpha(f) = \langle f, t^2\rangle = \int_0^1 f(t)t^2\,dt$. Then

$$\|\alpha\| = \|t^2\| = \left(\int_0^1 (t^2)^2\,dt\right)^{1/2} = \frac{1}{\sqrt{5}}.$$

5. Fourier Analysis

All bases are equal, but some are more equal then others.

As we saw already any separable Hilbert space posses an orthonormal basis (infinitely many of them indeed). Are they equally good? This depends from our purposes. For solution of differential equation which arose in mathematical physics (wave, heat, Laplace equations, etc.) there is a proffered choice. The fundamental formula: $\frac{d}{dx}e^{ax} = ae^{ax}$ reduces the derivative to a multiplication by a. We could benefit from this observation if the orthonormal basis will be constructed out of exponents. This helps to solve differential equations as was demonstrated in Subsection 0.2.

7.40pm Fourier series: Episode II
Today's TV listing

5.1. Fourier series.

Now we wish to address questions stated in Remark 0.9. Let us consider the space $L_2[-\pi,\pi]$. As we saw in Example 3.3(iii) there is an orthonormal sequence $e_n(t) = (2\pi)^{-1/2}e^{int}$ in $L_2[-\pi,\pi]$. We will show that it is an orthonormal basis, i.e.

$$f(t) \in L_2[-\pi,\pi] \quad\Leftrightarrow\quad f(t) = \sum_{k=-\infty}^{\infty} \langle f, e_k\rangle\, e_k(t),$$

with convergence in L_2 norm. To do this we show that $\mathrm{CLin}\{e_k : k \in \mathbb{Z}\} = L_2[-\pi,\pi]$.

Let $CP[-\pi,\pi]$ denote the continuous functions f on $[-\pi,\pi]$ such that $f(\pi) = f(-\pi)$. We also define f outside of the interval $[-\pi,\pi]$ by periodicity.

Lemma 5.1. *The space* $CP[-\pi,\pi]$ *is dense in* $L_2[-\pi,\pi]$.

Proof. Let $f \in L_2[-\pi,\pi]$. Given $\epsilon > 0$ there exists $g \in C[-\pi,\pi]$ such that $\|f - g\| < \epsilon/2$. From continuity of g on a compact set follows that there is M such that $|g(t)| < M$ for all $t \in [-\pi,\pi]$.

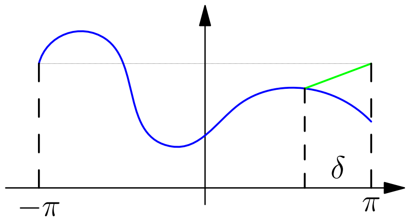


FIGURE 9. A modification of continuous function to periodic

We can now replace g by periodic $\tilde{g}$, which coincides with g on $[-\pi, \pi-\delta]$ for an arbitrary $\delta > 0$ and has the same bounds: $|\tilde{g}(t)| < M$, see Figure 9. Then

$$\|g-\tilde{g}\|_2^2 = \int_{\pi-\delta}^{\pi} |g(t)-\tilde{g}(t)|^2 \, dt \leqslant (2M)^2\delta.$$

So if $\delta < \epsilon^2/(4M)^2$ then $\|g-\tilde{g}\| < \epsilon/2$ and $\|f-\tilde{g}\| < \epsilon$. □

Now if we could show that $\mathrm{CLin}\{e_k : k \in \mathbb{Z}\}$ includes $CP[-\pi, \pi]$ then it also includes $L_2[-\pi, \pi]$.

Notation 5.2. Let $f \in CP[-\pi, \pi]$,write

$$f_n = \sum_{k=-n}^{n} \langle f, e_k \rangle \, e_k, \qquad \text{for } n = 0, 1, 2, \ldots \tag{5.1}$$

the *partial sum of the Fourier series* for f.

We want to show that $\|f - f_n\|_2 \to 0$. To this end we define nth *Fejér sum* by the formula

$$F_n = \frac{f_0 + f_1 + \cdots + f_n}{n+1}, \tag{5.2}$$

and show that

$$\|F_n - f\|_\infty \to 0.$$

Then we conclude

$$\|F_n - f\|_2 = \left(\int_{-\pi}^{\pi} |F_n(t) - f|^2 \right)^{1/2} \leqslant (2\pi)^{1/2} \|F_n - f\|_\infty \to 0.$$

Since $F_n \in \mathrm{Lin}((e_n))$ then $f \in \mathrm{CLin}((e_n))$ and hence $f = \sum_{-\infty}^{\infty} \langle f, e_k \rangle \, e_k$.

Remark 5.3. It is **not** always true that $\|f_n - f\|_\infty \to 0$ even for $f \in CP[-\pi, \pi]$.

Exercise 5.4. Find an example illustrating the above Remark.

The summation method used in (5.2) us useful not only in the context of Fourier series but for many other cases as well. In such a wider framework the method is known as Cesàro summation.

It took 19 years of his life to prove this theorem

5.2. Fejér's theorem.

Proposition 5.5 (Fejér, age 19). *Let* $f \in CP[-\pi, \pi]$. *Then*

$$F_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)\,dt, \qquad \textit{where} \tag{5.3}$$

$$K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{m=-k}^{k} e^{imt}, \tag{5.4}$$

is the Fejér kernel.

Proof. From notation (5.1):

$$\begin{aligned} f_k(x) &= \sum_{m=-k}^{k} \langle f, e_m\rangle\, e_m(x) \\ &= \sum_{m=-k}^{k} \int_{-\pi}^{\pi} f(t)\frac{e^{-imt}}{\sqrt{2\pi}}\,dt\,\frac{e^{imx}}{\sqrt{2\pi}} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{m=-k}^{k} e^{im(x-t)}\,dt. \end{aligned}$$

Then from (5.2):

$$\begin{aligned} F_n(x) &= \frac{1}{n+1}\sum_{k=0}^{n} f_k(x) \\ &= \frac{1}{n+1}\frac{1}{2\pi}\sum_{k=0}^{n}\int_{-\pi}^{\pi} f(t)\sum_{m=-k}^{k} e^{im(x-t)}\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{n+1}\sum_{k=0}^{n}\sum_{m=-k}^{k} e^{im(x-t)}\,dt, \end{aligned}$$

which finishes the proof. □

Lemma 5.6. *The Fejér kernel is* 2π*-periodic,* $K_n(0) = n+1$ *and can be expressed as:*

$$K_n(t) = \frac{1}{n+1} \frac{\sin^2 \frac{(n+1)t}{2}}{\sin^2 \frac{t}{2}}, \qquad \text{for } t \notin 2\pi\mathbb{Z}. \tag{5.5}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & z^{-1} & 1 & z & & \\ & z^{-2} & z^{-1} & 1 & z & z^2 & \\ \cdot^{\cdot^{\cdot}} & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

TABLE 1. Counting powers in rows and columns

Proof. Let $z = e^{it}$, then:

$$\begin{aligned} K_n(t) &= \frac{1}{n+1} \sum_{k=0}^{n} (z^{-k} + \cdots + 1 + z + \cdots + z^k) \\ &= \frac{1}{n+1} \sum_{j=-n}^{n} (n+1-|j|) z^j, \end{aligned}$$

by switch from counting in rows to counting in columns in Table 1. Let $w = e^{it/2}$, i.e. $z = w^2$, then

$$\begin{aligned} K_n(t) &= \frac{1}{n+1} (w^{-2n} + 2w^{-2n+2} + \cdots + (n+1) + nw^2 + \cdots + w^{2n}) \\ &= \frac{1}{n+1} (w^{-n} + w^{-n+2} + \cdots + w^{n-2} + w^n)^2 \qquad (5.6) \\ &= \frac{1}{n+1} \left(\frac{w^{-n-1} - w^{n+1}}{w^{-1} - w} \right)^2 \qquad \text{Could you sum a geometric progression?} \\ &= \frac{1}{n+1} \left(\frac{2i \sin \frac{(n+1)t}{2}}{2i \sin \frac{t}{2}} \right)^2, \end{aligned}$$

if $w \neq \pm 1$. For the value of $K_n(0)$ we substitute $w = 1$ into (5.6). □

The first eleven Fejér kernels are shown on Figure 10, we could observe that:

Lemma 5.7. *Fejér's kernel has the following properties:*

(i) $K_n(t) \geqslant 0$ *for all* $t \in \mathbb{R}$ *and* $n \in \mathbb{N}$.

(ii) $\int_{-\pi}^{\pi} K_n(t)\,dt = 2\pi.$

(iii) *For any* $\delta \in (0, \pi)$

$$\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} K_n(t)\,dt \to 0 \qquad \textit{as} \qquad n \to \infty.$$

Proof. The first property immediately follows from the explicit formula (5.5). In contrast the second property is easier to deduce from expression with double

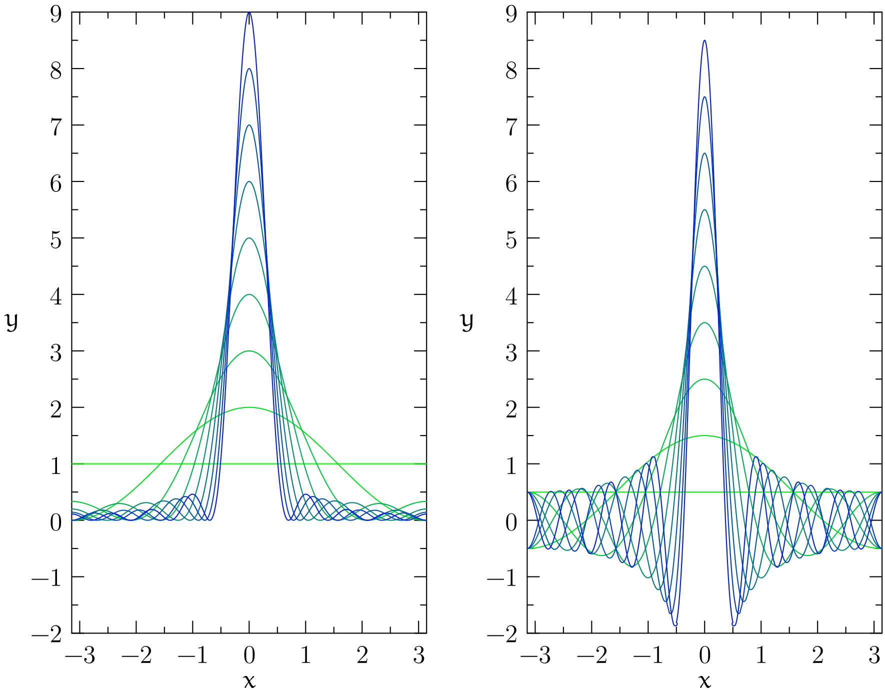


FIGURE 10. A family of Fejér kernels with the parameter m running from 0 to 9 is on the left picture. For a comparison unregularised Fourier kernels are on the right picture.

sum (5.4):

$$\begin{aligned}\int_{-\pi}^{\pi} K_n(t)\,dt &= \int_{-\pi}^{\pi} \frac{1}{n+1}\sum_{k=0}^{n}\sum_{m=-k}^{k} e^{imt}\,dt\\ &= \frac{1}{n+1}\sum_{k=0}^{n}\sum_{m=-k}^{k}\int_{-\pi}^{\pi} e^{imt}\,dt\\ &= \frac{1}{n+1}\sum_{k=0}^{n} 2\pi\\ &= 2\pi,\end{aligned}$$

since the formula (3.1).

Finally if $|t| > \delta$ then $\sin^2(t/2) \geqslant \sin^2(\delta/2) > 0$ by monotonicity of sinus on $[0, \pi/2]$, so:

$$0 \leqslant K_n(t) \leqslant \frac{1}{(n+1)\sin^2(\delta/2)}$$

implying:

$$0 \leqslant \int_{\delta\leqslant|t|\leqslant\pi} K_n(t)\,dt \leqslant \frac{1(\pi-\delta)}{(n+1)\sin^2(\delta/2)} \to 0 \quad \text{as} \quad n \to 0.$$

Therefore the third property follows from the squeeze rule. □

Theorem 5.8 (Fejér Theorem). *Let* $f \in CP[-\pi, \pi]$. *Then its Fejér sums* F_n (5.2) *converges in supremum norm to* f *on* $[-\pi, \pi]$ *and hence in* L_2 *norm as well.*

Proof. Idea of the proof: if in the formula (5.3)

$$F_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)\,dt,$$

t is long way from x, K_n is small (see Lemma 5.7 and Figure 10), for t near x, K_n is big with total "weight" 2π, so the weighted average of $f(t)$ is near $f(x)$.

Here are details. Using property 5.7(ii) and periodicity of f and K_n we could express trivially

$$f(x) = f(x)\frac{1}{2\pi}\int_{x-\pi}^{x+\pi} K_n(x-t)\,dt = \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x)K_n(x-t)\,dt.$$

Similarly we rewrite (5.3) as

$$F_n(x) = \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(t) K_n(x-t)\, dt,$$

then

$$\begin{aligned} |f(x) - F_n(x)| &= \frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} (f(x) - f(t)) K_n(x-t)\, dt \right| \\ &\leqslant \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} |f(x) - f(t)|\, K_n(x-t)\, dt. \end{aligned}$$

Given $\epsilon > 0$ split into three intervals: $I_1 = [x-\pi, x-\delta]$, $I_2 = [x-\delta, x+\delta]$, $I_3 = [x+\delta, x+\pi]$, where δ is chosen such that $|f(t) - f(x)| < \epsilon/2$ for $t \in I_2$, which is possible by continuity of f. So

$$\frac{1}{2\pi} \int_{I_2} |f(x) - f(t)|\, K_n(x-t)\, dt \leqslant \frac{\epsilon}{2} \frac{1}{2\pi} \int_{I_2} K_n(x-t)\, dt < \frac{\epsilon}{2}.$$

And

$$\begin{aligned} \frac{1}{2\pi} \int_{I_1 \cup I_3} |f(x) - f(t)|\, K_n(x-t)\, dt &\leqslant 2 \|f\|_\infty \frac{1}{2\pi} \int_{I_1 \cup I_3} K_n(x-t)\, dt \\ &= \frac{\|f\|_\infty}{\pi} \int_{\delta < |u| < \pi} K_n(u)\, du \\ &< \frac{\epsilon}{2}, \end{aligned}$$

if n is sufficiently large due to property 5.7(iii) of K_n. Hence $|f(x) - F_n(x)| < \epsilon$ for a large n independent of x. □

Remark 5.9. The above properties 5.7(i)–5.7(iii) and their usage in the last proof can be generalised to the concept of *approximation of the identity*. See § 15.4 for a further example.

We almost finished the demonstration that $e_n(t) = (2\pi)^{-1/2} e^{int}$ is an orthonormal basis of $L_2[-\pi, \pi]$:

Corollary 5.10 (Fourier series)**.** *Let* $f \in L_2[-\pi, \pi]$*, with Fourier series*

$$\sum_{n=-\infty}^{\infty} \langle f, e_n \rangle\, e_n = \sum_{n=-\infty}^{\infty} c_n e^{int} \qquad \textit{where} \quad c_n = \frac{\langle f, e_n \rangle}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt.$$

Then the series $\sum_{-\infty}^{\infty} \langle f, e_n \rangle e_n = \sum_{-\infty}^{\infty} c_n e^{int}$ *converges in* $L_2[-\pi, \pi]$ *to* f, *i.e*

$$\lim_{k\to\infty} \left\| f - \sum_{n=-k}^{k} c_n e^{int} \right\|_2 = 0.$$

Proof. This follows from the previous Theorem, Lemma 5.1 about density of CP in L_2, and Theorem 3.15 on orthonormal basis. □

Remark 5.11. There is a reason why we had used the Fejér kernel and the Cezàro summation F_n (5.2) instead of plain partial sums f_n (5.1) of the Fourier series. It can be shown that point-wise convergence $f_n \to f$ does not hold for every continuous function f, cf. Cor. 16.31.

5.3. **Parseval's formula.** The following result first appeared in the framework of $L_2[-\pi, \pi]$ and only later was understood to be a general property of inner product spaces.

Theorem 5.12 (Parseval's formula). *If* $f, g \in L_2[-\pi, \pi]$ *have Fourier series* $f = \sum_{n=-\infty}^{\infty} c_n e^{int}$ *and* $g = \sum_{n=-\infty}^{\infty} d_n e^{int}$, *then*

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt = 2\pi \sum_{-\infty}^{\infty} c_n \overline{d_n}. \tag{5.7}$$

More generally if f *and* g *are two vectors of a Hilbert space* H *with an orthonormal basis* $(e_n)_{-\infty}^{\infty}$ *then*

$$\langle f, g \rangle = \sum_{k=-\infty}^{\infty} c_n \overline{d_n}, \qquad \textit{where} \quad c_n = \langle f, e_n \rangle, \; d_n = \langle g, e_n \rangle,$$

are the Fourier coefficients of f *and* g.

Proof. In fact we could just prove the second, more general, statement—the first one is its particular realisation. Let $f_n = \sum_{k=-n}^{n} c_k e_k$ and $g_n = \sum_{k=-n}^{n} d_k e_k$ will be partial sums of the corresponding Fourier series. Then from orthonormality of (e_n) and linearity of the inner product:

$$\langle f_n, g_n \rangle = \left\langle \sum_{k=-n}^{n} c_k e_k, \sum_{k=-n}^{n} d_k e_k \right\rangle = \sum_{k=-n}^{n} c_k \overline{d_k}.$$

This formula together with the facts that $f_k \to f$ and $g_k \to g$ (following from Corollary 5.10) and Lemma about continuity of the inner product implies the assertion. □

Corollary 5.13. *A integrable function* f *belongs to* $L_2[-\pi, \pi]$ *if and only if its Fourier series is convergent and then* $\|f\|^2 = 2\pi \sum_{-\infty}^{\infty} |c_k|^2$.

Proof. The necessity, i.e. implication $f \in L_2 \Rightarrow \langle f, f \rangle = \|f\|^2 = 2\pi \sum |c_k|^2$, follows from the previous Theorem. The sufficiency follows by Riesz–Fisher Theorem. □

Remark 5.14. The actual rôle of the Parseval's formula is shadowed by the orthonormality and is rarely recognised until we meet the *wavelets* or *coherent states*. Indeed the equality (5.7) should be read as follows:

Theorem 5.15 (Modified Parseval). *The map* $\mathcal{W} : H \to \ell_2$ *given by the formula* $[\mathcal{W}f](n) = \langle f, e_n \rangle$ *is an isometry for any orthonormal basis* (e_n).

We could find many other systems of vectors (e_x), $x \in X$ (very different from orthonormal bases) such that the map $\mathcal{W} : H \to L_2(X)$ given by the simple universal formula

$$[\mathcal{W}f](x) = \langle f, e_x \rangle \tag{5.8}$$

will be an isometry of Hilbert spaces. The map (5.8) is oftenly called *wavelet transform* and most famous is the *Cauchy integral formula* in complex analysis. The majority of wavelets transforms are linked with *group representations*, see our postgraduate course *Wavelets in Applied and Pure Maths*.

Heat and noise but not a fire?

Answer: "Application of Fourier Series"

5.4. **Some Application of Fourier Series.** We are going to provide now few examples which demonstrate the importance of the Fourier series in many questions. The first two (Example 5.16 and Theorem 5.17) belong to pure mathematics and last two are of more applicable nature.

Example 5.16. Let $f(t) = t$ on $[-\pi, \pi]$. Then

$$\langle f, e_n \rangle = \int_{-\pi}^{\pi} t e^{-int}\, dt = \begin{cases} (-1)^n \frac{2\pi i}{n}, & n \neq 0 \\ 0, & n = 0 \end{cases} \qquad \text{(check!)},$$

so $f(t) \sim \sum_{-\infty}^{\infty}(-1)^n(i/n)e^{int}$. By a direct integration:

$$\|f\|_2^2 = \int_{-\pi}^{\pi} t^2\,dt = \frac{2\pi^3}{3}.$$

On the other hand by the previous Corollary:

$$\|f\|_2^2 = 2\pi \sum_{n\neq 0} \left|\frac{(-1)^n i}{n}\right|^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus we get a beautiful formula

$$\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Here is another important result.

Theorem 5.17 (Weierstrass Approximation Theorem). *For any function* $f \in C[a,b]$ *and any* $\epsilon > 0$ *there exists a polynomial* p *such that* $\|f-p\|_\infty < \epsilon$.

Proof. Change variable: $t = 2\pi(x - \frac{a+b}{2})/(b-a)$ this maps $x \in [a,b]$ onto $t \in [-\pi,\pi]$. Let P denote the subspace of polynomials in $C[-\pi,\pi]$. Then $e^{int} \in \bar{P}$ for any $n \in \mathbb{Z}$ since Taylor series converges uniformly in $[-\pi,\pi]$. Consequently P contains the closed linear span in (supremum norm) of e^{int}, any $n \in \mathbb{Z}$, which is $CP[-\pi,\pi]$ by the Fejér theorem. Thus $\bar{P} \supseteq CP[-\pi,\pi]$ and we extend that to non-periodic function as follows (why we could not make use of Lemma 5.1 here, by the way?).
For any $f \in C[-\pi,\pi]$ let $\lambda = (f(\pi) - f(-\pi))/(2\pi)$ then $f_1(t) = f(t) - \lambda t \in CP[-\pi,\pi]$ and could be approximated by a polynomial $p_1(t)$ from the above discussion. Then $f(t)$ is approximated by the polynomial $p(t) = p_1(t) + \lambda t$. □

It is easy to see, that the rôle of exponents e^{int} in the above prove is rather modest: they can be replaced by any functions which has a Taylor expansion. The real glory of the Fourier analysis is demonstrated in the two following examples.

Example 5.18. The modern history of the Fourier analysis starts from the works of Fourier on the heat equation. As was mentioned in the introduction to this part, the exceptional role of Fourier coefficients for differential equations is explained by the simple formula $\partial_x e^{inx} = ine^{inx}$. We shortly review a solution of the *heat equation* to illustrate this.
Let we have a rod of the length 2π. The temperature at its point $x \in [-\pi,\pi]$ and a moment $t \in [0,\infty)$ is described by a function $u(t,x)$ on $[0,\infty) \times [-\pi,\pi]$. The mathematical equation describing a dynamics of the temperature distribution

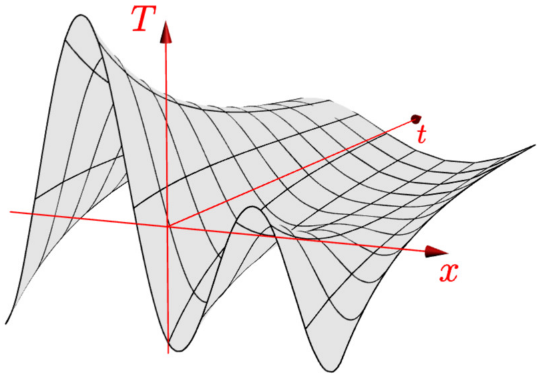


FIGURE 11. The dynamics of a heat equation:
x—coordinate on the rod,
t—time,
T—temperature.

is:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} \qquad \text{or, equivalently,} \qquad (\partial_t - \partial_x^2)\, u(t,x) = 0. \tag{5.9}$$

For any fixed moment t_0 the function $u(t_0, x)$ depends only from $x \in [-\pi, \pi]$ and according to Corollary 5.10 could be represented by its Fourier series:

$$u(t_0, x) = \sum_{n=-\infty}^{\infty} \langle u, e_n \rangle\, e_n = \sum_{n=-\infty}^{\infty} c_n(t_0) e^{inx},$$

where

$$c_n(t_0) = \frac{\langle u, e_n \rangle}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(t_0, x) e^{-inx}\, dx,$$

with Fourier coefficients $c_n(t_0)$ depending from t_0. We substitute that decomposition into the heat equation (5.9) to receive:

$$\begin{aligned} (\partial_t - \partial_x^2)\, u(t,x) &= (\partial_t - \partial_x^2) \sum_{n=-\infty}^{\infty} c_n(t) e^{inx} \\ &= \sum_{n=-\infty}^{\infty} (\partial_t - \partial_x^2)\, c_n(t) e^{inx} \\ &= \sum_{n=-\infty}^{\infty} (c_n'(t) + n^2 c_n(t)) e^{inx} = 0. \end{aligned} \tag{5.10}$$

Since function e^{inx} form a basis the last equation (5.10) holds if and only if

$$c_n'(t) + n^2 c_n(t) = 0 \qquad \text{for all } n \text{ and } t. \tag{5.11}$$

Equations from the system (5.11) have general solutions of the form:

$$c_n(t) = c_n(0)e^{-n^2 t} \qquad \text{for all } t \in [0, \infty), \tag{5.12}$$

producing a general solution of the heat equation (5.9) in the form:

$$u(t, x) = \sum_{n=-\infty}^{\infty} c_n(0)e^{-n^2 t}e^{inx} = \sum_{n=-\infty}^{\infty} c_n(0)e^{-n^2 t + inx}, \tag{5.13}$$

where constant $c_n(0)$ could be defined from boundary condition. For example, if it is known that the initial distribution of temperature was $u(0, x) = g(x)$ for a function $g(x) \in L_2[-\pi, \pi]$ then $c_n(0)$ is the n-th Fourier coefficient of $g(x)$.
The general solution (5.13) helps produce both the analytical study of the heat equation (5.9) and numerical simulation. For example, from (5.13) obviously follows that

- the temperature is rapidly relaxing toward the thermal equilibrium with the temperature given by $c_0(0)$, however never reach it within a finite time;
- the "higher frequencies" (bigger thermal gradients) have a bigger speed of relaxation; etc.

The example of numerical simulation for the initial value problem with $g(x) = 2\cos(2 * u) + 1.5\sin(u)$. It is clearly illustrate our above conclusions.

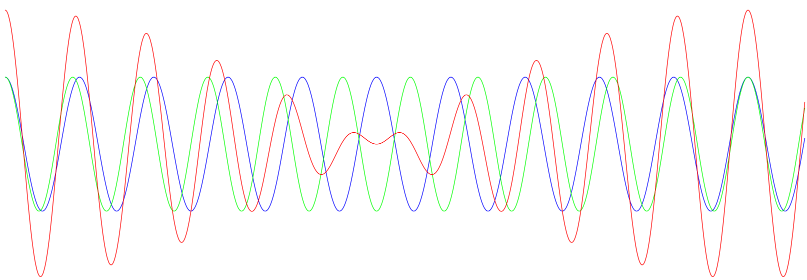


FIGURE 12. Two oscillation with unharmonious frequencies and the appearing dissonance. Click to listen the blue and green pure harmonics and red dissonance.

Example 5.19. Among the oldest periodic functions in human culture are acoustic waves of musical tones. The mathematical theory of musics (including rudiments of the Fourier analysis!) is as old as mathematics itself and was highly respected already in *Pythagoras' school* more 2500 years ago.

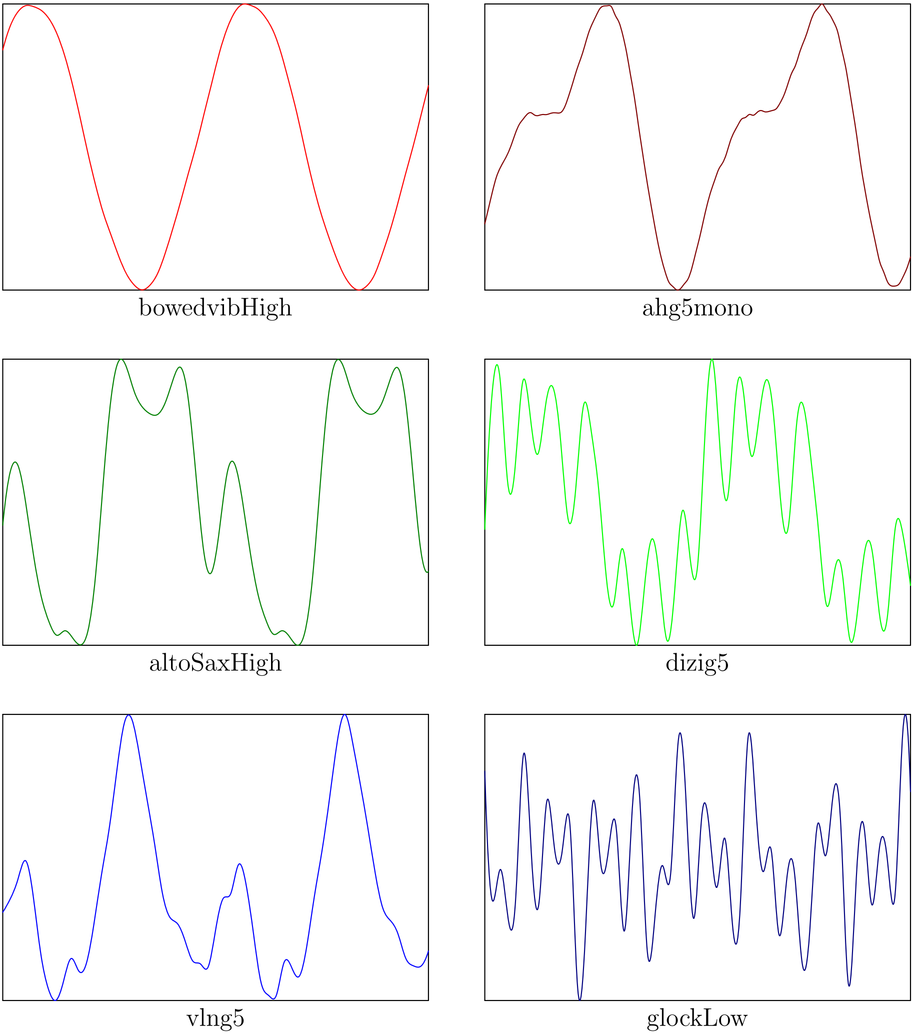


FIGURE 13. Graphics of G5 performed on different musical instruments (click on picture to hear the sound). Samples are taken from Sound Library.

The earliest observations are that

(i) The musical sounds are made of pure harmonics (see the blue and green graphs on the Figure 12), in our language $\cos$ and $\sin$ functions form a basis;

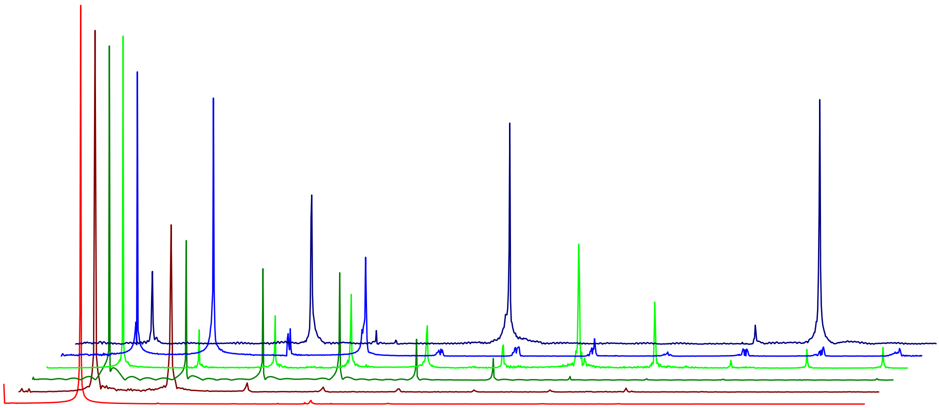


FIGURE 14. Fourier series for G5 performed on different musical instruments (same order and colour as on the previous Figure)

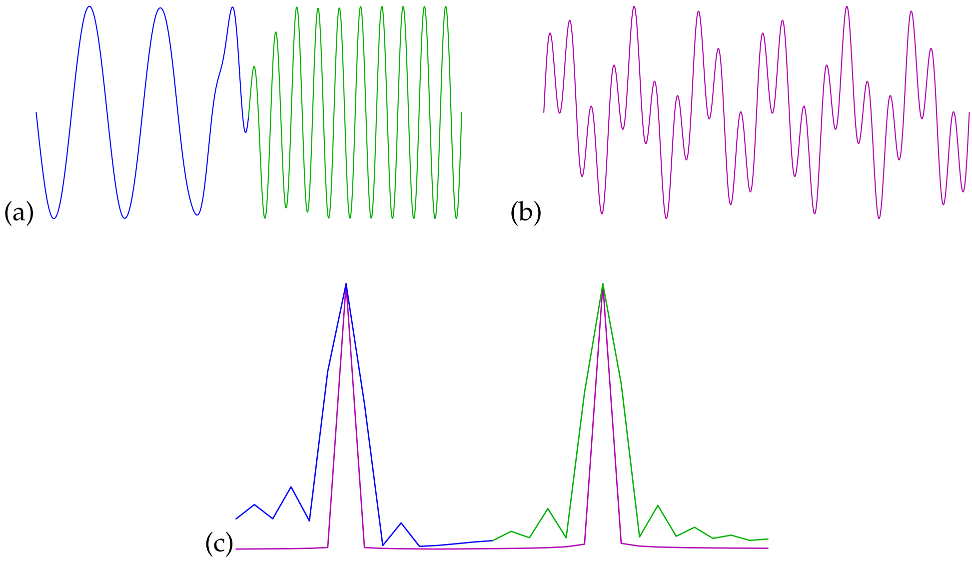


FIGURE 15. Limits of the Fourier analysis: different frequencies separated in time

(ii) Not every two pure harmonics are compatible, to be their frequencies should make a simple ratio. Otherwise the dissonance (red graph on Figure 12) appears.

The musical tone, say G5, performed on different instruments clearly has something in common and different, see Figure 13 for comparisons. The decomposition into the pure harmonics, i.e. finding Fourier coefficient for the signal, could provide the complete characterisation, see Figure 14.
The Fourier analysis tells that:

(i) All sound have the same base (i.e. the lowest) frequencies which corresponds to the G5 tone, i.e. 788 Gz.
(ii) The higher frequencies, which are necessarily are multiples of 788 Gz to avoid dissonance, appears with different weights for different instruments.

The Fourier analysis is very useful in the signal processing and is indeed the fundamental tool. However it is not universal and has very serious limitations. Consider the simple case of the signals plotted on the Figure 15(a) and (b). They are both made out of same two pure harmonics:

(i) On the first signal the two harmonics (drawn in blue and green) follow one after another in time on Figure 15(a);
(ii) They just blended in equal proportions over the whole interval on Figure 15(b).

This appear to be two very different signals. However the Fourier performed over the whole interval does not seems to be very different, see Figure 15(c). Both transforms (drawn in blue-green and pink) have two major pikes corresponding to the pure frequencies. It is not very easy to extract differences between signals from their Fourier transform (yet this should be possible according to our study).
Even a better picture could be obtained if we use *windowed Fourier transform*, namely use a sliding "window" of the constant width instead of the entire interval for the Fourier transform. Yet even better analysis could be obtained by means of *wavelets* already mentioned in Remark 5.14 in connection with Plancherel's formula. Roughly, wavelets correspond to a sliding window of a variable size—narrow for high frequencies and wide for low.

6. Operators

All the space's a stage,
and all functionals and operators merely players!

All our previous considerations were only a preparation of the stage and now the main actors come forward to perform a play. The vectors spaces are not so interesting while we consider them in statics, what really make them exciting is the their transformations. The natural first steps is to consider transformations which respect both linear structure and the norm.

6.1. **Linear operators.**

Definition 6.1. A *linear operator* T between two normed spaces X and Y is a mapping $T : X \to Y$ such that $T(\lambda v + \mu u) = \lambda T(v) + \mu T(u)$. The *kernel of linear operator* $\ker T$ and *image* are defined by

$$\ker T = \{x \in X : Tx = 0\} \qquad \operatorname{Im} T = \{y \in Y : y = Tx, \text{ for some } x \in X\}.$$

Exercise 6.2. Show that kernel of T is a linear subspace of X and image of T is a linear subspace of Y.

As usual we are interested also in connections with the second (topological) structure:

Definition 6.3. A *norm of linear operator* is defined:

(6.1) $$\|T\| = \sup\{\|Tx\|_Y : \|x\|_X \leqslant 1\}.$$

T is a *bounded linear operator* if $\|T\| = \sup\{\|Tx\| : \|x\|\} < \infty$.

Exercise 6.4. Show that $\|Tx\| \leqslant \|T\| \cdot \|x\|$ for all $x \in X$.

Example 6.5. Consider the following examples and determine kernel and images of the mentioned operators.

(i) On a normed space X define the *zero operator* to a space Y by $Z : x \to 0$ for all $x \in X$. Its norm is 0.

(ii) On a normed space X define the *identity operator* by $I_X : x \to x$ for all $x \in X$. Its norm is 1.

(iii) On a normed space X any linear functional define a linear operator from X to $\mathbb{C}$, its norm as operator is the same as functional.

(iv) The set of operators from $\mathbb{C}^n$ to $\mathbb{C}^m$ is given by $n \times m$ matrices which acts on vector by the matrix multiplication. All linear operators on finite-dimensional spaces are bounded.

(v) On ℓ_2, let $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ be the *right shift operator*. Clearly $\|Sx\| = \|x\|$ for all x, so $\|S\| = 1$.

(vi) On $L_2[a, b]$, let $w(t) \in C[a, b]$ and define *multiplication operator* $M_w f$ by $(M_w f)(t) = w(t)f(t)$. Now:

$$\begin{aligned} \|M_w f\|^2 &= \int_a^b |w(t)|^2 \, |f(t)|^2 \, dt \\ &\leqslant K^2 \int_a^b |f(t)|^2 \, dt, \quad \text{where} \quad K = \|w\|_\infty = \sup_{[a,b]} |w(t)|, \end{aligned}$$

so $\|M_w\| \leqslant K$.

Exercise 6.6. Show that for multiplication operator in fact there is the equality of norms $\|M_w\|_2 = \|w(t)\|_\infty$.

Theorem 6.7. *Let $T : X \to Y$ be a linear operator. The following conditions are equivalent:*

(i) *T is continuous on X;*
(ii) *T is continuous at the point 0.*
(iii) *T is a bounded linear operator.*

Proof. Proof essentially follows the proof of similar Theorem 4.4. □

6.2. **Orthoprojections.** Here we will use orthogonal complement, see § 3.5, to introduce a class of linear operators—orthogonal projections. Despite of (or rather due to) their extreme simplicity these operators are among most frequently used tools in the theory of Hilbert spaces.

Corollary 6.8 (of Thm. 3.23, about Orthoprojection)**.** *Let M be a closed linear subspace of a hilbert space H. There is a linear map P_M from H onto M (the* orthogonal projection *or* orthoprojection*) such that*

$$P_M^2 = P_M, \qquad \ker P_M = M^\perp, \qquad P_{M^\perp} = I - P_M. \tag{6.2}$$

Proof. Let us define $P_M(x) = m$ where $x = m + n$ is the decomposition from the previous theorem. The linearity of this operator follows from the fact that both M and $M^\perp$ are linear subspaces. Also $P_M(m) = m$ for all $m \in M$ and the image of P_M is M. Thus $P_M^2 = P_M$. Also if $P_M(x) = 0$ then $x \perp M$, i.e. $\ker P_M = M^\perp$. Similarly $P_{M^\perp}(x) = n$ where $x = m + n$ and $P_M + P_{M^\perp} = I$. □

Example 6.9. Let (e_n) be an orthonormal basis in a Hilber space and let $S \subset \mathbb{N}$ be fixed. Let $M = \mathrm{CLin}\{e_n : n \in S\}$ and $M^\perp = \mathrm{CLin}\{e_n : n \in \mathbb{N} \setminus S\}$. Then

$$\sum_{k=1}^{\infty} a_k e_k = \sum_{k \in S} a_k e_k + \sum_{k \notin S} a_k e_k.$$

Remark 6.10. In fact there is a one-to-one correspondence between closed linear subspaces of a Hilber space H and orthogonal projections defined by identities (6.2).

6.3. $B(H)$ **as a Banach space (and even algebra).**

Theorem 6.11. *Let* $B(X, Y)$ *be the* space of bounded linear operators *from* X *and* Y *with the norm defined above. If* Y *is complete, then* $B(X, Y)$ *is a Banach space.*

Proof. The proof repeat proof of the Theorem 4.8, which is a particular case of the present theorem for $Y = \mathbb{C}$, see Example 6.5(iii). □

Theorem 6.12. *Let* $T \in B(X, Y)$ *and* $S \in B(Y, Z)$*, where* X*,* Y*, and* Z *are normed spaces. Then* $ST \in B(X, Z)$ *and* $\|ST\| \leqslant \|S\| \|T\|$*.*

Proof. Clearly $(ST)x = S(Tx) \in Z$, and

$$\|STx\| \leqslant \|S\| \|Tx\| \leqslant \|S\| \|T\| \|x\| ,$$

which implies norm estimation if $\|x\| \leqslant 1$. □

Corollary 6.13. *Let* $T \in B(X, X) = B(X)$*, where* X *is a normed space. Then for any* $n \geqslant 1$*,* $T^n \in B(X)$ *and* $\|T^n\| \leqslant \|T\|^n$*.*

Proof. It is induction by n with the trivial base $n = 1$ and the step following from the previous theorem. □

Remark 6.14. Some texts use notations $L(X, Y)$ and $L(X)$ instead of ours $B(X, Y)$ and $B(X)$.

Definition 6.15. Let $T \in B(X, Y)$. We say T is an *invertible operator* if there exists $S \in B(Y, X)$ such that

$$ST = I_X \qquad \text{and} \qquad TS = I_Y .$$

Such an S is called the *inverse operator* of T.

Exercise 6.16. Show that

(i) for an invertible operator $T : X \to Y$ we have $\ker T = \{0\}$ and $\Im T = Y$.
(ii) the inverse operator is unique (if exists at all). (Assume existence of S and S', then consider operator STS'.)

Example 6.17. We consider inverses to operators from Exercise 6.5.

(i) The zero operator is never invertible unless the pathological spaces $X = Y = \{0\}$.
(ii) The identity operator I_X is the inverse of itself.

(iii) A linear functional is not invertible unless it is non-zero and X is one dimensional.
(iv) An operator $\mathbb{C}^n \to \mathbb{C}^m$ is invertible if and only if $m = n$ and corresponding square matrix is non-singular, i.e. has non-zero determinant.
(v) The right shift S is not invertible on ℓ_2 (it is one-to-one but is not onto). But the *left shift operator* $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ is its *left inverse*, i.e. $TS = I$ but $TS \neq I$ since $ST(1, 0, 0, \ldots) = (0, 0, \ldots)$. T is not invertible either (it is onto but not one-to-one), however S is its *right inverse*.
(vi) Operator of multiplication M_w is invertible if and only if $w^{-1} \in C[a, b]$ and inverse is $M_{w^{-1}}$. For example M_{1+t} is invertible $L_2[0, 1]$ and M_t is not.

6.4. **Adjoints.**

Theorem 6.18. *Let* H *and* K *be Hilbert Spaces and* $T \in B(H, K)$. *Then there exists operator* $T^* \in B(K, H)$ *such that*

$$\langle Th, k \rangle_K = \langle h, T^*k \rangle_H \qquad \text{for all} \quad h \in H,\ k \in K.$$

Such T^* *is called the* adjoint operator *of* T. *Also* $T^{**} = T$ *and* $\|T^*\| = \|T\|$.

Proof. For any fixed $k \in K$ the expression $h :\to \langle Th, k \rangle_K$ defines a bounded linear functional on H. By the Riesz–Fréchet lemma there is a *unique* $y \in H$ such that $\langle Th, k \rangle_K = \langle h, y \rangle_H$ for all $h \in H$. Define $T^*k = y$ then T^* is linear:

$$\begin{aligned}
\langle h, T^*(\lambda_1 k_1 + \lambda_2 k_2) \rangle_H &= \langle Th, \lambda_1 k_1 + \lambda_2 k_2 \rangle_K \\
&= \bar{\lambda}_1 \langle Th, k_1 \rangle_K + \bar{\lambda}_2 \langle Th, k_2 \rangle_K \\
&= \bar{\lambda}_1 \langle h, T^*k_1 \rangle_H + \bar{\lambda}_2 \langle h, T^*k_2 \rangle_K \\
&= \langle h, \lambda_1 T^*k_1 + \lambda_2 T^*k_2 \rangle_H
\end{aligned}$$

So $T^*(\lambda_1 k_1 + \lambda_2 k_2) = \lambda_1 T^*k_1 + \lambda_2 T^*k_2$. T^{**} is defined by $\langle k, T^{**}h \rangle = \langle T^*k, h \rangle$ and the identity $\langle T^{**}h, k \rangle = \langle h, T^*k \rangle = \langle Th, k \rangle$ for all h and k shows $T^{**} = T$. Also:

$$\begin{aligned}
\|T^*k\|^2 &= \langle T^*k, T^*k \rangle = \langle k, TT^*k \rangle \\
&\leqslant \|k\| \cdot \|TT^*k\| \leqslant \|k\| \cdot \|T\| \cdot \|T^*k\|,
\end{aligned}$$

which implies $\|T^*k\| \leqslant \|T\| \cdot \|k\|$, consequently $\|T^*\| \leqslant \|T\|$. The opposite inequality follows from the identity $\|T\| = \|T^{**}\|$. □

Exercise 6.19. (i) For operators T_1 and T_2 show that

$$(T_1 T_2)^* = T_2^* T_1^*, \qquad (T_1 + T_2)^* = T_1^* + T_2^* \qquad (\lambda T)^* = \bar{\lambda} T^*.$$

(ii) If A is an operator on a Hilbert space H then $(\ker A)^\perp = \operatorname{Im} A^*$.

6.5. Hermitian, unitary and normal operators.

Definition 6.20. An operator $T : H \to H$ is a *Hermitian operator* or *self-adjoint operator* if $T = T^*$, i.e. $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$.

Example 6.21. (i) On ℓ_2 the adjoint S^* to the right shift operator S is given by the left shift $S^* = T$, indeed:

$$\begin{aligned}\langle Sx, y\rangle &= \langle (0, x_1, x_2, \ldots), (y_1, y_2, \ldots)\rangle \\ &= x_1\bar{y}_2 + x_2\bar{y}_3 + \cdots = \langle (x_1, x_2, \ldots), (y_2, y_3, \ldots)\rangle \\ &= \langle x, Ty\rangle .\end{aligned}$$

Thus S is *not* Hermitian.

(ii) Let D be *diagonal operator* on ℓ_2 given by

$$D(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots).$$

where (λ_k) is any bounded complex sequence. It is easy to check that $\|D\| = \|(\lambda_n)\|_\infty = \sup_k |\lambda_k|$ and

$$D^*(x_1, x_2, \ldots) = (\bar{\lambda}_1 x_1, \bar{\lambda}_2 x_2, \ldots),$$

thus D is Hermitian if and only if $\lambda_k \in \mathbb{R}$ for all k.

(iii) If $T : \mathbb{C}^n \to \mathbb{C}^n$ is represented by multiplication of a column vector by a matrix A, then T^* is multiplication by the matrix A^*—transpose and conjugate to A.

Exercise 6.22. Show that for any bounded operator T operators $T_r = \frac{1}{2}(T + T^*)$, $T_i = \frac{1}{2i}(T - T^*)$, T^*T and TT^* are Hermitians. Note, that any operator is the linear combination of two hermitian operators: $T = T_r + iT_i$ (cf. $z = \Re z + i\Im z$ for $z \in \mathbb{C}$).

To appreciate the next Theorem the following exercise is useful:

Exercise 6.23. Let H be a Hilbert space. Show that

(i) For $x \in H$ we have $\|x\| = \sup\{\, |\langle x, y\rangle| \,$ for all $y \in H$ such that $\|y\| = 1\}$.

(ii) For $T \in B(H)$ we have

$$\|T\| = \sup\{\, |\langle Tx, y\rangle| \text{ for all } x, y \in H \text{ such that } \|x\| = \|y\| = 1\}. \tag{6.3}$$

The next theorem says, that for a Hermitian operator T the supremum in (6.3) may be taken over the "diagonal" $x = y$ only.

Theorem 6.24. *Let T be a Hermitian operator on a Hilbert space. Then*

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle| .$$

Proof. If $Tx = 0$ for all $x \in H$, both sides of the identity are 0. So we suppose that $\exists x \in H$ for which $Tx \neq 0$.
We see that $|\langle Tx, x\rangle| \leqslant \|Tx\| \|x\| \leqslant \|T\| \|x^2\|$, so $\sup_{\|x\|=1} |\langle Tx, x\rangle| \leqslant \|T\|$. To get the inequality the other way around, we first write $s := \sup_{\|x\|=1} |\langle Tx, x\rangle|$. Then for any $x \in H$, we have $|\langle Tx, x\rangle| \leqslant s \|x^2\|$.
We now consider

$$\langle T(x+y), x+y\rangle = \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle = \langle Tx, x\rangle + 2\Re \langle Tx, y\rangle + \langle Ty, y\rangle$$

(because T being Hermitian gives $\langle Ty, x\rangle = \langle y, Tx\rangle = \overline{\langle Tx, y\rangle}$) and, similarly,

$$\langle T(x-y), x-y\rangle = \langle Tx, x\rangle - 2\Re \langle Tx, y\rangle + \langle Ty, y\rangle .$$

Subtracting gives

$$\begin{aligned} 4\Re \langle Tx, y\rangle &= \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \\ &\leqslant s(\|x+y\|^2 + \|x-y\|^2) \\ &= 2s(\|x\|^2 + \|y\|^2), \end{aligned}$$

by the parallelogram identity.
Now, for $x \in H$ such that $Tx \neq 0$, we put $y = \|Tx\|^{-1} \|x\| Tx$. Then $\|y\| = \|x\|$ and when we substitute into the previous inequality, we get

$$4 \|Tx\| \|x\| = 4\Re \langle Tx, y\rangle \leqslant 4s \|x^2\| ,$$

So $\|Tx\| \leqslant s \|x\|$ and it follows that $\|T\| \leqslant s$, as required. □

Definition 6.25. We say that $U : H \to H$ is a *unitary operator* on a Hilbert space H if $U^* = U^{-1}$, i.e. $U^*U = UU^* = I$.

Example 6.26.
(i) If $D : \ell_2 \to \ell_2$ is a diagonal operator such that $De_k = \lambda_k e_k$, then $D^* e_k = \bar{\lambda}_k e_k$ and D is unitary if and only if $|\lambda_k| = 1$ for all k.
(ii) The shift operator S satisfies $S^*S = I$ but $SS^* \neq I$ thus S is **not** unitary.

Theorem 6.27. *For an operator* U *on a complex Hilbert space* H *the following are equivalent:*
(i) U *is unitary;*
(ii) U *is surjection and an* isometry, *i.e.* $\|Ux\| = \|x\|$ *for all* $x \in H$*;*
(iii) U *is a surjection and preserves the inner product, i.e.* $\langle Ux, Uy\rangle = \langle x, y\rangle$ *for all* $x, y \in H$.

Proof. 6.27(i)⇒6.27(ii). Clearly unitarity of operator implies its invertibility and hence surjectivity. Also

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \langle x, U^*Ux\rangle = \langle x, x\rangle = \|x\|^2 .$$

6.27(ii)⇒6.27(iii). Using the polarisation identity (cf. polarisation in equation (2.9)):

$$\begin{aligned}
4\langle Tx, y\rangle &= \langle T(x+y), x+y\rangle + i\langle T(x+iy), x+iy\rangle \\
&\quad - \langle T(x-y), x-y\rangle - i\langle T(x-iy), x-iy\rangle . \\
&= \sum_{k=0}^{3} i^k \langle T(x+i^k y), x+i^k y\rangle
\end{aligned}$$

Take $T = U^*U$ and $T = I$, then

$$\begin{aligned}
4\langle U^*Ux, y\rangle &= \sum_{k=0}^{3} i^k \langle U^*U(x+i^k y), x+i^k y\rangle \\
&= \sum_{k=0}^{3} i^k \langle U(x+i^k y), U(x+i^k y)\rangle \\
&= \sum_{k=0}^{3} i^k \langle (x+i^k y), (x+i^k y)\rangle \\
&= 4\langle x, y\rangle .
\end{aligned}$$

6.27(iii)⇒6.27(i). Indeed $\langle U^*Ux, y\rangle = \langle x, y\rangle$ implies $\langle (U^*U - I)x, y\rangle = 0$ for all $x,y \in H$, then $U^*U = I$. Since U is surjective, for any $y \in H$ there is $x \in H$ such that $y = Ux$. Then, using the already established fact $U^*U = I$ we get

$$UU^*y = UU^*(Ux) = U(U^*U)x = Ux = y.$$

Thus we have $UU^* = I$ as well and U is unitary. □

Definition 6.28. A *normal operator* T is one for which $T^*T = TT^*$.

Example 6.29.
(i) Any self-adjoint operator T is normal, since $T^* = T$.
(ii) Any unitary operator U is normal, since $U^*U = I = UU^*$.
(iii) Any diagonal operator D is normal , since $De_k = \lambda_k e_k$, $D^*e_k = \bar{\lambda}_k e_k$, and $DD^*e_k = D^*De_k = |\lambda_k|^2 e_k$.
(iv) The shift operator S is **not** normal.
(v) A finite matrix is normal (as an operator on ℓ_2^n) if and only if it has an orthonormal basis in which it is diagonal.

Remark 6.30. Theorems 6.24 and 6.27(ii) draw similarity between those types of operators and multiplications by complex numbers. Indeed Theorem 6.24 said that an operator which significantly change direction of vectors ("rotates") cannot be Hermitian, just like a multiplication by a real number scales but do not rotate. On the other hand Theorem 6.27(ii) says that unitary operator just rotate vectors but do not scale, as a multiplication by an unimodular complex number. We will see further such connections in Theorem 7.17.

7. Spectral Theory

Beware of ghosts[2] in this area!

As we saw operators could be added and multiplied each other, in some sense they behave like numbers, but are much more complicated. In this lecture we will associate to each operator a set of complex numbers which reflects certain (unfortunately not all) properties of this operator.

The analogy between operators and numbers become even more deeper since we could construct *functions of operators* (called *functional calculus*) in a way we build numeric functions. The most important functions of this sort is called *resolvent* (see Definition 7.5). The methods of analytical functions are very powerful in operator theory and students may wish to refresh their knowledge of complex analysis before this part.

7.1. **The spectrum of an operator on a Hilbert space.** An *eigenvalue of operator* $T \in B(H)$ is a complex number λ such that there exists a nonzero $x \in H$, called *eigenvector* with property $Tx = \lambda x$, in other words $x \in \ker(T - \lambda I)$.

In finite dimensions $T - \lambda I$ is invertible if and only if λ is **not** an eigenvalue. In infinite dimensions it is not the same: the right shift operator S is not invertible but 0 is not its eigenvalue because $Sx = 0$ implies $x = 0$ (check!).

Definition 7.1. The *resolvent set* $\rho(T)$ of an operator T is the set

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}.$$

The *spectrum of operator* $T \in B(H)$, denoted $\sigma(T)$, is the complement of the resolvent set $\rho(T)$:

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is } \textbf{not} \text{ invertible}\}.$$

Example 7.2. If H is finite dimensional the from previous discussion follows that $\sigma(T)$ is the set of eigenvalues of T for any T.
Even this example demonstrates that spectrum does not provide a complete description for operator even in finite-dimensional case. For example, both operators in $\mathbb{C}^2$ given by matrices $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ have a single point spectrum $\{0\}$, however are rather different. The situation became even worst in the infinite dimensional spaces.

Theorem 7.3. *The spectrum $\sigma(T)$ of a bounded operator T is a nonempty compact (i.e. closed and bounded) subset of $\mathbb{C}$.*

For the proof we will need several Lemmas.

Lemma 7.4. *Let $A \in B(H)$. If $\|A\| < 1$ then $I - A$ is invertible in $B(H)$ and inverse is given by the* Neumann series *(C. Neumann, 1877):*

$$(I - A)^{-1} = I + A + A^2 + A^3 + \ldots = \sum_{k=0}^{\infty} A^k. \tag{7.1}$$

Proof. Define the sequence of operators $B_n = I + A + \cdots + A^N$—the partial sums of the infinite series (7.1). It is a Cauchy sequence, indeed:

$$\begin{aligned} \|B_n - B_m\| &= \|A^{m+1} + A^{m+2} + \cdots + A^n\| && (\text{if } n < m)\\ &\leqslant \|A^{m+1}\| + \|A^{m+2}\| + \cdots + \|A^n\| \\ &\leqslant \|A\|^{m+1} + \|A\|^{m+2} + \cdots + \|A\|^n \\ &\leqslant \frac{\|A\|^{m+1}}{1 - \|A\|} < \epsilon \end{aligned}$$

for a large m. By the completeness of $B(H)$ there is a limit, say B, of the sequence B_n. It is a simple algebra to check that $(I-A)B_n = B_n(I-A) = I - A^{n+1}$, passing to the limit in the norm topology, where $A^{n+1} \to 0$ and $B_n \to B$ we get:

$$(I - A)B = B(I - A) = I \quad \Leftrightarrow \quad B = (I - A)^{-1}.$$

□

Definition 7.5. The *resolvent*of an operator T is the operator valued function defined on the resolvent set by the formula:

$$R(\lambda, T) = (T - \lambda I)^{-1}. \tag{7.2}$$

Corollary 7.6. (i) *If $|\lambda| > \|T\|$ then $\lambda \in \rho(T)$, hence the spectrum is bounded.*

(ii) *The resolvent set $\rho(T)$ is open, i.e for any $\lambda \in \rho(T)$ then there exist $\epsilon > 0$ such that all μ with $|\lambda - \mu| < \epsilon$ are also in $\rho(T)$, i.e. the resolvent set is open and the spectrum is closed.*

Both statements together imply that the spectrum is compact.

Proof. (i) If $|\lambda| > \|T\|$ then $\|\lambda^{-1}T\| < 1$ and the operator $T - \lambda I = -\lambda(I - \lambda^{-1}T)$ has the inverse

$$R(\lambda, T) = (T - \lambda I)^{-1} = -\sum_{k=0}^{\infty} \lambda^{-k-1}T^k. \tag{7.3}$$

by the previous Lemma.

(ii) Indeed:

$$\begin{aligned} T - \mu I &= T - \lambda I + (\lambda - \mu)I \\ &= (T - \lambda I)(I + (\lambda - \mu)(T - \lambda I)^{-1}). \end{aligned}$$

The last line is an invertible operator because $T - \lambda I$ is invertible by the assumption and $I + (\lambda - \mu)(T - \lambda I)^{-1}$ is invertible by the previous Lemma, since $\|(\lambda - \mu)(T - \lambda I)^{-1}\| < 1$ if $\epsilon < \|(T - \lambda I)^{-1}\|$. □

Exercise 7.7. (i) Prove the *first resolvent identity*:

$$R(\lambda, T) - R(\mu, T) = (\lambda - \mu)R(\lambda, T)R(\mu, T) \tag{7.4}$$

(ii) Use the identity (7.4) to show that $(T - \mu I)^{-1} \to (T - \lambda I)^{-1}$ as $\mu \to \lambda$.

(iii) Use the identity (7.4) to show that for $z \in \rho(t)$ the complex derivative $\frac{d}{dz}R(z, T)$ of the resolvent $R(z, T)$ is well defined, i.e. the resolvent is an analytic function operator valued function of z.

Lemma 7.8. *The spectrum is non-empty.*

Proof. Let us assume the opposite, $\sigma(T) = \varnothing$ then the resolvent function $R(\lambda, T)$ is well defined for all $\lambda \in \mathbb{C}$. As could be seen from the von Neumann series (7.3) $\|R(\lambda, T)\| \to 0$ as $\lambda \to \infty$. Thus for any vectors $x, y \in H$ the function $f(\lambda) = \langle R(\lambda, T)x, y\rangle$ is analytic (see Exercise 7.7(iii)) function tensing to zero at infinity. Then by the Liouville theorem from complex analysis $R(\lambda, T) = 0$, which is impossible. Thus the spectrum is not empty. □

Proof of Theorem 7.3. Spectrum is nonempty by Lemma 7.8 and compact by Corollary 7.6. □

Remark 7.9. Theorem 7.3 gives the maximal possible description of the spectrum, indeed any non-empty compact set could be a spectrum for some bounded operator, see Problem A.23.

7.2. **The spectral radius formula.** The following definition is of interest.

Definition 7.10. The *spectral radius* of T is

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

From the Lemma 7.6(i) immediately follows that $r(T) \leqslant \|T\|$. The more accurate estimation is given by the following theorem.

Theorem 7.11. *For a bounded operator* T *we have*

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}. \tag{7.5}$$

We start from the following general lemma:

Lemma 7.12. *Let a sequence* (a_n) *of positive real numbers satisfies inequalities:* $0 \leqslant a_{m+n} \leqslant a_m + a_n$ *for all* m *and* n*. Then there is a limit* $\lim_{n\to\infty}(a_n/n)$ *and its equal to* $\inf_n(a_n/n)$.

Proof. The statements follows from the observation that for any n and $m = nk+l$ with $0 \leqslant l \leqslant n$ we have $a_m \leqslant ka_n + la_1$ thus, for big m we got $a_m/m \leqslant a_n/n + la_1/m \leqslant a_n/n + \epsilon$. □

Proof of Theorem 7.11. The existence of the limit $\lim_{n\to\infty}\|T^n\|^{1/n}$ in (7.5) follows from the previous Lemma since by the Lemma 6.12 $\log\|T^{n+m}\| \leqslant \log\|T^n\| + \log\|T^m\|$. Now we are using some results from the complex analysis. The Laurent series for the resolvent $R(\lambda, T)$ in the neighbourhood of infinity is given by the von Neumann series (7.3). The radius of its convergence (which is equal, obviously, to $r(T)$) by the Hadamard theorem is exactly $\lim_{n\to\infty}\|T^n\|^{1/n}$. □

Corollary 7.13. *There exists* $\lambda \in \sigma(T)$ *such that* $|\lambda| = r(T)$.

Proof. Indeed, as its known from the complex analysis the boundary of the convergence circle of a Laurent (or Taylor) series contain a singular point, the singular point of the resolvent is obviously belongs to the spectrum. □

Example 7.14. Let us consider the left shift operator S^*, for any $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ the vector $(1, \lambda, \lambda^2, \lambda^3, \ldots)$ is in ℓ_2 and is an eigenvector of S^* with eigenvalue λ, so the open unit disk $|\lambda| < 1$ belongs to $\sigma(S^*)$. On the other hand spectrum of S^* belongs to the closed unit disk $|\lambda| \leqslant 1$ since $r(S^*) \leqslant \|S^*\| = 1$.

Because spectrum is closed it should coincide with the closed unit disk, since the open unit disk is dense in it. Particularly $1 \in \sigma(S^*)$, but it is easy to see that 1 is not an eigenvalue of S^*.

Proposition 7.15. *For any* $T \in B(H)$ *the spectrum of the adjoint operator is* $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$.

Proof. If $(T - \lambda I)V = V(T - \lambda I) = I$ the by taking adjoints $V^*(T^* - \bar{\lambda} I) = (T^* - \bar{\lambda} I)V^* = I$. So $\lambda \in \rho(T)$ implies $\bar{\lambda} \in \rho(T^*)$, using the property $T^{**} = T$ we could invert the implication and get the statement of proposition. □

Example 7.16. In continuation of Example 7.14 using the previous Proposition we conclude that $\sigma(S)$ is also the closed unit disk, but S does not have eigenvalues at all!

7.3. Spectrum of Special Operators.

Theorem 7.17. (i) *If* U *is a unitary operator then* $\sigma(U) \subseteq \{|z| = 1\}$.
(ii) *If* T *is Hermitian then* $\sigma(T) \subseteq \mathbb{R}$.

Proof. (i) If $|\lambda| > 1$ then $\left\|\lambda^{-1}U\right\| < 1$ and then $\lambda I - U = \lambda(I - \lambda^{-1}U)$ is invertible, thus $\lambda \notin \sigma(U)$. If $|\lambda| < 1$ then $\|\lambda U^*\| < 1$ and then $\lambda I - U = U(\lambda U^* - I)$ is invertible, thus $\lambda \notin \sigma(U)$. The remaining set is exactly $\{z : |z| = 1\}$.

(ii) Without lost of generality we could assume that $\|T\| < 1$, otherwise we could multiply T by a small real scalar. Let us consider the *Cayley transform* which maps real axis to the unit circle:

$$U = (T - iI)(T + iI)^{-1}.$$

Straightforward calculations show that U is unitary if T is Hermitian. Let us take $\lambda \notin \mathbb{R}$ and $\lambda \neq -i$ (this case could be checked directly by Lemma 7.4). Then the Cayley transform $\mu = (\lambda - i)(\lambda + i)^{-1}$ of λ is not on the unit circle and thus the operator

$$U - \mu I = (T - iI)(T + iI)^{-1} - (\lambda - i)(\lambda + i)^{-1} I = 2i(\lambda + i)^{-1}(T - \lambda I)(T + iI)^{-1},$$

is invertible, which implies invertibility of $T - \lambda I$. So $\lambda \notin \mathbb{R}$.

□

The above reduction of a self-adjoint operator to a unitary one (it can be done on the opposite direction as well!) is an important tool which can be applied in other questions as well, e.g. in the following exercise.

Exercise 7.18. (i) Show that an operator $U : f(t) \mapsto e^{it}f(t)$ on $L_2[0, 2\pi]$ is unitary and has the entire unit circle $\{|z| = 1\}$ as its spectrum .
(ii) Find a self-adjoint operator T with the entire real line as its spectrum.

8. Compactness

It is not easy to study linear operators "in general" and there are many questions about operators in Hilbert spaces raised many decades ago which are still unanswered. Therefore it is reasonable to single out classes of operators which have (relatively) simple properties. Such a class of operators more closed to finite dimensional ones will be studied here.

These operators are so compact that we even can fit them in our course

8.1. Compact operators.

Let us recall some topological definition and results.

Definition 8.1. A *compact set* in a metric space is defined by the property that any its covering by a family of open sets contains a subcovering by a finite subfamily.

In the finite dimensional vector spaces $\mathbb{R}^n$ or $\mathbb{C}^n$ there is the following equivalent definition of compactness (equivalence of 8.2(i) and 8.2(ii) is known as *Heine–Borel theorem*):

Theorem 8.2. *If a set E in $\mathbb{R}^n$ or $\mathbb{C}^n$ has any of the following properties then it has other two as well:*
(i) *E is bounded and closed;*
(ii) *E is compact;*
(iii) *Any infinite subset of E has a limiting point belonging to E.*

Exercise* 8.3. Which equivalences from above are not true any more in the infinite dimensional spaces?

Definition 8.4. Let X and Y be normed spaces, $T \in B(X, Y)$ is a *finite rank operator* if $\operatorname{Im} T$ is a finite dimensional subspace of Y. T is a *compact operator* if whenever $(x_i)_1^\infty$ is a bounded sequence in X then its image $(Tx_i)_1^\infty$ has a convergent subsequence in Y.
The set of finite rank operators is denote by $F(X, Y)$ and the set of compact operators—by $K(X, Y)$

Exercise 8.5. Show that both $F(X,Y)$ and $K(X,Y)$ are linear subspaces of $B(X,Y)$.

We intend to show that $F(X,Y) \subset K(X,Y)$.

Lemma 8.6. *Let Z be a finite-dimensional normed space. Then there is a number N and a mapping $S : \ell_2^N \to Z$ which is invertible and such that S and S^{-1} are bounded.*

Proof. The proof is given by an explicit construction. Let $N = \dim Z$ and z_1, z_2, …, z_N be a basis in Z. Let us define

$$S : \ell_2^N \to Z \quad \text{by} \quad S(a_1, a_2, \ldots, a_N) = \sum_{k=1}^{N} a_k z_k,$$

then we have an estimation of norm:

$$\begin{aligned} \|Sa\| &= \left\| \sum_{k=1}^{N} a_k z_k \right\| \leqslant \sum_{k=1}^{N} |a_k| \, \|z_k\| \\ &\leqslant \left(\sum_{k=1}^{N} |a_k|^2 \right)^{1/2} \left(\sum_{k=1}^{N} \|z_k\|^2 \right)^{1/2}. \end{aligned}$$

So $\|S\| \leqslant \left(\sum_1^N \|z_k\|^2 \right)^{1/2}$ and S is continuous.
Clearly S has the trivial kernel, particularly $\|Sa\| > 0$ if $\|a\| = 1$. By the Heine–Borel theorem the unit sphere in ℓ_2^N is compact, consequently the continuous function $a \mapsto \left\| \sum_1^N a_k z_k \right\|$ attains its lower bound, which has to be positive. This means there exists $\delta > 0$ such that $\|a\| = 1$ implies $\|Sa\| > \delta$, or, equivalently if $\|z\| < \delta$ then $\left\| S^{-1} z \right\| < 1$. The later means that $\left\| S^{-1} \right\| \leqslant \delta^{-1}$ and boundedness of S^{-1}. □

Corollary 8.7. *For any two metric spaces X and Y we have $F(X,Y) \subset K(X,Y)$.*

Proof. Let $T \in F(X,Y)$, if $(x_n)_1^\infty$ is a bounded sequence in X then $((Tx_n)_1^\infty \subset Z = \operatorname{Im} T$ is also bounded. Let $S : \ell_2^N \to Z$ be a map constructed in the above Lemma. The sequence $(S^{-1} T x_n)_1^\infty$ is bounded in ℓ_2^N and thus has a limiting point, say a_0. Then Sa_0 is a limiting point of $(Tx_n)_1^\infty$. □

There is a simple condition which allows to determine which diagonal operators are compact (particularly the identity operator I_X is *not* compact if $\dim X = \infty$):

Proposition 8.8. *Let* T *is a diagonal operator and given by identities* $Te_n = \lambda_n e_n$ *for all* n *in a basis* e_n. T *is compact if and only if* $\lambda_n \to 0$.

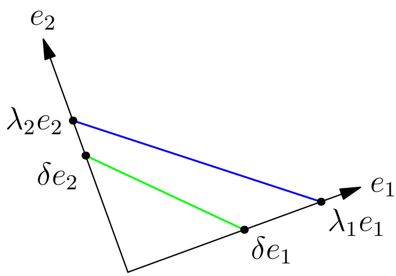


FIGURE 16. Distance between scales of orthonormal vectors

Proof. If $\lambda_n \not\to 0$ then there exists a subsequence λ_{n_k} and $\delta > 0$ such that $|\lambda_{n_k}| > \delta$ for all k. Now the sequence (e_{n_k}) is bounded but its image $Te_{n_k} = \lambda_{n_k} e_{n_k}$ has no convergent subsequence because for any $k \neq l$:

$$\|\lambda_{n_k} e_{n_k} - \lambda_{n_l} e_{n_l}\| = (|\lambda_{n_k}|^2 + |\lambda_{n_l}|^2)^{1/2} \geqslant \sqrt{2}\delta,$$

i.e. Te_{n_k} is not a Cauchy sequence, see Figure 16. For the converse, note that if $\lambda_n \to 0$ then we can define a finite rank operator T_m, $m \geqslant 1$—m-"truncation" of T by:

$$T_m e_n = \begin{cases} Te_n = \lambda_n e_n, & 1 \leqslant n \leqslant m; \\ 0, & n > m. \end{cases} \tag{8.1}$$

Then obviously

$$(T - T_m)e_n = \begin{cases} 0, & 1 \leqslant n \leqslant m; \\ \lambda_n e_n, & n > m, \end{cases}$$

and $\|T - T_m\| = \sup_{n>m} |\lambda_n| \to 0$ if $m \to \infty$. All T_m are finite rank operators (so are compact) and T is also compact as their limit—by the next Theorem. □

Theorem 8.9. *Let* T_m *be a sequence of compact operators convergent to an operator* T *in the norm topology (i.e.* $\|T - T_m\| \to 0$*) then* T *is compact itself. Equivalently* $K(X, Y)$ *is a closed subspace of* $B(X, Y)$.

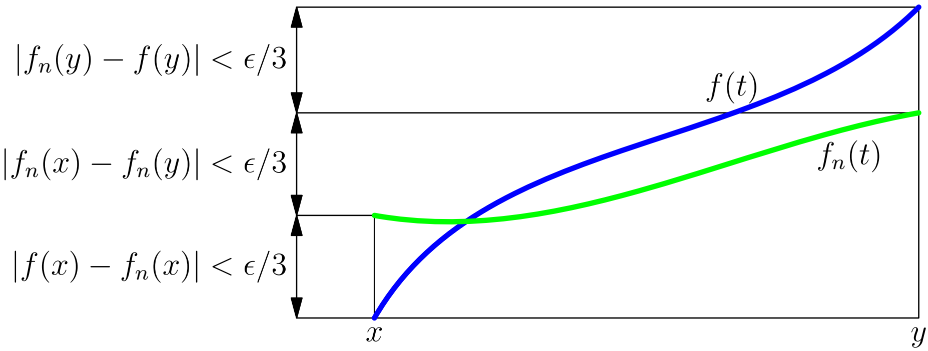


FIGURE 17. The $\epsilon/3$ argument to estimate $|f(x) - f(y)|$.

$$
\begin{array}{cccccccc}
\mathbf{T_1 x_1^{(1)}} & T_1 x_2^{(1)} & T_1 x_3^{(1)} & \ldots & T_1 x_n^{(1)} & \ldots & \to & a_1 \\
T_2 x_1^{(2)} & \mathbf{T_2 x_2^{(2)}} & T_2 x_3^{(2)} & \ldots & T_2 x_n^{(2)} & \ldots & \to & a_2 \\
T_3 x_1^{(3)} & T_3 x_2^{(3)} & \mathbf{T_3 x_3^{(3)}} & \ldots & T_3 x_n^{(3)} & \ldots & \to & a_3 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\
T_n x_1^{(n)} & T_n x_2^{(n)} & T_n x_3^{(n)} & \ldots & \mathbf{T_n x_n^{(n)}} & \ldots & \to & a_n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \downarrow \\
 & & & & & & \searrow & \\
 & & & & & & & \mathbf{a}
\end{array}
$$

TABLE 2. The "diagonal argument".

Proof. Take a bounded sequence $(x_n)_1^\infty$. From compactness

$$
\begin{array}{lllll}
\text{of } T_1 & \Rightarrow \exists & \text{subsequence } (x_n^{(1)})_1^\infty \text{ of } (x_n)_1^\infty & \text{s.t.} & (T_1 x_n^{(1)})_1^\infty \text{ is convergent.} \\
\text{of } T_2 & \Rightarrow \exists & \text{subsequence } (x_n^{(2)})_1^\infty \text{ of } (x_n^{(1)})_1^\infty & \text{s.t.} & (T_2 x_n^{(2)})_1^\infty \text{ is convergent.} \\
\text{of } T_3 & \Rightarrow \exists & \text{subsequence } (x_n^{(3)})_1^\infty \text{ of } (x_n^{(2)})_1^\infty & \text{s.t.} & (T_3 x_n^{(3)})_1^\infty \text{ is convergent.} \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
$$

Could we find a subsequence which converges for all T_m simultaneously? The first guess "take the intersection of all above sequences $(x_n^{(k)})_1^\infty$" does not work because the intersection could be empty. The way out is provided by the *diagonal argument* (see Table 2): a subsequence $(T_m x_k^{(k)})_1^\infty$ is convergent for all m, because at latest after the term $x_m^{(m)}$ it is a subsequence of $(x_k^{(m)})_1^\infty$.

We are claiming that a subsequence $(T x_k^{(k)})_1^\infty$ of $(T x_n)_1^\infty$ is convergent as well. We use here $\epsilon/3$ *argument* (see Figure 17): for a given $\epsilon > 0$ choose $p \in \mathbb{N}$ such that $\|T - T_p\| < \epsilon/3$. Because $(T_p x_k^{(k)}) \to 0$ it is a Cauchy sequence, thus there exists $n_0 > p$ such that $\left\|T_p x_k^{(k)} - T_p x_l^{(l)}\right\| < \epsilon/3$ for all $k, l > n_0$. Then:

$$\begin{aligned} \left\|Tx_k^{(k)} - Tx_l^{(l)}\right\| &= \left\|(Tx_k^{(k)} - T_p x_k^{(k)}) + (T_p x_k^{(k)} - T_p x_l^{(l)}) + (T_p x_l^{(l)} - Tx_l^{(l)})\right\| \\ &\leqslant \left\|Tx_k^{(k)} - T_p x_k^{(k)}\right\| + \left\|T_p x_k^{(k)} - T_p x_l^{(l)}\right\| + \left\|T_p x_l^{(l)} - Tx_l^{(l)}\right\| \\ &\leqslant \epsilon \end{aligned}$$

Thus T is compact. □

8.2. Hilbert–Schmidt operators.

Definition 8.10. Let $T : H \to K$ be a bounded linear map between two Hilbert spaces. Then T is said to be *Hilbert–Schmidt operator* if there exists an orthonormal basis in H such that the series $\sum_{k=1}^{\infty} \|Te_k\|^2$ is convergent.

Example 8.11. (i) Let $T : \ell_2 \to \ell_2$ be a diagonal operator defined by $Te_n = e_n/n$, for all $n \geqslant 1$. Then $\sum \|Te_n\|^2 = \sum n^{-2} = \pi^2/6$ (see Example 5.16) is finite.
(ii) The identity operator I_H is **not** a Hilbert–Schmidt operator, unless H is finite dimensional.

A relation to compact operator is as follows.

Theorem 8.12. *All Hilbert–Schmidt operators are compact. (The opposite inclusion is false, give a counterexample!)*

Proof. Let $T \in B(H, K)$ have a convergent series $\sum \|Te_n\|^2$ in an orthonormal basis $(e_n)_1^\infty$ of H. We again (see (8.1)) define the m-truncation of T by the formula

$$T_m e_n = \begin{cases} Te_n, & 1 \leqslant n \leqslant m; \\ 0, & n > m. \end{cases} \tag{8.2}$$

Then $T_m(\sum_1^\infty a_k e_k) = \sum_1^m a_k e_k$ and each T_m is a finite rank operator because its image is spanned by the finite set of vectors $Te_1, \ldots, Te_n$. We claim that $\|T - T_m\| \to 0$. Indeed by linearity and definition of T_m:

$$(T - T_m)\left(\sum_{n=1}^{\infty} a_n e_n\right) = \sum_{n=m+1}^{\infty} a_n (Te_n).$$

Thus:

$$\left\|(T - T_m)\left(\sum_{n=1}^{\infty} a_n e_n\right)\right\| = \left\|\sum_{n=m+1}^{\infty} a_n (Te_n)\right\| \tag{8.3}$$

$$\leqslant \sum_{n=m+1}^{\infty} |a_n| \, \|(Te_n)\|$$

$$\leqslant \left(\sum_{n=m+1}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=m+1}^{\infty} \|(Te_n)\|^2\right)^{1/2}$$

$$\leqslant \left\|\sum_{n=1}^{\infty} a_n e_n\right\| \left(\sum_{n=m+1}^{\infty} \|(Te_n)\|^2\right)^{1/2} \tag{8.4}$$

so $\|T - T_m\| \to 0$ and by the previous Theorem T is compact as a limit of compact operators. □

Corollary 8.13 (from the above proof). *For a Hilbert–Schmidt operator*

$$\|T\| \leqslant \left(\sum_{n=m+1}^{\infty} \|(Te_n)\|^2\right)^{1/2}.$$

Proof. Just consider difference of T and $T_0 = 0$ in (8.3)–(8.4). □

Example 8.14. An *integral operator* T on $L_2[0,1]$ is defined by the formula:

$$(Tf)(x) = \int_0^1 K(x,y) f(y)\, dy, \qquad f(y) \in L_2[0,1], \tag{8.5}$$

where the continuous on $[0,1] \times [0,1]$ function K is called the *kernel of integral operator*.

Theorem 8.15. *Integral operator* (8.5) *is Hilbert–Schmidt.*

Proof. Let $(e_n)_{-\infty}^{\infty}$ be an orthonormal basis of $L_2[0,1]$, e.g. $(e^{2\pi i n t})_{n\in\mathbb{Z}}$. Let us consider the kernel $K_x(y) = K(x,y)$ as a function of the argument y depending from the parameter x. Then:

$$(Te_n)(x) = \int_0^1 K(x,y) e_n(y)\, dy = \int_0^1 K_x(y) e_n(y)\, dy = \langle K_x, \bar{e}_n \rangle .$$

So $\|Te_n\|^2 = \int_0^1 |\langle K_x, \bar{e}_n\rangle|^2 \, dx$. Consequently:

$$\begin{aligned}\sum_{-\infty}^{\infty} \|Te_n\|^2 &= \sum_{-\infty}^{\infty} \int_0^1 |\langle K_x, \bar{e}_n\rangle|^2 \, dx \\ &= \int_0^1 \sum_{1}^{\infty} |\langle K_x, \bar{e}_n\rangle|^2 \, dx \qquad (8.6) \\ &= \int_0^1 \|K_x\|^2 \, dx \\ &= \int_0^1 \int_0^1 |K(x,y)|^2 \, dx \, dy < \infty\end{aligned}$$

Exercise 8.16. Justify the exchange of summation and integration in (8.6).

□

Remark 8.17. The definition 8.14 and Theorem 8.15 work also for any $T : L_2[a,b] \to L_2[c,d]$ with a continuous kernel $K(x,y)$ on $[c,d] \times [a,b]$.

Definition 8.18. Define *Hilbert–Schmidt norm* of a Hilbert–Schmidt operator A by $\|A\|_{HS}^2 = \sum_{n=1}^{\infty} \|Ae_n\|^2$ (it is independent of the choice of orthonormal basis $(e_n)_1^{\infty}$, see Question A.27).

Exercise* 8.19. Show that set of Hilbert–Schmidt operators with the above norm is a Hilbert space and find the an expression for the inner product.

Example 8.20. Let $K(x,y) = x - y$, then

$$(Tf)(x) = \int_0^1 (x-y) f(y) \, dy = x \int_0^1 f(y) \, dy - \int_0^1 y f(y) \, dy$$

is a rank 2 operator. Furthermore:

$$\begin{aligned}\|T\|_{HS}^2 &= \int_0^1 \int_0^1 (x-y)^2 \, dx \, dy = \int_0^1 \left[\frac{(x-y)^3}{3} \right]_{x=0}^1 dy \\ &= \int_0^1 \frac{(1-y)^3}{3} + \frac{y^3}{3} \, dy = \left[-\frac{(1-y)^4}{12} + \frac{y^4}{12} \right]_0^1 = \frac{1}{6}.\end{aligned}$$

On the other hand there is an orthonormal basis such that

$$Tf = \frac{1}{\sqrt{12}} \langle f, e_1 \rangle e_1 - \frac{1}{\sqrt{12}} \langle f, e_2 \rangle e_2,$$

and $\|T\| = \frac{1}{\sqrt{12}}$ and $\sum_1^2 \|Te_k\|^2 = \frac{1}{6}$ and we get $\|T\| \leqslant \|T\|_{HS}$ in agreement with Corollary 8.13.

9. THE SPECTRAL THEOREM FOR COMPACT NORMAL OPERATORS

Recall from Section 6.5 that an operator T is normal if $TT^* = T^*T$; Hermitian ($T^* = T$) and unitary ($T^* = T^{-1}$) operators are normal.

9.1. Spectrum of normal operators.

Theorem 9.1. *Let $T \in B(H)$ be a normal operator then*

(i) $\ker T = \ker T^*$, *so* $\ker(T - \lambda I) = \ker(T^* - \bar{\lambda} I)$ *for all* $\lambda \in \mathbb{C}$

(ii) *Eigenvectors corresponding to distinct eigenvalues are orthogonal.*

(iii) $\|T\| = r(T)$.

Proof. (i) Obviously:

$$\begin{aligned} x \in \ker T &\Leftrightarrow \langle Tx, Tx \rangle = 0 \Leftrightarrow \langle T^*Tx, x \rangle = 0 \\ &\Leftrightarrow \langle TT^*x, x \rangle = 0 \Leftrightarrow \langle T^*x, T^*x \rangle = 0 \\ &\Leftrightarrow x \in \ker T^*. \end{aligned}$$

The second part holds because normalities of T and $T - \lambda I$ are equivalent.

(ii) If $Tx = \lambda x$, $Ty = \mu y$ then from the previous statement $T^* y = \bar{\mu} y$. If $\lambda \neq \mu$ then the identity

$$\lambda \langle x, y \rangle = \langle Tx, y \rangle = \langle x, T^* y \rangle = \mu \langle x, y \rangle$$

implies $\langle x, y \rangle = 0$.

(iii) Let $S = T^*T$, then S is Hermitian (check!). Consequently, inequality

$$\|Sx\|^2 = \langle Sx, Sx \rangle = \langle S^2 x, x \rangle \leqslant \|S^2\| \|x\|^2$$

implies $\|S\|^2 \leqslant \|S^2\|$. But the opposite inequality follows from the Theorem 6.12, thus we have the equality $\|S^2\| = \|S\|^2$ and more generally by induction: $\|S^{2^m}\| = \|S\|^{2^m}$ for all m.

Now we claim $\|S\| = \|T\|^2$. From Theorem 6.12 and 6.18 we get $\|S\| = \|T^*T\| \leqslant \|T\|^2$. On the other hand if $\|x\| = 1$ then

$$\|T^*T\| \geqslant |\langle T^*Tx, x \rangle| = \langle Tx, Tx \rangle = \|Tx\|^2$$

implies the opposite inequality $\|S\| \geqslant \|T\|^2$. Only now we use normality of T to obtain $(T^{2^m})^* T^{2^m} = (T^*T)^{2^m}$ and get the equality

$$\left\|T^{2^m}\right\|^2 = \left\|(T^*T)^{2^m}\right\| = \|T^*T\|^{2^m} = \|T\|^{2^{m+1}}.$$

Thus:

$$r(T) = \lim_{m\to\infty} \left\|T^{2^m}\right\|^{1/2^m} = \lim_{m\to\infty} \|T\|^{2^{m+1}/2^{m+1}} = \|T\|.$$

by the spectral radius formula (7.5).

□

Example 9.2. It is easy to see that normality is important in 9.1(iii), indeed the non-normal operator T given by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $\mathbb{C}$ has one-point spectrum $\{0\}$, consequently $r(T) = 0$ but $\|T\| = 1$.

Lemma 9.3. *Let T be a compact normal operator then*

(i) *The set of of eigenvalues of T is either finite or a countable sequence tending to zero.*

(ii) *All the eigenspaces, i.e.* $\ker(T - \lambda I)$, *are finite-dimensional for all* $\lambda \neq 0$.

Remark 9.4. This Lemma is true for any compact operator, but we will not use that in our course.

Proof. (i) Let H_0 be the closed linear span of eigenvectors of T. Then T restricted to H_0 is a diagonal compact operator with the same set of eigenvalues λ_n as in H. Then $\lambda_n \to 0$ from Proposition 8.8 .

Exercise 9.5. Use the proof of Proposition 8.8 to give a direct demonstration.

Solution. Or straightforwardly assume opposite: there exist an $\delta > 0$ and infinitely many eigenvalues λ_n such that $|\lambda_n| > \delta$. By the previous Theorem there is an orthonormal sequence v_n of corresponding eigenvectors $Tv_n = \lambda_n v_n$. Now the sequence (v_n) is bounded but its image $Tv_n = \lambda_n e_n$ has no convergent subsequence because for any $k \neq l$:

$$\|\lambda_k v_k - \lambda_l e_l\| = (|\lambda_k|^2 + |\lambda_l|^2)^{1/2} \geqslant \sqrt{2}\delta,$$

i.e. Te_{n_k} is not a Cauchy sequence, see Figure 16. □

(ii) Similarly if $H_0 = \ker(T - \lambda I)$ is infinite dimensional, then restriction of T on H_0 is λI—which is non-compact by Proposition 8.8. Alternatively consider the infinite orthonormal sequence (v_n), $Tv_n = \lambda v_n$ as in Exercise 9.5.

□

Lemma 9.6. *Let T be a compact normal operator. Then all non-zero points $\lambda \in \sigma(T)$ are eigenvalues and there exists an eigenvalue of modulus $\|T\|$.*

Proof. Assume without lost of generality that $T \neq 0$. Let $\lambda \in \sigma(T)$, without lost of generality (multiplying by a scalar) $\lambda = 1$.
We claim that if 1 is not an eigenvalue then there exist $\delta > 0$ such that

$$\|(I - T)x\| \geqslant \delta \|x\| . \tag{9.1}$$

Otherwise there exists a sequence of vectors (x_n) with unit norm such that $(I - T)x_n \to 0$. Then from the compactness of T for a subsequence (x_{n_k}) there is $y \in H$ such that $Tx_{n_k} \to y$, then $x_n \to y$ implying $Ty = y$ and $y \neq 0$—i.e. y is eigenvector with eigenvalue 1.
Now we claim $\operatorname{Im}(I - T)$ is closed, i.e. $y \in \overline{\operatorname{Im}(I - T)}$ implies $y \in \operatorname{Im}(I - T)$. Indeed, if $(I - T)x_n \to y$, then there is a subsequence (x_{n_k}) such that $Tx_{n_k} \to z$ implying $x_{n_k} \to y + z$, then $(I - T)(z + y) = y$ by continuity of $I - T$.
Finally $I - T$ is injective, i.e $\ker(I - T) = \{0\}$, by (9.1). By the property 9.1(i), $\ker(I - T^*) = \{0\}$ as well. But because always $\ker(I - T^*) = \operatorname{Im}(I - T)^\perp$ (by 6.19(ii)) we got surjectivity, i.e. $\operatorname{Im}(I - T)^\perp = \{0\}$, of $I - T$. Thus $(I - T)^{-1}$ exists and is bounded because (9.1) implies $\|y\| > \delta \left\|(I - T)^{-1}y\right\|$. Thus $1 \notin \sigma(T)$.
The existence of eigenvalue λ such that $|\lambda| = \|T\|$ follows from combination of Lemma 7.13 and Theorem 9.1(iii). □

9.2. Compact normal operators.

Theorem 9.7 (The spectral theorem for compact normal operators)**.** *Let T be a compact normal operator on a Hilbert space H. Then there exists an orthonormal sequence (e_n) of eigenvectors of T and corresponding eigenvalues (λ_n) such that:*

$$Tx = \sum_n \lambda_n \langle x, e_n \rangle e_n, \qquad \textit{for all } x \in H. \tag{9.2}$$

If (λ_n) is an infinite sequence it tends to zero.
Conversely, if T is given by a formula (9.2) then it is compact and normal.

Proof. Suppose $T \neq 0$. Then by the previous Theorem there exists an eigenvalue λ_1 such that $|\lambda_1| = \|T\|$ with corresponding eigenvector e_1 of the unit norm. Let $H_1 = \operatorname{Lin}(e_1)^\perp$. If $x \in H_1$ then

$$\langle Tx, e_1 \rangle = \langle x, T^* e_1 \rangle = \left\langle x, \bar{\lambda}_1 e_1 \right\rangle = \lambda_1 \langle x, e_1 \rangle = 0, \tag{9.3}$$

thus $Tx \in H_1$ and similarly $T^*x \in H_1$. Write $T_1 = T|_{H_1}$ which is again a normal compact operator with a norm does not exceeding $\|T\|$. We could inductively repeat this procedure for T_1 obtaining sequence of eigenvalues $\lambda_2, \lambda_3, \ldots$ with eigenvectors $e_2, e_3, \ldots$. If $T_n = 0$ for a finite n then theorem is already proved. Otherwise we have an infinite sequence $\lambda_n \to 0$. Let

$$x = \sum_1^n \langle x, e_k \rangle e_k + y_n \quad \Rightarrow \quad \|x\|^2 = \sum_1^n |\langle x, e_k \rangle|^2 + \|y_n\|^2, \qquad y_n \in H_n,$$

from Pythagoras's theorem. Then $\|y_n\| \leqslant \|x\|$ and $\|Ty_n\| \leqslant \|T_n\| \|y_n\| \leqslant |\lambda_n| \|x\| \to 0$ by Lemma 9.3. Thus

$$Tx = \lim_{n\to\infty} \left(\sum_1^n \langle x, e_n \rangle Te_n + Ty_n \right) = \sum_1^\infty \lambda_n \langle x, e_n \rangle e_n$$

Conversely, if $Tx = \sum_1^\infty \lambda_n \langle x, e_n \rangle e_n$ then

$$\langle Tx, y \rangle = \sum_1^\infty \lambda_n \langle x, e_n \rangle \langle e_n, y \rangle = \sum_1^\infty \langle x, e_n \rangle \lambda_n \overline{\langle y, e_n \rangle},$$

thus $T^*y = \sum_1^\infty \bar{\lambda}_n \langle y, e_n \rangle e_n$. Then we got the normality of T: $T^*Tx = TT^*x = \sum_1^\infty |\lambda_n|^2 \langle y, e_n \rangle e_n$. Also T is compact because it is a uniform limit of the finite rank operators $T_n x = \sum_1^n \lambda_n \langle x, e_n \rangle e_n$. □

Corollary 9.8. *Let* T *be a compact normal operator on a separable Hilbert space* H, *then there exists a orthonormal basis* g_k *such that*

$$Tx = \sum_1^\infty \lambda_n \langle x, g_n \rangle g_n,$$

and λ_n *are eigenvalues of* T *including zeros.*

Proof. Let (e_n) be the orthonormal sequence constructed in the proof of the previous Theorem. Then x is perpendicular to all e_n if and only if its in the kernel of T. Let (f_n) be any orthonormal basis of $\ker T$. Then the union of (e_n) and (f_n) is the orthonormal basis (g_n) we have looked for. □

Exercise 9.9. Finish all details in the above proof.

Corollary 9.10 (Singular value decomposition). *If* T *is any compact operator on a separable Hilbert space then there exists orthonormal sequences* (e_k) *and* (f_k) *such that* $Tx = \sum_k \mu_k \langle x, e_k \rangle f_k$ *where* (μ_k) *is a sequence of positive numbers such that* $\mu_k \to 0$ *if it is an infinite sequence.*

Proof. Operator T^*T is compact and Hermitian (hence normal). From the previous Corollary there is an orthonormal basis (e_k) such that $T^*Tx =$

$\sum_n \lambda_n \langle x, e_k \rangle e_k$ for some positive $\lambda_n = \|Te_n\|^2$. Let $\mu_n = \|Te_n\|$ and $f_n = Te_n/\mu_n$. Then f_n is an orthonormal sequence (check!) and

$$Tx = \sum_n \langle x, e_n \rangle Te_n = \sum_n \langle x, e_n \rangle \mu_n f_n.$$

□

Corollary 9.11. *A bounded operator in a Hilber space is compact if and only if it is a uniform limit of the finite rank operators.*

Proof. Sufficiency follows from 8.9.
Necessity: by the previous Corollary $Tx = \sum_n \langle x, e_n \rangle \mu_n f_n$ thus T is a uniform limit of operators $T_m x = \sum_{n=1}^{m} \langle x, e_n \rangle \mu_n f_n$ which are of finite rank. □

10. APPLICATIONS TO INTEGRAL EQUATIONS

In this lecture we will study the *Fredholm equation* defined as follows. Let the *integral operator* with a *kernel* $K(x, y)$ defined on $[a, b] \times [a, b]$ be defined as before:

$$(T\phi)(x) = \int_a^b K(x, y)\phi(y)\, dy. \tag{10.1}$$

The Fredholm equation of the *first* and *second* kinds correspondingly are:

$$T\phi = f \qquad \text{and} \qquad \phi - \lambda T\phi = f, \tag{10.2}$$

for a function f on $[a, b]$. A special case is given by *Volterra equation* by an operator integral operator (10.1) T with a kernel $K(x, y) = 0$ for all $y > x$ which could be written as:

$$(T\phi)(x) = \int_a^x K(x, y)\phi(y)\, dy. \tag{10.3}$$

We will consider integral operators with kernels K such that $\int_a^b \int_a^b K(x, y)\, dx\, dy < \infty$, then by Theorem 8.15 T is a Hilbert–Schmidt operator and in particular bounded.

As a reason to study Fredholm operators we will mention that solutions of differential equations in mathematical physics (notably heat and wave equations) requires a decomposition of a function f as a linear combination of functions $K(x, y)$ with "coefficients" ϕ. This is an continuous analog of a discrete decomposition into Fourier series.

Using ideas from the proof of Lemma 7.4 we define *Neumann series* for the resolvent:

$$(I - \lambda T)^{-1} = I + \lambda T + \lambda^2 T^2 + \cdots, \tag{10.4}$$

which is valid for all $\lambda < \|T\|^{-1}$.

Example 10.1. Solve the Volterra equation

$$\phi(x) - \lambda \int_0^x y\phi(y)\,dy = x^2, \qquad \text{on } L_2[0,1].$$

In this case $I - \lambda T\phi = f$, with $f(x) = x^2$ and:

$$K(x,y) = \begin{cases} y, & 0 \leqslant y \leqslant x; \\ 0, & x < y \leqslant 1. \end{cases}$$

Straightforward calculations shows:

$$\begin{aligned} (Tf)(x) &= \int_0^x y \cdot y^2\,dy = \frac{x^4}{4}, \\ (T^2f)(x) &= \int_0^x y\frac{y^4}{4}\,dy = \frac{x^6}{24}, \ldots \end{aligned}$$

and generally by induction:

$$(T^nf)(x) = \int_0^x y\frac{y^{2n}}{2^{n-1}n!}\,dy = \frac{x^{2n+2}}{2^n(n+1)!}.$$

Hence:

$$\begin{aligned} \phi(x) &= \sum_0^\infty \lambda^n T^n f = \sum_0^\infty \frac{\lambda^n x^{2n+2}}{2^n(n+1)!} \\ &= \frac{2}{\lambda}\sum_0^\infty \frac{\lambda^{n+1}x^{2n+2}}{2^{n+1}(n+1)!} \\ &= \frac{2}{\lambda}(e^{\lambda x^2/2} - 1) \qquad \text{for all } \lambda \in \mathbb{C} \setminus \{0\}, \end{aligned}$$

because in this case $r(T) = 0$. For the Fredholm equations this is not always the case, see Tutorial problem A.29.

Among other integral operators there is an important subclass with *separable kernel*, namely a kernel which has a form:

$$K(x,y) = \sum_{j=1}^n g_j(x)h_j(y). \tag{10.5}$$

In such a case:

$$\begin{aligned}(T\phi)(x) &= \int_a^b \sum_{j=1}^n g_j(x)h_j(y)\phi(y)\,dy \\ &= \sum_{j=1}^n g_j(x)\int_a^b h_j(y)\phi(y)\,dy,\end{aligned}$$

i.e. the image of T is spanned by $g_1(x), \ldots, g_n(x)$ and is finite dimensional, consequently the solution of such equation reduces to linear algebra.

Example 10.2. Solve the Fredholm equation (actually find eigenvectors of T):

$$\begin{aligned}\phi(x) &= \lambda\int_0^{2\pi} \cos(x+y)\phi(y)\,dy \\ &= \lambda\int_0^{2\pi} (\cos x\cos y - \sin x\sin y)\phi(y)\,dy.\end{aligned}$$

Clearly $\phi(x)$ should be a linear combination $\phi(x) = A\cos x + B\sin x$ with coefficients A and B satisfying to:

$$\begin{aligned}A &= \lambda\int_0^{2\pi} \cos y(A\cos y + B\sin y)\,dy, \\ B &= -\lambda\int_0^{2\pi} \sin y(A\cos y + B\sin y)\,dy.\end{aligned}$$

Basic calculus implies $A = \lambda\pi A$ and $B = -\lambda\pi B$ and the only nonzero solutions are:

$$\begin{array}{lll}\lambda = \pi^{-1} & A \neq 0 & B = 0 \\ \lambda = -\pi^{-1} & A = 0 & B \neq 0\end{array}$$

We develop some Hilbert–Schmidt theory for integral operators.

Theorem 10.3. *Suppose that $K(x, y)$ is a continuous function on $[a, b] \times [a, b]$ and $K(x, y) = \overline{K(y, x)}$ and operator T is defined by (10.1). Then*

(i) *T is a self-adjoint Hilbert–Schmidt operator.*
(ii) *All eigenvalues of T are real and satisfy $\sum_n \lambda_n^2 < \infty$.*

(iii) *The eigenvectors* v_n *of* T *can be chosen as an orthonormal basis of* $L_2[a, b]$*, are continuous for nonzero* λ_n *and*

$$T\phi = \sum_{n=1}^{\infty} \lambda_n \langle \phi, v_n \rangle v_n \qquad \textit{where} \quad \phi = \sum_{n=1}^{\infty} \langle \phi, v_n \rangle v_n$$

Proof. (i) The condition $K(x, y) = \overline{K(y, x)}$ implies the Hermitian property of T:

$$\begin{aligned}
\langle T\phi, \psi \rangle &= \int_a^b \left(\int_a^b K(x, y)\phi(y)\, dy \right) \bar{\psi}(x)\, dx \\
&= \int_a^b \int_a^b K(x, y)\phi(y)\bar{\psi}(x)\, dx\, dy \\
&= \int_a^b \phi(y) \left(\int_a^b \overline{K(y, x)\psi(x)}\, dx \right) dy \\
&= \langle \phi, T\psi \rangle .
\end{aligned}$$

The Hilbert–Schmidt property (and hence compactness) was proved in Theorem 8.15.

(ii) Spectrum of T is real as for any Hermitian operator, see Theorem 7.17(ii) and finiteness of $\sum_n \lambda_n^2$ follows from Hilbert–Schmidt property

(iii) The existence of orthonormal basis consisting from eigenvectors (v_n) of T was proved in Corollary 9.8. If $\lambda_n \neq 0$ then:

$$\begin{aligned}
v_n(x_1) - v_n(x_2) &= \lambda_n^{-1}((Tv_n)(x_1) - (Tv_n)(x_2)) \\
&= \frac{1}{\lambda_n} \int_a^b (K(x_1, y) - K(x_2, y))v_n(y)\, dy
\end{aligned}$$

and by Cauchy–Schwarz-Bunyakovskii inequality:

$$|v_n(x_1) - v_n(x_2)| \leqslant \frac{1}{|\lambda_n|} \|v_n\|_2 \int_a^b |K(x_1, y) - K(x_2, y)|\, dy$$

which tense to 0 due to (uniform) continuity of $K(x, y)$.

□

Theorem 10.4. *Let* T *be as in the previous Theorem. Then if* $\lambda \neq 0$ *and* $\lambda^{-1} \notin \sigma(T)$, *the unique solution* ϕ *of the* Fredholm equation of the second kind $\phi - \lambda T\phi = f$ *is*

$$\phi = \sum_{1}^{\infty} \frac{\langle f, v_n \rangle}{1 - \lambda\lambda_n} v_n. \tag{10.6}$$

Proof. Let $\phi = \sum_1^\infty a_n v_n$ where $a_n = \langle \phi, v_n \rangle$, then

$$\phi - \lambda T\phi = \sum_{1}^{\infty} a_n(1 - \lambda\lambda_n)v_n = f = \sum_{1}^{\infty} \langle f, v_n \rangle v_n$$

if and only if $a_n = \langle f, v_n \rangle / (1-\lambda\lambda_n)$ for all n. Note $1-\lambda\lambda_n \neq 0$ since $\lambda^{-1} \notin \sigma(T)$. Because $\lambda_n \to 0$ we got $\sum_1^\infty |a_n|^2$ by its comparison with $\sum_1^\infty |\langle f, v_n \rangle|^2 = \|f\|^2$, thus the solution exists and is unique by the Riesz–Fisher Theorem. □

See Exercise A.30 for an example.

Theorem 10.5 (Fredholm alternative)**.** *Let* $T \in \mathcal{K}(H)$ *be compact normal and* $\lambda \in \mathbb{C} \setminus \{0\}$. *Consider the equations:*

$$\phi - \lambda T\phi = 0 \tag{10.7}$$

$$\phi - \lambda T\phi = f \tag{10.8}$$

then ***either***

(A) *the only solution to* (10.7) *is* $\phi = 0$ *and* (10.8) *has a unique solution for any* $f \in H$; ***or***

(B) *there exists a nonzero solution to* (10.7) *and* (10.8) *can be solved if and only if* f *is orthogonal all solutions to* (10.7).

Proof. (A) If $\phi = 0$ is the only solution of (10.7), then λ^{-1} is not an eigenvalue of T and then by Lemma 9.6 is neither in spectrum of T. Thus $I - \lambda T$ is invertible and the unique solution of (10.8) is given by $\phi = (I - \lambda T)^{-1} f$.

(B) A nonzero solution to (10.7) means that $\lambda^{-1} \in \sigma(T)$. Let (v_n) be an orthonormal basis of eigenvectors of T for eigenvalues (λ_n). By Lemma 9.3(ii) only a finite number of λ_n is equal to λ^{-1}, say they are $\lambda_1, \ldots, \lambda_N$, then

$$(I - \lambda T)\phi = \sum_{n=1}^{\infty} (1 - \lambda\lambda_n) \langle \phi, v_n \rangle v_n = \sum_{n=N+1}^{\infty} (1 - \lambda\lambda_n) \langle \phi, v_n \rangle v_n.$$

If $f = \sum_1^\infty \langle f, v_n \rangle v_n$ then the identity $(I - \lambda T)\phi = f$ is only possible if $\langle f, v_n \rangle = 0$ for $1 \leqslant n \leqslant N$. Conversely from that condition we could give a solution

$$\phi = \sum_{n=N+1}^{\infty} \frac{\langle f, v_n \rangle}{1 - \lambda\lambda_n} v_n + \phi_0, \qquad \text{for any } \phi_0 \in \mathrm{Lin}(v_1, \ldots, v_N),$$

which is again in H because $f \in H$ and $\lambda_n \to 0$.

□

Example 10.6. Let us consider

$$(T\phi)(x) = \int_0^1 (2xy - x - y + 1)\phi(y)\, dy.$$

Because the kernel of T is real and symmetric $T = T^*$, the kernel is also separable:

$$(T\phi)(x) = x \int_0^1 (2y - 1)\phi(y)\, dy + \int_0^1 (-y + 1)\phi(y)\, dy,$$

and T of the rank 2 with image of T spanned by 1 and x. By direct calculations:

$$\begin{array}{lll} T: & 1 & \mapsto \frac{1}{2} \\ T: & x & \mapsto \frac{1}{6}x + \frac{1}{6}, \end{array} \qquad \text{or T is given by the matrix} \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} \end{pmatrix}$$

According to linear algebra decomposition over eigenvectors is:

$$\lambda_1 = \frac{1}{2} \quad \text{with vector} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\lambda_2 = \frac{1}{6} \quad \text{with vector} \quad \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

with normalisation $v_1(y) = 1$, $v_2(y) = \sqrt{12}(y - 1/2)$ and we complete it to an orthonormal basis (v_n) of $L_2[0,1]$. Then

- If $\lambda \neq 2$ or 6 then $(I - \lambda T)\phi = f$ has a unique solution (cf. equation (10.6)):

$$\begin{aligned} \phi &= \sum_{n=1}^{2} \frac{\langle f, v_n \rangle}{1 - \lambda\lambda_n} v_n + \sum_{n=3}^{\infty} \langle f, v_n \rangle v_n \\ &= \sum_{n=1}^{2} \frac{\langle f, v_n \rangle}{1 - \lambda\lambda_n} v_n + \left(f - \sum_{n=1}^{2} \langle f, v_n \rangle v_n \right) \\ &= f + \sum_{n=1}^{2} \frac{\lambda\lambda_n}{1 - \lambda\lambda_n} \langle f, v_n \rangle v_n. \end{aligned}$$

- If $\lambda = 2$ then the solutions exist provided $\langle f, v_1 \rangle = 0$ and are:

$$\phi = f + \frac{\lambda\lambda_2}{1 - \lambda\lambda_2} \langle f, v_2 \rangle v_2 + Cv_1 = f + \frac{1}{2} \langle f, v_2 \rangle v_2 + Cv_1, \qquad C \in \mathbb{C}.$$

- If $\lambda = 6$ then the solutions exist provided $\langle f, v_2 \rangle = 0$ and are:

$$\phi = f + \frac{\lambda\lambda_1}{1 - \lambda\lambda_1} \langle f, v_1 \rangle v_1 + Cv_2 = f - \frac{3}{2} \langle f, v_2 \rangle v_2 + Cv_2, \qquad C \in \mathbb{C}.$$

11. Banach and Normed Spaces

We will work with either the field of real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. To avoid repetition, we use $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

11.1. **Normed spaces.** Recall, see Defn. 2.3, a *norm* on a vector space V is a map $\|\cdot\| : V \to [0, \infty)$ such that

(i) $\|u\| = 0$ only when $u = 0$;
(ii) $\|\lambda u\| = |\lambda| \, \|u\|$ for $\lambda \in \mathbb{K}$ and $u \in V$;
(iii) $\|u + v\| \leqslant \|u\| + \|v\|$ for $u, v \in V$.

Note, that the second and third conditions imply that linear operations—multiplication by a scalar and addition of vectors respectively—are continuous in the topology defined by the norm.

A norm induces a *metric*, see Defn. 2.1, on V by setting $d(u, v) = \|u - v\|$. When V is *complete*, see Defn. 2.6, for this metric, we say that V is a *Banach space*.

Theorem 11.1. *Every finite-dimensional normed vector space is a Banach space.*

We will use the following simple inequality:

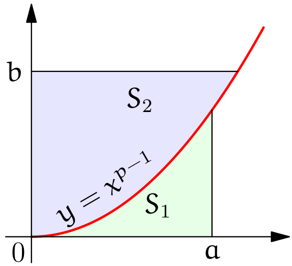
Lemma 11.2 (Young's inequality). *Let two real numbers* $1 < p, q < \infty$ *are related through* $\frac{1}{p} + \frac{1}{q} = 1$ *then*

$$|ab| \leqslant \frac{|a|^p}{p} + \frac{|b|^q}{q}, \tag{11.1}$$

for any complex a *and* b.

First proof: analytic. Obviously, it is enough to prove inequality for positive reals $a = |a|$ and $b = |b|$. If $p > 1$ then $0 < \frac{1}{p} < 1$. Consider the function $\phi(t) = t^m - mt$ for an $0 < m < 1$. From its derivative $\phi(t) = m(t^{m-1} - 1)$ we find the only critical point $t = 1$ on $[0, \infty)$, which is its maximum for $m = \frac{1}{p} < 1$. Thus write the inequality $\phi(t) \leqslant \phi(1)$ for $t = a^p/b^q$ and $m = 1/p$. After a transformation we get $a \cdot b^{-q/p} - 1 \leqslant \frac{1}{p}(a^p b^{-q} - 1)$ and multiplication by b^q with rearrangements lead to the desired result. □

Second proof: geometric. Consider the plane with coordinates (x, y) and take the curve $y = x^{p-1}$ which is the same as $x = y^{q-1}$. Comparing areas on the figure:



we see that $S_1 + S_2 \geqslant ab$ for any positive reals a and b. Elementary integration shows:

$$S_1 = \int_0^a x^{p-1}\,dx = \frac{a^p}{p}, \qquad S_2 = \int_0^b y^{q-1}\,dy = \frac{b^q}{q}.$$

This finishes the demonstration. □

Remark 11.3. You may notice, that the both proofs introduced some specific auxiliary functions related to x^p/p. It is a fruitful generalisation to conduct the proofs for more functions and derive respective forms of Young's inequality.

Proposition 11.4 (Hölder's Inequality). *For $1 < p < \infty$, let $q \in (1, \infty)$ be such that $1/p + 1/q = 1$. For $n \geqslant 1$ and $u, v \in \mathbb{K}^n$, we have that*

$$\sum_{j=1}^{n} |u_j v_j| \leqslant \left(\sum_{j=1}^{n} |u_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |v_j|^q \right)^{\frac{1}{q}}.$$

Proof. For reasons become clear soon we use the notation $\|u\|_p = \left(\sum_{j=1}^{n} |u_j|^p\right)^{\frac{1}{p}}$ and $\|v\|_q = \left(\sum_{j=1}^{n} |v_j|^q\right)^{\frac{1}{q}}$ and define for $1 \leqslant i \leqslant n$:

$$a_i = \frac{u_i}{\|u\|_p} \qquad \text{and} \qquad b_i = \frac{v_i}{\|v\|_q}.$$

Summing up for $1 \leqslant i \leqslant n$ all inequalities obtained from (11.1):

$$|a_i b_i| \leqslant \frac{|a_i|^p}{p} + \frac{|b_i|^q}{q},$$

we get the result. □

Using Hölder inequality we can derive the following one:

Proposition 11.5 (Minkowski's Inequality). *For $1 < p < \infty$, and $n \geqslant 1$, let $u, v \in \mathbb{K}^n$. Then*

$$\left(\sum_{j=1}^{n} |u_j + v_j|^p\right)^{1/p} \leqslant \left(\sum_{j=1}^{n} |u_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |v_j|^p\right)^{1/p}.$$

Proof. For $p > 1$ we have:

$$\sum_{1}^{n} |u_k + v_k|^p = \sum_{1}^{n} |u_k| \, |u_k + v_k|^{p-1} + \sum_{1}^{n} |v_k| \, |u_k + v_k|^{p-1}. \tag{11.2}$$

By Hölder inequality

$$\sum_{1}^{n} |u_k| \, |u_k + v_k|^{p-1} \leqslant \left(\sum_{1}^{n} |u_k|^p\right)^{\frac{1}{p}} \left(\sum_{1}^{n} |u_k + v_k|^{q(p-1)}\right)^{\frac{1}{q}}.$$

Adding a similar inequality for the second term in the right hand side of (11.2) and division by $\left(\sum_{1}^{n} |u_k + v_k|^{q(p-1)}\right)^{\frac{1}{q}}$ yields the result. □

Minkowski's inequality shows that for $1 \leqslant p < \infty$ (the case $p = 1$ is easy) we can define a norm $\|\cdot\|_p$ on $\mathbb{K}^n$ by

$$\|u\|_p = \left(\sum_{j=1}^{n} |u_j|^p\right)^{1/p} \qquad (u = (u_1, \cdots, u_n) \in \mathbb{K}^n).$$

See, Figure 2 for illustration of various norms of this type defined in $\mathbb{R}^2$.

We can define an infinite analogue of this. Let $1 \leqslant p < \infty$, let ℓ_p be the space of all scalar sequences (x_n) with $\sum_n |x_n|^p < \infty$. A careful use of Minkowski's inequality shows that ℓ_p is a vector space. Then ℓ_p becomes a normed space for the $\|\cdot\|_p$ norm. Note also, that ℓ_2 is the Hilbert space introduced before in Example 2.12(ii).

Recall that a Cauchy sequence, see Defn. 2.5, in a normed space is bounded: if (x_n) is Cauchy then we can find N with $\|x_n - x_m\| < 1$ for all $n, m \geqslant N$. Then $\|x_n\| \leqslant \|x_n - x_N\| + \|x_N\| < \|x_N\| + 1$ for $n \geqslant N$, so in particular, $\|x_n\| \leqslant \max(\|x_1\|, \|x_2\|, \cdots, \|x_{N-1}\|, \|x_N\| + 1)$.

Theorem 11.6. *For $1 \leqslant p < \infty$, the space ℓ_p is a Banach space.*

Remark 11.7. Most completeness proofs (in particular, *all* completeness proof in this course) are similar to the next one, see also Thm. 2.24. The general scheme of those proofs has three steps:

(i) For a general Cauchy sequence we build "limit" in some point-wise sense.
(ii) At this stage it is not clear either the constructed "limit" is at our space at all, that is shown on the second step.
(iii) From the construction it does not follows that the "limit" is really the limit in the topology of our space, that is the third step of the proof.

Proof. We repeat the proof of Thm. 2.24 changing 2 to p. Let $(x^{(n)})$ be a Cauchy-sequence in ℓ_p; we wish to show this converges to some vector in ℓ_p.
For each n, $x^{(n)} \in \ell_p$ so is a sequence of scalars, say $(x_k^{(n)})_{k=1}^{\infty}$. As $(x^{(n)})$ is Cauchy, for each $\epsilon > 0$ there exists N_ϵ so that $\left\|x^{(n)} - x^{(m)}\right\|_p \leqslant \epsilon$ for $n, m \geqslant N_\epsilon$.
For k fixed,

$$\left|x_k^{(n)} - x_k^{(m)}\right| \leqslant \left(\sum_j \left|x_j^{(n)} - x_j^{(m)}\right|^p\right)^{1/p} = \left\|x^{(n)} - x^{(m)}\right\|_p \leqslant \epsilon,$$

when $n, m \geqslant N_\epsilon$. Thus the scalar sequence $(x_k^{(n)})_{n=1}^{\infty}$ is Cauchy in $\mathbb{K}$ and hence converges, to x_k say. Let $x = (x_k)$, so that x is a candidate for the limit of $(x^{(n)})$.
Firstly, we check that $x - x^{(n)} \in \ell_p$ for some n. Indeed, for a given $\epsilon > 0$ find n_0 such that $\left\|x^{(n)} - x^{(m)}\right\| < \epsilon$ for all $n, m > n_0$. For any K and m:

$$\sum_{k=1}^{K} \left|x_k^{(n)} - x_k^{(m)}\right|^p \leqslant \left\|x^{(n)} - x^{(m)}\right\|^p < \epsilon^p.$$

Let $m \to \infty$ then $\sum_{k=1}^{K} \left|x_k^{(n)} - x_k\right|^p \leqslant \epsilon^p$.
Let $K \to \infty$ then $\sum_{k=1}^{\infty} \left|x_k^{(n)} - x_k\right|^p \leqslant \epsilon^p$. Thus $x^{(n)} - x \in \ell_p$ and because ℓ_p is a linear space then $x = x^{(n)} - (x^{(n)} - x)$ is also in ℓ_p.
Finally, we saw above that for any $\epsilon > 0$ there is n_0 such that $\left\|x^{(n)} - x\right\| < \epsilon$ for all $n > n_0$. Thus $x^{(n)} \to x$. □

For $p = \infty$, there are two analogies to the ℓ_p spaces. First, we define ℓ_∞ to be the vector space of all bounded scalar sequences, with the sup-norm ($\|\cdot\|_\infty$-norm):

$$\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}} |x_n| \qquad ((x_n) \in \ell_\infty). \tag{11.3}$$

Second, we define c_0 to be the space of all scalar sequences (x_n) which converge to 0. We equip c_0 with the sup norm (11.3). This is defined, as if $x_n \to 0$, then (x_n) is bounded. Hence c_0 is a subspace of ℓ_∞, and we can check (exercise!) that c_0 is closed.

Theorem 11.8. *The spaces c_0 and ℓ_∞ are Banach spaces.*

Proof. This is another variant of the previous proof of Thm. 11.6. We do the ℓ_∞ case. Again, let $(x^{(n)})$ be a Cauchy sequence in ℓ_∞, and for each n, let $x^{(n)} = (x_k^{(n)})_{k=1}^\infty$. For $\epsilon > 0$ we can find N such that $\left\|x^{(n)} - x^{(m)}\right\|_\infty < \epsilon$ for $n, m \geqslant N$. Thus, for any k, we see that $\left|x_k^{(n)} - x_k^{(m)}\right| < \epsilon$ when $n, m \geqslant N$. So $(x_k^{(n)})_{n=1}^\infty$ is Cauchy, and hence converges, say to $x_k \in \mathbb{K}$. Let $x = (x_k)$.
Let $m \geqslant N$, so that for any k, we have that

$$\left|x_k - x_k^{(m)}\right| = \lim_{n\to\infty} \left|x_k^{(n)} - x_k^{(m)}\right| \leqslant \epsilon.$$

As k was arbitrary, we see that $\sup_k \left|x_k - x_k^{(m)}\right| \leqslant \epsilon$. So, firstly, this shows that $(x - x^{(m)}) \in \ell_\infty$, and so also $x = (x - x^{(m)}) + x^{(m)} \in \ell_\infty$. Secondly, we have shown that $\left\|x - x^{(m)}\right\|_\infty \leqslant \epsilon$ when $m \geqslant N$, so $x^{(m)} \to x$ in norm. □

Example 11.9. We can also consider a Banach space of functions $L_p[a, b]$ with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt\right)^{1/p}.$$

See the discussion after Defn. 2.22 for a realisation of such spaces.

11.2. **Bounded linear operators.** Recall what a *linear* map is, see Defn. 6.1. A linear map is often called an *operator*. A linear map $T : E \to F$ between normed spaces is *bounded* if there exists $M > 0$ such that $\|T(x)\| \leqslant M \|x\|$ for $x \in E$, see Defn. 6.3. We write $B(E, F)$ for the set of operators from E to F. For the natural operations, $B(E, F)$ is a vector space. We norm $B(E, F)$ by setting

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \in E, x \neq 0\right\}. \tag{11.4}$$

Exercise 11.10. Show that

(i) The expression (11.4) *is* a norm in the sense of Defn. 2.3.
(ii) We equivalently have

$$\|T\| = \sup\{\|T(x)\| : x \in E, \|x\| \leqslant 1\} = \sup\{\|T(x)\| : x \in E, \|x\| = 1\}.$$

Proposition 11.11. *For a linear map $T : E \to F$ between normed spaces, the following are equivalent:*

(i) *T is continuous (for the metrics induced by the norms on E and F);*
(ii) *T is continuous at 0;*
(iii) *T is bounded.*

Proof. Proof essentially follows the proof of similar Theorem 4.4. See also discussion about usefulness of this theorem there. □

Theorem 11.12. *Let* E *be a normed space, and let* F *be a Banach space. Then* $B(E,F)$ *is a Banach space.*

Proof. In the essence, we follows the same three-step procedure as in Thms. 2.24, 11.6 and 11.8. Let (T_n) be a Cauchy sequence in $\mathcal{B}(E,F)$. For $x \in E$, check that $(T_n(x))$ is Cauchy in F, and hence converges to, say, $T(x)$, as F is complete. Then check that $T : E \to F$ is linear, bounded, and that $\|T_n - T\| \to 0$. □

We write $B(E)$ for $B(E,E)$. For normed spaces E, F and G, and for $T \in B(E,F)$ and $S \in B(F,G)$, we have that $ST = S \circ T \in \mathcal{B}(E,G)$ with $\|ST\| \leqslant \|S\| \|T\|$.

For $T \in B(E,F)$, if there exists $S \in B(F,E)$ with $ST = I_E$, the identity of E, and $TS = I_F$, then T is said to be *invertible*, and write $T = S^{-1}$. In this case, we say that E and F are *isomorphic* spaces, and that T is an *isomorphism*.

If $\|T(x)\| = \|x\|$ for each $x \in E$, we say that T is an *isometry*. If additionally T is an isomorphism, then T is an *isometric isomorphism*, and we say that E and F are *isometrically isomorphic*.

11.3. **Dual Spaces.** Let E be a normed vector space, and let E^* (also written E') be $B(E,\mathbb{K})$, the space of bounded linear maps from E to $\mathbb{K}$, which we call *functionals*, or more correctly, *bounded linear functionals*, see Defn. 4.1. Notice that as $\mathbb{K}$ is complete, the above theorem shows that E^* is always a Banach space.

Theorem 11.13. *Let* $1 < p < \infty$*, and again let* q *be such that* $1/p + 1/q = 1$*. Then the map* $\ell_q \to (\ell_p)^* : u \mapsto \phi_u$*, is an isometric isomorphism, where* ϕ_u *is defined, for* $u = (u_j) \in \ell_q$*, by*

$$\phi_u(x) = \sum_{j=1}^{\infty} u_j x_j \qquad \left(x = (x_j) \in \ell_p\right).$$

Proof. By Hölder's inequality, we see that

$$|\phi_u(x)| \leqslant \sum_{j=1}^{\infty} |u_j|\,|x_j| \leqslant \left(\sum_{j=1}^{\infty} |u_j|^q\right)^{1/q} \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} = \|u\|_q \|x\|_p .$$

So the sum converges, and hence ϕ_u is defined. Clearly ϕ_u is linear, and the above estimate also shows that $\|\phi_u\| \leqslant \|u\|_q$. The map $u \mapsto \phi_u$ is also clearly linear, and we've just shown that it is norm-decreasing.

Now let $\phi \in (\ell_p)^*$. For each n, let $e_n = (0, \cdots, 0, 1, 0, \cdots)$ with the 1 in the nth position. Then, for $x = (x_n) \in \ell_p$,

$$\left\| x - \sum_{k=1}^{n} x_k e_k \right\|_p = \left(\sum_{k=n+1}^{\infty} |x_k|^p \right)^{1/p} \to 0,$$

as $n \to \infty$. As ϕ is continuous, we see that

$$\phi(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \phi(x_k e_k) = \sum_{k=1}^{\infty} x_k \phi(e_k).$$

Let $u_k = \phi(e_k)$ for each k. If $u = (u_k) \in \ell_q$ then we would have that $\phi = \phi_u$. Let us fix $N \in \mathbb{N}$, and define

$$x_k = \begin{cases} 0, & \text{if } u_k = 0 \text{ or } k > N; \\ \overline{u_k}\, |u_k|^{q-2}, & \text{if } u_k \neq 0 \text{ and } k \leqslant N. \end{cases}$$

Then we see that

$$\sum_{k=1}^{\infty} |x_k|^p = \sum_{k=1}^{N} |u_k|^{p(q-1)} = \sum_{k=1}^{N} |u_k|^q,$$

as $p(q-1) = q$. Then, by the previous paragraph,

$$\phi(x) = \sum_{k=1}^{\infty} x_k u_k = \sum_{k=1}^{N} |u_k|^q.$$

Hence

$$\|\phi\| \geqslant \frac{|\phi(x)|}{\|x\|_p} = \left(\sum_{k=1}^{N} |u_k|^q \right)^{1-1/p} = \left(\sum_{k=1}^{N} |u_k|^q \right)^{1/q}.$$

By letting $N \to \infty$, it follows that $u \in \ell_q$ with $\|u\|_q \leqslant \|\phi\|$. So $\phi = \phi_u$ and $\|\phi\| = \|\phi_u\| \leqslant \|u\|_q$. Hence every element of $(\ell_p)^*$ arises as ϕ_u for some u, and also $\|\phi_u\| = \|u\|_q$. □

Loosely speaking, we say that $\ell_q = (\ell_p)^*$, although we should always be careful to keep in mind the exact map which gives this.

Corollary 11.14 (Riesz–Frechet Self-duality Lemma 4.11). *ℓ_2 is self-dual: $\ell_2 = \ell_2{}^*$.*

Similarly, we can show that $c_0^* = \ell_1$ and that $(\ell_1)^* = \ell_\infty$ (the implementing isometric isomorphism is giving by the same summation formula).

11.4. **Hahn–Banach Theorem.** Mathematical induction is a well known method to prove statements depending from a natural number. The mathematical induction is based on the following property of natural numbers: any subset of $\mathbb{N}$ has the least element. This observation can be generalised to the transfinite induction described as follows.

A *poset* is a set X with a relation $\preceq$ such that $a \preceq a$ for all $a \in X$, if $a \preceq b$ and $b \preceq a$ then $a = b$, and if $a \preceq b$ and $b \preceq c$, then $a \preceq c$. We say that $(X, \preceq)$ is *total* if for every $a, b \in X$, either $a \preceq b$ or $b \preceq a$. For a subset $S \subseteq X$, an element $a \in X$ is an *upper bound* for S if $s \preceq a$ for every $s \in S$. An element $a \in X$ is *maximal* if whenever $b \in X$ is such that $a \preceq b$, then also $b \preceq a$.

Then *Zorn's Lemma* tells us that if X is a non-empty poset such that every total subset has an upper bound, then X has a maximal element. Really this is an *axiom* which we have to assume, in addition to the usual axioms of set-theory. Zorn's Lemma is equivalent to the *axiom of choice* and *Zermelo's theorem*.

Theorem 11.15 (Hahn–Banach Theorem). *Let E be a normed vector space, and let $F \subseteq E$ be a subspace. Let $\phi \in F^*$. Then there exists $\psi \in E^*$ with $\|\psi\| \leqslant \|\phi\|$ and $\psi(x) = \phi(x)$ for each $x \in F$.*

Proof. We do the real case. An "extension" of ϕ is a bounded linear map $\phi_G : G \to \mathbb{R}$ such that $F \subseteq G \subseteq E$, $\phi_G(x) = \phi(x)$ for $x \in F$, and $\|\phi_G\| \leqslant \|\phi\|$. We introduce a partial order on the pairs (G, ϕ_G) of subspaces and functionals as follows: $(G_1, \phi_{G_1}) \preceq (G_2, \phi_{G_2})$ if and only if $G_1 \subseteq G_2$ and $\phi_{G_1}(x) = \phi_{G_2}(x)$ for all $x \in G_1$. A Zorn's Lemma argument shows that a maximal extension $\phi_G : G \to \mathbb{R}$ exists. We shall show that if $G \neq E$, then we can extend ϕ_G, a contradiction.

Let $x \notin G$, so an extension ϕ_1 of ϕ to the linear span of G and x must have the form

$$\phi_1(\tilde{x} + ax) = \phi(x) + a\alpha \qquad (\tilde{x} \in G, a \in \mathbb{R}),$$

for some $\alpha \in \mathbb{R}$. Under this, ϕ_1 is linear and extends ϕ, but we also need to ensure that $\|\phi_1\| \leqslant \|\phi\|$. That is, we need

$$|\phi(\tilde{x}) + a\alpha| \leqslant \|\phi\| \, \|\tilde{x} + ax\| \qquad (\tilde{x} \in G, a \in \mathbb{R}). \tag{11.5}$$

It is straightforward for $a = 0$, otherwise to simplify proof put $-ay = \tilde{x}$ in (11.5) and divide both sides of the identity by a. Thus we need to show that there exist such α that

$$|\alpha - \phi(y)| \leqslant \|\phi\| \, \|x - y\| \qquad \text{for all} \quad y \in G, a \in \mathbb{R},$$

or

$$\phi(y) - \|\phi\| \, \|x - y\| \leqslant \alpha \leqslant \phi(y) + \|\phi\| \, \|x - y\| \, .$$

For any y_1 and y_2 in G we have:

$$\phi(y_1) - \phi(y_2) \leqslant \|\phi\| \, \|y_1 - y_2\| \leqslant \|\phi\| \, (\|x - y_2\| + \|x - y_1\|).$$

Thus
$$\phi(y_1) - \|\phi\| \, \|x - y_1\| \leqslant \phi(y_2) + \|\phi\| \, \|x - y_2\| \, .$$
As y_1 and y_2 were arbitrary,
$$\sup_{y \in G} (\phi(y) - \|\phi\| \, \|y + x\|) \leqslant \inf_{y \in G} (\phi(y) + \|\phi\| \, \|y + x\|).$$
Hence we can choose α between the $\inf$ and the $\sup$.
The complex case follows by "complexification". □

The Hahn-Banach theorem tells us that a functional from a subspace can be extended to the whole space without increasing the norm. In particular, extending a functional on a one-dimensional subspace yields the following.

Corollary 11.16. *Let* E *be a normed vector space, and let* $x \in E$*. Then there exists* $\phi \in E^*$ *with* $\|\phi\| = 1$ *and* $\phi(x) = \|x\|$.

Another useful result which can be proved by Hahn-Banach is the following.

Corollary 11.17. *Let* E *be a normed vector space, and let* F *be a subspace of* E*. For* $x \in E$*, the following are equivalent:*

(i) $x \in \overline{F}$ *the closure of* F*;*
(ii) *for each* $\phi \in E^*$ *with* $\phi(y) = 0$ *for each* $y \in F$*, we have that* $\phi(x) = 0$.

Proof. 11.17(i)⇒11.17(ii) follows because we can find a sequence (y_n) in F with $y_n \to x$; then it's immediate that $\phi(x) = 0$, because ϕ is continuous. Conversely, we show that if 11.17(i) doesn't hold then 11.17(ii) doesn't hold (that is, the contrapositive to 11.17(ii)⇒11.17(i)).
So, $x \notin \overline{F}$. Define $\psi : \mathrm{lin}\{F, x\} \to \mathbb{K}$ by
$$\psi(y + tx) = t \qquad (y \in F, t \in \mathbb{K}).$$
This is well-defined, for $y, y' \in F$ if $y + tx = y' + t'x$ then either $t = t'$, or otherwise $x = (t - t')^{-1}(y' - y) \in F$ which is a contradiction. The map ψ is obviously linear, so we need to show that it is bounded. Towards a contradiction, suppose that ψ is not bounded, so we can find a sequence $(y_n + t_n x)$ with $\|y_n + t_n x\| \leqslant 1$ for each n, and yet $|\psi(y_n + t_n x)| = |t_n| \to \infty$. Then $\left\|t_n^{-1} y_n + x\right\| \leqslant 1/|t_n| \to 0$, so that the sequence $(-t_n^{-1} y_n)$, which is in F, converges to x. So x is in the closure of F, a contradiction. So ψ is bounded. By Hahn-Banach theorem, we can find some $\phi \in E^*$ extending ψ. For $y \in F$, we have $\phi(y) = \psi(y) = 0$, while $\phi(x) = \psi(x) = 1$, so 11.17(ii) doesn't hold, as required. □

We define $E^{**} = (E^*)^*$ to be the bidual of E, and define $J : E \to E^{**}$ as follows. For $x \in E$, $J(x)$ should be in E^{**}, that is, a map $E^* \to \mathbb{K}$. We define this to be the map $\phi \mapsto \phi(x)$ for $\phi \in E^*$. We write this as

$$J(x)(\phi) = \phi(x) \qquad (x \in E, \phi \in E^*).$$

The Corollary 11.16 shows that J is an isometry; when J is surjective (that is, when J is an isomorphism), we say that E is *reflexive*. For example, ℓ_p is reflexive for $1 < p < \infty$. On the other hand c_0 is not reflexive.

11.5. $C(X)$ **Spaces.** This section is not examinable. Standard facts about topology will be used in later sections of the course.

All our topological spaces are assumed *Hausdorff*. Let X be a compact space, and let $C_{\mathbb{K}}(X)$ be the space of continuous functions from X to $\mathbb{K}$, with pointwise operations, so that $C_{\mathbb{K}}(X)$ is a vector space. We norm $C_{\mathbb{K}}(X)$ by setting

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \qquad (f \in C_{\mathbb{K}}(X)).$$

Theorem 11.18. *Let X be a compact space. Then $C_{\mathbb{K}}(X)$ is a Banach space.*

Let E be a vector space, and let $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ be norms on E. These norms are *equivalent* if there exists $m > 0$ with

$$m^{-1} \|x\|_{(2)} \leqslant \|x\|_{(1)} \leqslant m \|x\|_{(2)} \qquad (x \in E).$$

Theorem 11.19. *Let E be a finite-dimensional vector space with basis $\{e_1, \ldots, e_n\}$, so we can identify E with $\mathbb{K}^n$ as vector spaces, and hence talk about the norm $\|\cdot\|_2$ on E. If $\|\cdot\|$ is any norm on E, then $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent.*

Corollary 11.20. *Let E be a finite-dimensional normed space. Then a subset $X \subseteq E$ is compact if and only if it is closed and bounded.*

Lemma 11.21. *Let E be a normed vector space, and let F be a closed subspace of E with $E \neq F$. For $0 < \theta < 1$, we can find $x_0 \in E$ with $\|x_0\| \leqslant 1$ and $\|x_0 - y\| > \theta$ for $y \in F$.*

Theorem 11.22. *Let E be an infinite-dimensional normed vector space. Then the* closed unit ball *of E, the set $\{x \in E : \|x\| \leqslant 1\}$, is not compact.*

Proof. Use the above lemma to construct a sequence (x_n) in the closed unit ball of E with, say, $\|x_n - x_m\| \geqslant 1/2$ for each $n \neq m$. Then (x_n) can have no convergent subsequence, and so the closed unit ball cannot be compact. □

12. Measure Theory

The presentation in this section is close to [3,7,8].

12.1. **Basic Measure Theory.** The following object will be the cornerstone of our construction.

Definition 12.1. Let X be a set. A *σ-algebra* R on X is a collection of subsets of X, written $R \subseteq 2^X$, such that

(i) $X \in R$;
(ii) if $A, B \in R$, then $A \setminus B \in R$;
(iii) if (A_n) is any sequence in R, then $\cup_n A_n \in R$.

Note, that in the third condition we admit any *countable* unions. The usage of "σ" in the names of σ-algebra and σ-ring is a reference to this. If we replace the condition by

(iii') if $(A_n)_1^m$ is any finite family in R, then $\cup_{n=1}^m A_n \in R$;

then we obtain definitions of an *algebra*.

For a σ-algebra R and $A, B \in R$, we have

$$A \cap B = X \setminus (X \setminus (A \cap B)) = X \setminus ((X \setminus A) \cup (X \setminus B)) \in R.$$

Similarly, R is closed under taking (countably) infinite intersections.

If we drop the first condition from the definition of (σ-)algebra (but keep the above conclusion from it!) we got a (σ-)*ring*, that is a (σ-)ring is closed under (countable) unions, (countable) intersections and subtractions of sets.

Exercise 12.2. (i) Use the above comments to write in full the three missing definitions: of set algebra, set ring and set σ-ring.
(ii) Show that the empty set belongs to any non-empty ring.

Sets A_k are *pairwise disjoint* if $A_n \cap A_m = \varnothing$ for $n \neq m$. We denote the union of pairwise disjoint sets by $\sqcup$, e.g. $A \sqcup B \sqcup C$.

It is easy to work with a vector space through its basis. For a ring of sets the following notion works as a helpful "basis".

Definition 12.3. A *semiring* S of sets is a collection such that

(i) it is closed under intersection;
(ii) for $A, B \in S$ we have $A \setminus B = C_1 \sqcup \ldots \sqcup C_N$ with $C_k \in S$.

Again, any non-empty semiring contain the empty set.

Example 12.4. The following are semirings but not rings:

(i) The collection of intervals $[a, b)$ on the real line;
(ii) The collection of all rectangles $\{a \leqslant x < b, c \leqslant y < d\}$ on the plane.

As the intersection of a family of σ-algebras is again a σ-algebra, and the power set 2^X is a σ-algebra, it follows that given any collection $D \subseteq 2^X$, there is a σ-algebra R such that $D \subseteq R$, such that if S is any other σ-algebra, with $D \subseteq S$, then $R \subseteq S$. We call R the σ-*algebra generated by* D.

Exercise 12.5. Let S be a semiring. Show that

(i) The collection of all finite disjoint unions $\sqcup_{k=1}^{n} A_k$, where $A_k \in S$, is a ring. We call it the ring $R(S)$ *generated by* the semiring S.
(ii) Any ring containing S contains $R(S)$ as well.
(iii) The collection of all finite (not necessarily disjoint!) unions $\sqcup_{k=1}^{n} A_k$, where $A_k \in S$, coincides with $R(S)$.

We introduce the symbols $+\infty, -\infty$, and treat these as being "extended real numbers", so $-\infty < t < \infty$ for $t \in \mathbb{R}$. We define $t + \infty = \infty$, $t\infty = \infty$ if $t > 0$ and so forth. We do not (and cannot, in a consistent manner) define $\infty - \infty$ or 0∞.

Definition 12.6. A *measure* is a map $\mu : R \to [0, \infty]$ defined on a (semi-)ring (or σ-algebra) R, such that if $A = \sqcup_n A_n$ for $A \in R$ and a finite subset (A_n) of R, then $\mu(A) = \sum_n \mu(A_n)$. This property is called *additivity* of a measure.

Exercise 12.7. Show that the following two conditions are equivalent:

(i) $\mu(\varnothing) = 0$.
(ii) There is a set $A \in R$ such that $\mu(A) < \infty$.

The first condition often (but not always) is included in the definition of a measure.

In analysis we are interested in infinities and limits, thus the following extension of additivity is very important.

Definition 12.8. In terms of the previous definition we say that μ is *countably additive* (or σ-*additive*) if for any countable infinite family (A_n) of pairwise disjoint sets from R such that $A = \sqcup_n A_n \in R$ we have $\mu(A) = \sum_n \mu(A_n)$. If the sum diverges, then as it will be the sum of positive numbers, we can, without problem, define it to be $+\infty$.

Note, that this property may be stated as a sort of *continuity* of an additive measure, cf. (1.1):

$$\mu\left(\lim_{n\to\infty}\bigsqcup_{k=1}^{n}A_k\right)=\lim_{n\to\infty}\mu\left(\bigsqcup_{k=1}^{n}A_k\right).$$

Example 12.9. (i) Fix a point $a\in\mathbb{R}$ and define a measure μ by the condition $\mu(A)=1$ if $a\in A$ and $\mu(A)=0$ otherwise.

(ii) For the ring obtained in Exercise 12.5 from semiring S in Example 12.4(i) define $\mu([a,b))=b-a$ on S. This is a measure, and we will show its σ-additivity.

(iii) For ring obtained in Exercise 12.5 from the semiring in Example 12.4(ii), define $\mu(V)=(b-a)(d-c)$ for the rectangle $V=\{a\leqslant x<b, c\leqslant y<d\}$ S. It will be again a σ-additive measure.

(iv) Let $X=\mathbb{N}$ and $R=2^{\mathbb{N}}$, we define $\mu(A)=0$ if A is a finite subset of $X=\mathbb{N}$ and $\mu(A)=+\infty$ otherwise. Let $A_n=\{n\}$, then $\mu(A_n)=0$ and $\mu(\sqcup_n A_n)=\mu(\mathbb{N})=+\infty\neq\sum_n\mu(A_n)=0$. Thus, this measure is *not* σ-additive.

We will see further examples of measures which are not σ-additive in Section 12.4.

Definition 12.10. A measure μ is *finite* if $\mu(A)<\infty$ for all $A\in R$.
A measure μ is σ-*finite* if X is a union of countable number of sets X_k, such that for any $A\in R$ and any $k\in\mathbb{N}$ the intersection $A\cap X_k$ is in R and $\mu(A\cap X_k)<\infty$.

Exercise 12.11. Modify the example 12.9(i) to obtain

(i) a measure which is not finite, but is σ-finite. (*Hint:* let the measure count the number of integer points in a set).

(ii) a measure which is not σ-finite. (*Hint:* assign $\mu(A)=+\infty$ if $a\in A$.)

Proposition 12.12. *Let μ be a σ-additive measure on a σ-algebra R. Then:*

(i) *If $A,B\in R$ with $A\subseteq B$, then $\mu(A)\leqslant\mu(B)$ [we call this property "monotonicity of a measure"];*

(ii) *If $A,B\in R$ with $A\subseteq B$ and $\mu(B)<\infty$, then $\mu(B\setminus A)=\mu(B)-\mu(A)$;*

(iii) *If (A_n) is a sequence in R, with $A_1\subseteq A_2\subseteq A_3\subseteq\cdots$. Then*

$$\lim_{n\to\infty}\mu(A_n)=\mu\left(\cup_n A_n\right).$$

(iv) *If (A_n) is a sequence in R, with $A_1\supseteq A_2\supseteq A_3\supseteq\cdots$. If $\mu(A_m)<\infty$ for some m, then*

$$\lim_{n\to\infty}\mu(A_n)=\mu\left(\cap_n A_n\right). \tag{12.1}$$

Proof. The two first properties are easy to see. For the third statement, define $A = \cup_n A_n$, $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$, $n > 1$. Then $A_n = \sqcup_{k=1}^n B_n$ and $A = \sqcup_{k=1}^\infty B_n$. Using the σ-additivity of measures $\mu(A) = \sum_{k=1}^\infty \mu(B_k)$ and $\mu(A_n) = \sum_{k=1}^n \mu(B_k)$. From the theorem in real analysis that any monotonic sequence of real numbers converges (recall that we admit $+\infty$ as limits' value) we have $\mu(A) = \sum_{k=1}^\infty \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^n \mu(B_k) = \lim_{n\to\infty} \mu(A_n)$. The last statement can be shown similarly. □

Exercise 12.13. Let a measure μ on $\mathbb{N}$ be defined by $\mu(A) = 0$ for finite A and $\mu(A) = \infty$ for infinite A. Check that μ is additive but not σ-additive. Therefore give an example that μ does not satisfies 12.12(iii).

12.2. **Extension of Measures.** From now on we consider only finite measures, an extension to σ-finite measures will be done later.

Proposition 12.14. *Any measure μ' on a semiring S is uniquely extended to a measure μ on the generated ring $R(S)$, see Ex. 12.5. If the initial measure was σ-additive, then the extension is σ-additive as well.*

Proof. If an extension exists it shall satisfy $\mu(A) = \sum_{k=1}^n \mu'(A_k)$, where $A_k \in S$. We need to show for this definition two elements:

(i) Consistency, i.e. independence of the value from a presentation of $A \in R(S)$ as $A = \sqcup_{k=1}^n A_k$, where $A_k \in S$. For two different presentation $A = \sqcup_{j=1}^n A_j$ and $A = \sqcup_{k=1}^m B_k$ define $C_{jk} = A_j \cap B_k$, which will be pairwise disjoint. By the additivity of μ' we have $\mu'(A_j) = \sum_k \mu'(C_{jk})$ and $\mu'(B_k) = \sum_j \mu'(C_{jk})$. Then

$$\sum_j \mu'(A_j) = \sum_j \sum_k \mu'(C_{jk}) = \sum_k \sum_j \mu'(C_{jk}) = \sum_k \mu'(B_k).$$

(ii) Additivity. For $A = \sqcup_{k=1}^n A_k$, where $A_k \in R(S)$ we can present $A_k = \sqcup_{j=1}^{n(k)} C_{jk}$, $C_{jk} \in S$. Thus $A = \sqcup_{k=1}^n \sqcup_{j=1}^{n(k)} C_{jk}$ and:

$$\mu(A) = \sum_{k=1}^n \sum_{j=1}^{n(k)} \mu'(C_{jk}) = \sum_{k=1}^n \mu(A_k).$$

Finally, show the σ-additivity. For a set $A = \sqcup_{k=1}^\infty A_k$, where A and $A_k \in R(S)$, find presentations $A = \sqcup_{j=1}^n B_j$, $B_j \in S$ and $A_k = \sqcup_{l=1}^{m(k)} B_{lk}$, $B_{lk} \in S$. Define $C_{jlk} = B_j \cap B_{lk} \in S$, then $B_j = \sqcup_{k=1}^\infty \sqcup_{l=1}^{m(k)} C_{jlk}$ and $A_k = \sqcup_{j=1}^n \sqcup_{l=1}^{m(k)} C_{jlk}$. Then, from σ-additivity of μ':

$$\mu(A) = \sum_{j=1}^{n} \mu'(B_j) = \sum_{j=1}^{n} \sum_{k=1}^{\infty} \sum_{l=1}^{m(k)} \mu'(C_{jlk}) = \sum_{k=1}^{\infty} \sum_{j=1}^{n} \sum_{l=1}^{m(k)} \mu'(C_{jlk}) = \sum_{k=1}^{\infty} \mu(A_k),$$

where we changed the summation order in series with non-negative terms. □

In a similar way we can extend a measure from a semiring to corresponding σ-ring, however it can be done even for a larger family. The procedure recall the famous story on Baron Munchausen saves himself from being drowned in a swamp by pulling on his own hair. Indeed, initially we knew measure for elements of semiring S or their finite disjoint unions from $R(S)$. For an arbitrary set A we may assign a measure from an element of $R(S)$ which "approximates" A. But how to measure such approximation? Well, to this end we use the measure on $R(S)$ again (pulling on his own hair)!

Coming back to exact definitions, we introduce the following notion.

Definition 12.15. Let S be a semi-ring of subsets in X, and μ be a measure defined on S. An *outer measure* μ^* on X is a map $\mu^* : 2^X \to [0, \infty]$ defined by:

$$\mu^*(A) = \inf\left\{ \sum_k \mu(A_k), \text{ such that } A \subseteq \cup_k A_k, \quad A_k \in S \right\}.$$

Proposition 12.16. *An outer measure has the following properties:*

(i) $\mu^*(\varnothing) = 0$;

(ii) *if* $A \subseteq B$ *then* $\mu^*(A) \leqslant \mu^*(B)$, *this is called* monotonicity *of the outer measure;*

(iii) *if* (A_n) *is any sequence in* 2^X, *then* $\mu^*(\cup_n A_n) \leqslant \sum_n \mu^*(A_n)$.

The final condition says that an outer measure is *countably sub-additive*. Note, that an outer measure may be not a measure in the sense of Defn. 12.6 due to a luck of additivity.

Example 12.17. The *Lebesgue outer measure* on $\mathbb{R}$ is defined out of the measure from Example 12.9(ii), that is, for $A \subseteq \mathbb{R}$, as

$$\mu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} (b_j - a_j) : A \subseteq \cup_{j=1}^{\infty} [a_j, b_j) \right\}.$$

We make this definition, as intuitively, the "length", or measure, of the interval $[a, b)$ is $(b - a)$.

For example, for outer Lebesgue measure we have $\mu^*(A) = 0$ for any countable set, which follows, as clearly $\mu^*(\{x\}) = 0$ for any $x \in \mathbb{R}$.

Lemma 12.18. *Let* $a < b$. *Then* $\mu^*([a, b]) = b - a$.

Proof. For $\epsilon > 0$, as $[a, b] \subseteq [a, b + \epsilon)$, we have that $\mu^*([a, b]) \leqslant (b - a) + \epsilon$. As $\epsilon > 0$, was arbitrary, $\mu^*([a, b]) \leqslant b - a$.
To show the opposite inequality we observe that $[a, b) \subset [a, b]$ and $\mu^*[a, b) = b - a$ (because $[a, b)$ is in the semi-ring) so $\mu^*[a, b] \geqslant b - a$ by 12.16(ii). □

Our next aim is to construct measures from outer measures. We use the notation $A \bigtriangleup B = (A \cup B) \setminus (A \cap B)$ for *symmetric difference of sets*.

Definition 12.19. Given an outer measure μ^* defined by a measure μ on a semiring S, we define $A \subseteq X$ to be *Lebesgue measurable* if for any $\varepsilon > 0$ there is a finite union B of elements in S (in other words: $B \in R(S)$ by Lem. 12.5(iii)), such that $\mu^*(A \bigtriangleup B) < \varepsilon$.

Obviously all elements of S and $R(S)$ are measurable.

Exercise 12.20. (i) Define a function of pairs of elements $A, B \in L$ as the outer measure of the symmetric difference of A and B:

$$d(A, B) = \mu^*(A \bigtriangleup B). \tag{12.2}$$

Show that d is a metric on the collection of cosets with respect to the equivalence relation: $A \sim B$ if $d(A, B) = 0$. *Hint:* to show the triangle inequality use the inclusion:

$$A \bigtriangleup B \subseteq (A \bigtriangleup C) \cup (C \bigtriangleup B)$$

(ii) Let a sequence $(\varepsilon_n) \to 0$ be monotonically decreasing. For a Lebesgue measurable A there exists a sequence $(A_n) \subset R(S)$ such that $d(A, A_n) < \varepsilon_n$ for each n. Show that (A_n) is a Cauchy sequence for the distance d (12.2).

An alternative definition of a measurable set is due to Carathéodory.

Definition 12.21. Given an outer measure μ^*, we define $E \subseteq X$ to be *Carathéodory measurable* if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E),$$

for any $A \subseteq X$.

As μ^* is sub-additive, this is equivalent to

$$\mu^*(A) \geqslant \mu^*(A \cap E) + \mu^*(A \setminus E) \qquad (A \subseteq X),$$

as the other inequality is automatic.

Exercise* 12.22. Show that measurability by Lebesgue and Carathéodory are equivalent.

Suppose now that the ring $R(S)$ is an algebra (i.e., contains the maximal element X). Then, the outer measure of any set is finite, and the following theorem holds:

Theorem 12.23 (Lebesgue)**.** *Let μ^* be an outer measure on X defined by a semiring S, and let L be the collection of all Lebesgue measurable sets for μ^*. Then L is a σ-algebra, and if $\tilde{\mu}$ is the restriction of μ^* to L, then $\tilde{\mu}$ is a measure. Furthermore, $\tilde{\mu}$ is σ-additive on L if μ is σ-additive on S.*

Sketch of proof. Clearly, $R(S) \subset L$. Now we show that $\mu^*(A) = \mu(A)$ for a set $A \in R(S)$. If $A \subset \cup_k A_k$ for $A_k \in S$, then $\mu(A) \leqslant \sum_k \mu(A_k)$, taking the infimum we get $\mu(A) \leqslant \mu^*(A)$. For the opposite inequality, any $A \in R(S)$ has a disjoint representation $A = \sqcup_k A_k$, $A_k \in S$, thus $\mu^*(A) \leqslant \sum_k \mu(A_k) = \mu(A)$.
Now we will show that $R(S)$ with the distance d (12.2) is an incomplete metric space, with the measure μ being uniformly continuous functions. Measurable sets make the completion of $R(S)$ (cf. Ex. 12.20(ii)) with μ being continuation of μ^* to the completion by continuity, cf. Ex. 1.61.
Then, by the definition, Lebesgue measurable sets make the closure of $R(S)$ with respect to this distance.
We can check that measurable sets form an algebra. To this end we need to make estimations, say, of $\mu^*((A_1 \cap A_2) \triangle (B_1 \cap B_2))$ in terms of $\mu^*(A_i \triangle B_i)$. A demonstration for any finite number of sets is performed through mathematical inductions. The above two-sets case provide both: the base and the step of the induction.
Now, we show that L is σ-algebra. Let $A_k \in L$ and $A = \cup_k A_k$. Then for any $\varepsilon > 0$ there exists $B_k \in R(S)$, such that $\mu^*(A_k \triangle B_k) < \frac{\varepsilon}{2^k}$. Define $B = \cup_k B_k$. Then

$$(\cup_k A_k) \triangle (\cup_k B_k) \subset \cup_k (A_k \triangle B_k) \text{ implies } \mu^*(A \triangle B) < \varepsilon.$$

We cannot stop at this point since $B = \cup_k B_k$ may be not in $R(S)$. Thus, define $B'_1 = B_1$ and $B'_k = B_k \setminus \cup_{i=1}^{k-1} B_i$, so B'_k are pair-wise disjoint. Then $B = \sqcup_k B'_k$ and $B'_k \in R(S)$. From the convergence of the series there is N such that $\sum_{k=N}^{\infty} \mu(B'_k) < \varepsilon$. Let $B' = \cup_{k=1}^{N} B'_k$, which is in $R(S)$. Then $\mu^*(B \triangle B') \leqslant \varepsilon$ and, thus, $\mu^*(A \triangle B') \leqslant 2\varepsilon$.
To check that μ^* is measure on L we use the following

Lemma 12.24. $|\mu^*(A) - \mu^*(B)| \leqslant \mu^*(A \bigtriangleup B)$, *that is* μ^* *is uniformly continuous in the metric* $d(A, B)$ (12.2).

Proof of the Lemma. Use inclusions $A \subset B \cup (A \bigtriangleup B)$ and $B \subset A \cup (A \bigtriangleup B)$. □

To show additivity take $A_{1,2} \in L$, $A = A_1 \sqcup A_2$, $B_{1,2} \in R(S)$ and $\mu^*(A_i \bigtriangleup B_i) < \varepsilon$. Then $\mu^*(A \bigtriangleup (B_1 \cup B_2)) < 2\varepsilon$ and $|\mu^*(A) - \mu^*(B_1 \cup B_2)| < 2\varepsilon$. Thus $\mu^*(B_1 \cup B_2) = \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2) - \mu(B_1 \cap B_2)$, but $\mu(B_1 \cap B_2) = d(B_1 \cap B_2, \varnothing) = d(B_1 \cap B_2, A_1 \cap A_2) < 2\varepsilon$. Therefore

$$|\mu^*(B_1 \cup B_2) - \mu(B_1) - \mu(B_2)| < 2\varepsilon.$$

Combining everything together we get (this is a sort of $\varepsilon/3$-argument):

$$\begin{aligned}
&|\mu^*(A) - \mu^*(A_1) - \mu^*(A_2)| \\
&\quad = |\mu^*(A) - \mu^*(B_1 \cup B_2) + \mu^*(B_1 \cup B_2) \\
&\qquad - (\mu(B_1) + \mu(B_2)) + \mu(B_1) + \mu(B_2) - \mu^*(A_1) - \mu^*(A_2)| \\
&\quad \leqslant |\mu^*(A) - \mu^*(B_1 \cup B_2)| + |\mu^*(B_1 \cup B_2) - (\mu(B_1) + \mu(B_2))| \\
&\qquad\quad + |\mu(B_1) + \mu(B_2) - \mu^*(A_1) - \mu^*(A_2)| \\
&\quad \leqslant 6\varepsilon.
\end{aligned}$$

Thus μ^* is additive on L.
Check the countable additivity for $A = \sqcup_k A_k$. The inequality $\mu^*(A) \leqslant \sum_k \mu^*(A_k)$ follows from countable sub-additivity. The opposite inequality is the limiting case of the finite inequality $\mu^*(A) \geqslant \mu^*(\sqcup_{k=1}^N A_k) = \sum_{k=1}^N \mu^*(A_k)$ following from additivity and monotonicity of μ^*. □

Corollary 12.25. *Let* $E \subseteq \mathbb{R}$ *be open or closed. Then* E *is Lebesgue measurable.*

Proof. This is a common trick, using the density and the countability of the rationals. As σ-algebras are closed under taking complements, we need only show that open sets are Lebesgue measurable.
Intervals (a, b) are Lebesgue measurable by the very definition. Now let $U \subseteq \mathbb{R}$ be open. For each $x \in U$, there exists $a_x < b_x$ with $x \in (a_x, b_x) \subseteq U$. By making a_x slightly larger, and b_x slightly smaller, we can ensure that $a_x, b_x \in \mathbb{Q}$. Thus $U = \cup_x (a_x, b_x)$. Each interval is measurable, and there are at most a countable number of them (endpoints make a countable set) thus U is the countable (or finite) union of Lebesgue measurable sets, and hence U is Lebesgue measurable itself. □

We perform now an extension of finite measure to σ-finite one. Let μ be a σ-additive and σ-finite measure defined on a semiring in $X = \sqcup_k X_k$, such that the restriction of μ to every X_k is finite. Consider the Lebesgue extension μ_k of μ defined within X_k. A set $A \subset X$ is measurable if every intersection $A \cap X_k$ is μ_k measurable. For a such measurable set A we define its measure by the identity:

$$\mu(A) = \sum_k \mu_k(A \cap X_k).$$

We call a measure μ defined on L *complete* if whenever $E \subseteq X$ is such that there exists $F \in L$ with $\mu(F) = 0$ and $E \subseteq F$, we have that $E \in L$. Measures constructed from outer measures by the above theorem are always complete. On the example sheet, we saw how to form a complete measure from a given measure. We call sets like E *null sets*: complete measures are useful, because it is helpful to be able to say that null sets are in our σ-algebra. Null sets can be quite complicated. For the Lebesgue measure, all countable subsets of $\mathbb{R}$ are null, but then so is the *Cantor set*, which is uncountable.

Definition 12.26. If we have a property $P(x)$ which is true except possibly $x \in A$ and $\mu(A) = 0$, we say $P(x)$ is *almost everywhere* or *a.e.*.

12.3. **Complex-Valued Measures and Charges.** We start from the following observation.

Exercise 12.27. Let μ_1 and μ_2 be measures on a same σ-algebra. Define $\mu_1 + \mu_2$ and $\lambda\mu_1, \lambda > 0$ by $(\mu_1 + \mu_2)(A) = \mu_1(A) + \mu_2(A)$ and $(\lambda\mu_1)(A) = \lambda(\mu_1(A))$. Then $\mu_1 + \mu_2$ and $\lambda\mu_1$ are measures on the same σ-algebra as well.

In view of this, it will be helpful to extend the notion of a measure to obtain a linear space.

Definition 12.28. Let X be a set, and R be a σ-ring. A real- (complex-) valued function ν on R is called a *charge* (or *signed measure*) if it is countably additive as follows: for any $A_k \in R$ the identity $A = \sqcup_k A_k$ implies the series $\sum_k \nu(A_k)$ is absolute convergent and has the sum $\nu(A)$.

In the following "charge" means "real charge".

Example 12.29. Any linear combination of σ-additive measures on $\mathbb{R}$ with real (complex) coefficients is real (complex) charge.

The opposite statement is also true:

Theorem 12.30. *Any real (complex) charge ν has a representation $\nu = \mu_1 - \mu_2$ ($\nu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$), where μ_k are σ-additive measures.*

To prove the theorem we need the following definition.

Definition 12.31. The *variation of a charge* on a set A is $|\nu|(A) = \sup \sum_k |\nu(A_k)|$ for all disjoint splitting $A = \sqcup_k A_k$.

Example 12.32. If $\nu = \mu_1 - \mu_2$, then $|\nu|(A) \leqslant \mu_1(A) + \mu_2(A)$. The inequality becomes an identity for *disjunctive measures* on A (that is there is a partition $A = A_1 \sqcup A_2$ such that $\mu_2(A_1) = \mu_1(A_2) = 0$).

The relation of variation to charge is as follows:

Theorem 12.33. *For any charge ν the function $|\nu|$ is a σ-additive measure.*

Finally to prove the Thm. 12.30 we use the following

Proposition 12.34. *For any charge ν the function $|\nu| - \nu$ is a σ-additive measure as well.*

From the Thm. 12.30 we can deduce

Corollary 12.35. *The collection of all charges on a σ-algebra R is a linear space which is complete with respect to the distance:*

$$d(\nu_1, \nu_2) = \sup_{A \in R} |\nu_1(A) - \nu_2(A)| .$$

The following result is also important:

Theorem 12.36 (Hahn Decomposition)**.** *Let ν be a charge. There exist $A, B \in L$, called a* Hahn decomposition *of (X, ν), with $A \cap B = \varnothing$, $A \cup B = X$ and such that for any $E \in L$,*

$$\nu(A \cap E) \geqslant 0, \quad \nu(B \cap E) \leqslant 0.$$

This need not be unique.

Sketch of proof. We only sketch this. We say that $A \in L$ is *positive* if

$$\nu(E \cap A) \geqslant 0 \qquad (E \in L),$$

and similiarly define what it means for a measurable set to be *negative*. Suppose that ν never takes the value $-\infty$ (the other case follows by considering the charge $-\nu$).
Let $\beta = \inf \nu(B_0)$ where we take the infimum over all negative sets B_0. If $\beta = -\infty$ then for each n, we can find a negative B_n with $\nu(B_n) \leqslant -n$. But then $B = \cup_n B_n$ would be negative with $\nu(B) \leqslant -n$ for any n, so that $\nu(B) = -\infty$ a contradiction.
So $\beta > -\infty$ and so for each n we can find a negative B_n $\nu(B_n) < \beta + 1/n$. Then we can show that $B = \cup_n B_n$ is negative, and argue that $\nu(B) \leqslant \beta$. As B is negative, actually $\nu(B) = \beta$.
There then follows a very tedious argument, by contradiction, to show that $A = X \setminus B$ is a positive set. Then (A, B) is the required decomposition. □

12.4. **Constructing Measures, Products.** Consider the semiring S of intervals $[a, b)$. There is a simple description of all measures on it. For a measure μ define

$$F_\mu(t) = \begin{cases} \mu([0,t)) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -\mu([t,0)) & \text{if } t < 0, \end{cases} \tag{12.3}$$

F_μ is monotonic and any monotonic function F defines a measure μ on S by the by $\mu([a, b)) = F(b) - F(a)$. The correspondence is one-to-one with the additional assumption $F(0) = 0$.

Theorem 12.37. *The above measure μ is σ-additive on S if and only if F is continuous from the left: $F(t - 0) = F(t)$ for all $t \in \mathbb{R}$.*

Proof. Necessity: $F(t) - F(t-0) = \lim_{\varepsilon\to 0} \mu([t-\varepsilon, t)) = \mu(\lim_{\varepsilon\to 0}[t-\varepsilon, t)) = \mu(\emptyset) = 0$, by the continuity of a σ-additive measure, see 12.12(iv).
For sufficiency assume $[a, b) = \sqcup_k [a_k, b_k)$. The inequality $\mu([a, b)) \geqslant \sum_k \mu([a_k, b_k))$ follows from additivity and monotonicity. For the opposite inequality take δ_k s.t. $F(b) - F(b-\delta) < \varepsilon$ and $F(a_k) - F(a_k - \delta_k) < \varepsilon/2^k$ (use left continuity of F). Then the interval $[a, b-\delta]$ is covered by $(a_k - \delta_k, b_k)$, due to compactness of $[a, b-\delta]$ there is a finite subcovering. Thus $\mu([a, b-\delta)) \leqslant \sum_{j=1}^N \mu([a_{k_j} - \delta_{k_j}, b_{k_j}))$ and $\mu([a, b)) \leqslant \sum_{j=1}^N \mu([a_{k_j}, b_{k_j})) + 2\varepsilon$. □

Exercise 12.38.
(i) Give an example of function discontinued from the left at 1 and show that the resulting measure is additive but not σ-additive.
(ii) Check that, if a function F is continuous at point a then $\mu(\{a\}) = 0$.

Example 12.39. (i) Take $F(t) = t$, then the corresponding measure is the Lebesgue measure on $\mathbb{R}$.
(ii) Take $F(t)$ be the integer part of t, then μ counts the number of integer within the set.
(iii) Define the *Cantor function* as follows $\alpha(x) = 1/2$ on $(1/3, 2/3)$; $\alpha(x) = 1/4$ on $(1/9, 2/9)$; $\alpha(x) = 3/4$ on $(7/9, 8/9)$, and so for. This function is monotonic and can be continued to $[0, 1]$ by continuity, it is know as *Cantor ladder*. The resulting measure has the following properties:
- The measure of the entire interval is 1.
- Measure of every point is zero.
- The measure of the Cantor set is 1, while its Lebesgue measure is 0.

Another possibility to build measures is their product. In particular, it allows to expand various measures defined through (12.3) on the real line to $\mathbb{R}^n$.

Definition 12.40. Let X and Y be spaces, and let S and T be semirings on X and Y respectively. Then $S \times T$ is the semiring consisting of $\{A \times B : A \in S, B \in T\}$ ("generalised rectangles"). Let μ and ν be measures on S and T respectively. Define the *product measure* $\mu \times \nu$ on $S \times T$ by the rule $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$.

Example 12.41. The measure from Example 12.9(iii) is the product of two copies of pre-Lebesgue measures from Example 12.9(ii).

13. Integration

We now come to the main use of measure theory: to define a general theory of integration.

13.1. Measurable functions.

From now on, by a *measure space* we shall mean a triple (X, L, μ), where X is a set, L is a σ-algebra on X, and μ is a σ-additive measure defined on L. We say that the members of L are *measurable*, or L-measurable, if necessary to avoid confusion.

Definition 13.1. A function $f : X \to \mathbb{R}$ is *measurable* if

$$E_c(f) = \{x \in X : f(x) < c\} \qquad \text{that is} \qquad E_c(f) = f^{-1}((-\infty, c))$$

is in L (that is $E_c(f)$ is a measurable set) for any $c \in \mathbb{R}$.
A complex-valued function is measurable if its real and imaginary parts are measurable.

Lemma 13.2. *The following are equivalent:*

(i) *A function* f *is measurable;*
(ii) *For any* $a < b$ *the set* $f^{-1}((a, b))$ *is measurable;*
(iii) *For any open set* $U \subset \mathbb{R}$ *the set* $f^{-1}(U)$ *is measurable.*

Proof. To show 13.2(i) ⇒ 13.2(ii) we note that

$$f^{-1}((a, b)) = E_b(f) \setminus \left(\bigcap_n E_{a+1/n}(f)\right).$$

For 13.2(ii) ⇒ 13.2(iii) use that any open set $U \subset \mathbb{R}$ is a union of countable set of intervals (a, b), cf. proof of Cor. 12.25.
The final implication 13.2(iii) ⇒ 13.2(i) directly follows from openness of $(-\infty, a)$. □

Corollary 13.3. *Let* $f : X \to \mathbb{R}$ *be measurable and* $g : \mathbb{R} \to \mathbb{R}$ *be continuous, then the composition* $g(f(x))$ *is measurable.*

Proof. The preimage of the open set $(-\infty, c)$ under a continuous g is an open set, say U. The preimage of U under f is measurable by Lem. 13.2(iii). Thus, the preimage of $(-\infty, c)$ under the composition $g \circ f$ is measurable, thereafter $g \circ f$ is a measurable function. □

Theorem 13.4. *Let* $f, g : X \to \mathbb{R}$ *be measurable. Then* af *(*$a \in \mathbb{R}$*),* $f + g$*,* fg*,* $\max(f, g)$ *and* $\min(f, g)$ *are all measurable. That is measurable functions form an algebra and this algebra is closed under convergence a.e.*

Proof. Use Cor. 13.3 to show measurability of λf, $|f|$ and f^2. The measurability of a sum $f_1 + f_2$ follows from the relation

$$E_c(f_1 + f_2) = \cup_{r\in\mathbb{Q}}(E_r(f_1) \cap E_{c-r}(f_2)).$$

Next use the following identities:

$$f_1 f_2 = \frac{(f_1 + f_2)^2 - (f_1 - f_2)^2}{4},$$

$$\max(f_1, f_2) = \frac{(f_1 + f_2) + |f_1 - f_2|}{2}.$$

If (f_n) is a non-increasing sequence of measurable functions converging to f. Than $E_c(f) = \cup_n E_c(f_n)$.

Moreover *any limit can be replaced by two monotonic limits*:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{k\to\infty} \max(f_n(x), f_{n+1}(x), \ldots, f_{n+k}(x)). \tag{13.1}$$

Finally if f_1 is measurable and $f_2 = f_1$ almost everywhere, then f_2 is measurable as well. □

We can define several types of convergence for measurable functions.

Definition 13.5. We say that sequence (f_n) of functions converges

(i) *uniformly* to f (notated $f_n \rightrightarrows f$) if

$$\sup_{x\in X} |f_n(x) - f(x)| \to 0;$$

(ii) *almost everywhere* to f (notated $f_n \xrightarrow{a.e.} f$) if

$$f_n(x) \to f(x) \quad \text{for all } x \in X \setminus A,\ \mu(A) = 0;$$

(iii) *in measure* μ to f (notated $f_n \xrightarrow{\mu} f$) if for all $\varepsilon > 0$

$$\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \to 0. \tag{13.2}$$

Clearly uniform convergence implies both convergences a.e and in measure.

Theorem 13.6. *On finite measures convergence a.e. implies convergence in measure.*

Proof. Define $A_n(\varepsilon) = \{x \in X : |f_n(x) - f(x)| \geqslant \varepsilon\}$. Let $B_n(\varepsilon) = \cup_{k\geqslant n} A_k(\varepsilon)$. Clearly $B_n(\varepsilon) \supset B_{n+1}(\varepsilon)$, let $B(\varepsilon) = \cap_1^\infty B_n(\varepsilon)$. If $x \in B(\varepsilon)$ then $f_n(x) \not\to f(x)$. Thus $\mu(B(\varepsilon)) = 0$, but $\mu(B(\varepsilon)) = \lim_{n\to\infty} \mu(B_n(\varepsilon))$, cf. (12.1). Since $A_n(\varepsilon) \subset B_n(\varepsilon)$ we see that $\mu(A_n(\varepsilon)) \to 0$ as required for (13.2) □

Note, that the construction of sets $B_n(\varepsilon)$ is just another implementation of the "two monotonic limits" trick (13.1) for sets.

Exercise 13.7. Present examples of sequences (f_n) and functions f such that:

(i) $f_n \xrightarrow{\mu} f$ but not $f_n \xrightarrow{a.e.} f$.
(ii) $f_n \xrightarrow{a.e.} f$ but not $f_n \rightrightarrows f$.

However we can slightly "fix" either the set or the sequence to "upgrade" the convergence as shown in the following two theorems.

Theorem 13.8 (Egorov). *If* $f_n \xrightarrow{a.e.} f$ *on a finite measure set* X *then for any* $\sigma > 0$ *there is* $E_\sigma \subset X$ *with* $\mu(E_\sigma) < \sigma$ *and* $f_n \rightrightarrows f$ *on* $X \setminus E_\sigma$.

Proof. We use $A_n(\varepsilon)$ and $B_n(\varepsilon)$ from the proof of Thm. 13.6. Observe that $|f(x) - f_k(x)| < \varepsilon$ uniformly for all $x \in X \setminus B_n(\varepsilon)$ and $k > n$. For every $\varepsilon > 0$ we seen that $\mu(B_n(\varepsilon)) \to 0$, thus for each k there is $N(k)$ such that $\mu(B_{N(k)}(1/k)) < \sigma/2^k$. Put $E_\sigma = \cup_k B_{N(k)}(1/k)$. □

Theorem 13.9. *If* $f_n \xrightarrow{\mu} f$ *then there is a subsequence* (n_k) *such that* $f_{n_k} \xrightarrow{a.e.} f$ *for* $k \to \infty$.

Proof. In the notations of two previous proofs: for every natural k take n_k such that $\mu(A_{n_k}(1/k)) < 1/2^k$, which is possible since $\mu(A_n(\varepsilon)) \to 0$. Define $C_m = \cup_{k=m}^{\infty} A_{n_k}(1/k)$ and $C = \cap C_m$. Then, $\mu(C_m) = 1/2^{m-1}$ and, thus, $\mu(C) = 0$ by (12.1). If $x \notin C$ then there is such N that $x \notin A_{n_k}(1/k)$ for all $k > N$. That means that $|f_{n_k}(x) - f(x)| < 1/k$ for all such k, i.e $f_{n_k}(x) \to f(x)$. Thus, we have the point-wise convergence everywhere except the zero-measure set C. □

It is worth to note, that we can use the last two theorem subsequently and upgrade the convergence in measure to the uniform convergence of a subsequence on a subset.

Exercise 13.10. For your counter examples from Exercise 13.7, find

(i) a subsequence f_{n_k} of the sequence from 13.7(i) which converges to f a.e.;
(ii) a subset such that sequence from 13.7(ii) converges uniformly.

Exercise 13.11. Read about Luzin's C-property.

13.2. **Lebesgue Integral.** First we define a sort of "basis" for the space of integral functions.

Definition 13.12. For $A \subseteq X$, we define χ_A to be the *indicator function* of A, by

$$\chi_A(x) = \begin{cases} 1 & : x \in A, \\ 0 & : x \notin A. \end{cases}$$

Then, if χ_A is measurable, then $\chi_A^{-1}((\frac{1}{2}, \frac{3}{2})) = A \in L$; conversely, if $A \in L$, then $X \setminus A \in L$, and we see that for any $U \subseteq \mathbb{R}$ open, $\chi_A^{-1}(U)$ is either $\varnothing$, A, $X \setminus A$, or X, all of which are in L. So χ_A is measurable if and only if $A \in L$.

Definition 13.13. A measurable function $f: X \to \mathbb{R}$ is *simple* if it attains only a countable number of values.

Lemma 13.14. *A function $f: X \to \mathbb{R}$ is simple if and only if*

$$f = \sum_{k=1}^{\infty} t_k \chi_{A_k} \tag{13.3}$$

for some $(t_k)_{k=1}^{\infty} \subseteq \mathbb{R}$ and $A_k \in L$. That is, simple functions are linear combinations of indicator functions of measurable sets.
Moreover in the above representation the sets A_k can be pair-wise disjoint and all $t_k \neq 0$ pair-wise different. In this case the representation is unique.

Notice that it is now obvious that

Corollary 13.15. *The collection of simple functions forms a vector space: this wasn't clear from the original definition.*

Definition 13.16. A simple function in the form (13.3) with disjoint A_k is called *summable* if the following series converges:

$$\sum_{k=1}^{\infty} |t_k| \, \mu(A_k) \quad \text{if } f \text{ has the above unique representation} \quad f = \sum_{k=1}^{\infty} t_k \chi_{A_k} \tag{13.4}$$

It is another combinatorial exercise to show that this definition is independent of the way we write f.

Definition 13.17. We define the *integral* of a simple function $f = \sum_k t_k \chi_{A_k}$ (13.3) over a measurable set A by setting

$$\int_A f \, d\mu = \sum_{k=1}^{\infty} t_k \mu(A_k \cap A).$$

Clearly the series converges for any simple summable function f. Moreover

Lemma 13.18. *The value of integral of a simple summable function is independent from its representation by the sum of indicators* (13.3)*. In particular, we can evaluate the integral taking the canonical representation over pair-wise disjoint sets having pair-wise different values.*

Proof. This is another slightly tedious combinatorial exercise. You need to prove that the integral of a simple function is well-defined, in the sense that it is independent of the way we choose to write the simple function. □

Exercise 13.19. Let f be the function on $[0,1]$ which take the value 1 in all rational points and 0—everywhere else. Find the value of the Lebesgue integral $\int_{[0,1]} f, d\mu$ with respect to the Lebesgue measure on $[0,1]$. Show that the Riemann upper- and lower sums for f converges to different values, so f is not Riemann-integrable.

Remark 13.20. The previous exercise shows that the Lebesgue integral does not have those problems of the Riemann integral related to discontinuities. Indeed, most of function which are not Riemann-integrable are *integrable* in the sense of Lebesgue. The only reason, why a measurable function is not integrable by Lebesgue is divergence of the series (13.4). Therefore, we prefer to speak that the function is *summable* rather than *integrable*. However, those terms are used interchangeably in the mathematical literature.

We will denote by $S(X)$ the collection of all simple summable functions on X.

Proposition 13.21. *Let* $f, g : X \to \mathbb{R}$ *be in* $S(X)$ *(that is simple summable), let* a, $b \in \mathbb{R}$ *and* A *is a measurable set. Then:*

(i) $\int_A af + bg \, d\mu = a \int_A f \, d\mu + b \int_A g \, d\mu$, *that is* $S(X)$ *is a linear space;*
(ii) *The correspondence* $f \to \int_A f \, d\mu$ *is a linear functional on* $S(X)$*;*
(iii) *The correspondence* $A \to \int_A f \, d\mu$ *is a charge;*
(iv) *If* $f \leqslant g$ *then* $\int_X f \, d\mu \leqslant \int_X g \, d\mu$, *that is* integral is monotonic;
(v) *The function*

$$d_1(f,g) = \int_X |f(x) - g(x)| \, d\mu(x) \tag{13.5}$$

has all properties of a metric (distance) on $S(X)$ *probably except separation, but see the next item.*
(vi) *For* $f \geqslant 0$ *we have* $\int_X f \, d\mu = 0$ *if and only if* $\mu(\{x \in X : f(x) \neq 0\}) = 0$. *Therefore for the function* d_1 (13.5)*:*

$$d_1(f,g) = 0 \quad \text{if and only if} \quad f \overset{a.e.}{=} g.$$

(vii) *The integral is* uniformly *continuous with respect the above metric* d_1 (13.5):

$$\left|\int_A f(x)\,d\mu(x) - \int_A g(x)\,d\mu(x)\right| \leqslant d_1(f,g).$$

Proof. The proof is almost obvious, for example the Property 13.21(i) easily follows from Lem. 13.18.

We will outline 13.21(iii) only. Let f is an indicator function of a set B, then $A \to \int_A f\,d\mu = \mu(A \cap B)$ is a σ-additive measure (and thus—a charge). By the Cor. 12.35 the same is true for finite linear combinations of indicator functions and their limits in the sense of distance d_1. □

We can identify functions which has the same values a.e. Then $S(X)$ becomes a metric space with the distance d_1 (13.5). The space may be incomplete and we may wish to look for its completion. However, if we will simply try to assign a limiting point to *every* Cauchy sequence in $S(X)$, then the resulting space becomes so huge that it will be impossible to realise it as a space of functions on X.

Exercise 13.22. Use ideas of Ex. 13.7(i) to present a sequence of simple functions which has the Cauchy property in metric d_1 (13.5) but does not have point-wise limits anywhere.

To reduce the number of Cauchy sequences in $S(X)$ eligible to have a limit, we shall ask an additional condition. A convenient reduction to functions on X appears if we ask both the convergence in d_1 metric *and* the point-wise convergence on X a.e.

Definition 13.23. A function f is *summable* by a measure μ if there is a sequence $(f_n) \subset S(X)$ such that

(i) the sequence (f_n) is a Cauchy sequence in $S(X)$;

(ii) $f_n \stackrel{a.e.}{\to} f$.

Clearly, if a function is summable, then any equivalent function is summable as well. Set of equivalent classes of summable functions will be denoted by $L_1(X)$.

Lemma 13.24. *If the measure μ is finite then any bounded measurable function is summable.*

Proof. Define $E_{kn}(f) = \{x \in X : k/n \leqslant f(x) < (k+1)/n\}$ and $f_n = \sum_k \frac{k}{n}\chi_{E_{kn}}$ (note that the sum is finite due to boundedness of f).

Since $|f_n(x) - f(x)| < 1/n$ we have uniform convergence (thus convergence a.e.) and (f_n) is the Cauchy sequence: $d_1(f_n, f_m) = \int_X |f_n - f_m|\, d\mu \leqslant (\frac{1}{n} + \frac{1}{m})\mu(X)$. □

Remark 13.25. This Lemma can be extended to the space of *essentially bounded* functions $L_\infty(X)$, that is functions which are bounded a.e. In other words, $L_\infty(X) \subset L_1(X)$ for finite measures.

Another simple result, which is useful on many occasions is as follows.

Lemma 13.26. *If the measure μ is finite and $f_n \rightrightarrows f$ then $d_1(f_n, f) \to 0$.*

Corollary 13.27. *For a convergent sequence $f_n \xrightarrow{a.e.} f$, which admits the uniform bound $|f_n(x)| < M$ for all n and x, we have $d_1(f_n, f) \to 0$.*

Proof. For any $\varepsilon > 0$, by the Egorov's theorem 13.8 we can find E, such that

(i) $\mu(E) < \frac{\varepsilon}{2M}$; and

(ii) from the uniform convergence on $X \setminus E$ there exists N such that for any $n > N$ we have $|f(x) - f_n(x)| < \frac{\varepsilon}{2\mu(X)}$.

Combining this we found that for $n > N$, $d_1(f_n, f) < M\frac{\varepsilon}{2M} + \mu(X)\frac{\varepsilon}{2\mu(X)} < \varepsilon$. □

Exercise 13.28. Convergence in the metric d_1 and a.e. do not imply each other:

(i) Give an example of $f_n \xrightarrow{a.e.} f$ such that $d_1(f_n, f) \not\to 0$.

(ii) Give an example of the sequence (f_n) and function f in $L_1(X)$ such that $d_1(f_n, f) \to 0$ but f_n does not converge to f a.e.

To build integral we need the following

Lemma 13.29. *Let (f_n) and (g_n) be two Cauchy sequences in $S(X)$ with the same limit a.e., then $d_1(f_n, g_n) \to 0$.*

Proof. Let $\phi_n = f_n - g_n$, then this is a Cauchy sequence with zero limit a.e. Assume the opposite to the statement: there exist $\delta > 0$ and sequence (n_k) such that $\int_x |\phi_{n_k}|\, d\mu > \delta$. Rescaling-renumbering we can obtain $\int_x |\phi_n|\, d\mu > 1$. Take *quickly convergent subsequence* using the Cauchy property:

$$d_1(\phi_{n_k}, \phi_{n_{k+1}}) \leqslant 1/2^{k+2}.$$

Renumbering agian assume $d_1(\phi_k, \phi_{k+1}) \leqslant 1/2^{k+2}$.

Since ϕ_1 is a simple, take the canonical presentation $\phi_1 = \sum_k t_k \chi_{A_k}$, then $\sum_k |t_k| \mu(A_k) = \int_X |\phi_1| \, d\mu \geqslant 1$. Thus, there exists N, such that $\sum_{k=1}^{N} |t_k| \mu(A_k) \geqslant 3/4$. Put $A = \sqcup_{k=1}^{N} A_k$ and $C = \max_{1 \leqslant k \leqslant N} |t_k| = \max_{x \in A} |\phi_1(x)|$.

By the Egorov's Theorem 13.8 there is $E \subset A$ such that $\mu(E) < 1/(4C)$ and $\phi_n \rightrightarrows 0$ on $B = A \setminus E$. Then

$$\int_B |\phi_1| \, d\mu = \int_A |\phi_1| \, d\mu - \int_E |\phi_1| \, d\mu \geqslant \frac{3}{4} - \frac{1}{4C} \cdot C = \frac{1}{2}.$$

By the triangle inequality for d_1:

$$\left| \int_B |\phi_n| \, d\mu - \int_B |\phi_{n+1}| \, d\mu \right| \leqslant d_1(\phi_n, \phi_{n+1}) \leqslant \frac{1}{2^{n+2}}$$

we get

$$\int_B |\phi_n| \, d\mu \geqslant \int_B |\phi_1| \, d\mu - \sum_{k=1}^{n-1} \left| \int_B |\phi_n| \, d\mu - \int_B |\phi_{n+1}| \, d\mu \right| \geqslant \frac{1}{2} - \sum_{1}^{n-1} \frac{1}{2^{k+2}} > \frac{1}{4}.$$

But this contradicts to the fact $\int_B |\phi_n| \, d\mu \to 0$, which follows from the uniform convergence $\phi_n \rightrightarrows 0$ on B. □

It follows from the Lemma that we can use any Cauchy sequence of simple functions for the extension of integral.

Corollary 13.30. *The functional* $I_A(f) = \int_A f(x) \, d\mu(x)$, *defined on any* $A \in \mathfrak{L}$ *on the space of simple functions* $S(X)$ *can be extended by continuity to the functional on* $L_1(X, \mu)$.

Definition 13.31. For an arbitrary summable $f \in L_1(X)$, we define the *Lebesgue integral*

$$\int_A f \, d\mu = \lim_{n \to \infty} \int_A f_n \, d\mu,$$

where the Cauchy sequence f_n of summable simple functions converges to f a.e.

Theorem 13.32. (i) $L_1(X)$ *is a linear space.*

(ii) *For any measurable set* $A \subset X$ *the correspondence* $f \mapsto \int_A f \, d\mu$ *is a linear functional on* $L_1(X)$.

(iii) *For any* $f \in L_1(X)$ *the value* $\nu(A) = \int_A f \, d\mu$ *is a charge.*

(iv) $d_1(f, g) = \int_A |f - g| \, d\mu$ *is a distance on* $L_1(X)$.

Proof. The proof is follows from Prop. 13.21 and continuity of extension. □

Summing up: we build $L_1(X)$ as a completion of $S(X)$ with respect to the distance d_1 such that elements of $L_1(X)$ are associated with (equivalence classes of) measurable functions on X.

13.3. **Properties of the Lebesgue Integral.** The space L_1 was defined from dual convergence—in d_1 metric and point-wise a.e. Can we get the continuity of the integral from the convergence almost everywhere alone? No, in general. However, we will state now some results on continuity of the integral under convergence a.e. with some additional assumptions. Finally, we show that $L_1(X)$ is closed in d_1 metric.

Theorem 13.33 (Lebesgue on dominated convergence). *Let (f_n) be a sequence of μ-summable functions on X, and there is $\phi \in L_1(X)$ such that $|f_n(x)| \leqslant \phi(x)$ for all $x \in X$, $n \in \mathbb{N}$.*
If $f_n \overset{a.e.}{\to} f$, then $f \in L_1(X)$ and for any measurable A:

$$\lim_{n\to\infty} \int_A f_n \, d\mu = \int_A f \, d\mu.$$

Proof. For any measurable A the expression $\nu(A) = \int_A \phi \, d\mu$ defines a finite measure on X due to non-negativeness of ϕ and Thm. 13.32.

Lemma 13.34 (Change of variable). *If g is measurable and bounded then $f = \phi g$ is μ-summable and for any μ-measurable set A we have*

$$\int_A f \, d\mu = \int_A g \, d\nu. \tag{13.6}$$

Proof of the Lemma. Let M be the set of all g such that the Lemma is true. M includes any indicator functions $g = \chi_B$ of a measurable B:

$$\int_A f\,d\mu = \int_A \phi\chi_B\,d\mu = \int_{A\cap B} \phi\,d\mu = \nu(A\cap B) = \int_A \chi_B\,d\nu = \int_A g\,d\nu.$$

Thus M contains also finite linear combinations of indicators. For any $n \in \mathbb{N}$ and a bounded g two functions $g_-(x) = \frac{1}{n}[ng(x)]$ and $g_+(x) = g_- + \frac{1}{n}$ are finite linear combinations of indicators and are in M. Since $g_-(x) \leqslant g(x) \leqslant g_+(x)$ we have

$$\int_A g_-\,d\nu = \int_A \phi\, g_-\,d\mu \leqslant \int_A \phi\, g\,d\mu \leqslant \int_A \phi\, g_+\,d\mu = \int_A g_+\,d\nu.$$

By squeeze rule for $n \to \infty$ we have the middle term tenses to $\int_A g\,d\nu$, that is $g \in M$.

Note, that formula (13.6) is a change of variable in the Lebesgue integral of the type: $\int f(\sin x)\cos x\,dx = \int f(\sin x)\,d(\sin x)$. □

For the proof of the theorem define:

$$g_n(x) = \begin{cases} f_n(x)/\phi(x), & \text{if } \phi(x) \neq 0,\\ 0, & \text{if } \phi(x) = 0,\end{cases}$$
$$g(x) = \begin{cases} f(x)/\phi(x), & \text{if } \phi(x) \neq 0,\\ 0, & \text{if } \phi(x) = 0.\end{cases}$$

Then g_n is bounded by 1 and $g_n \stackrel{a.e.}{\to} g$. To show the theorem it will be enough to show $\lim_{n\to\infty} \int_A g_n\,d\nu = \int_A g\,d\nu$. For the uniformly bounded functions on the finite measure set this can be derived from the Egorov's Thm. 13.8, see an example of this in the proof of Lemma 13.29. □

Note, that in the above proof summability of ϕ was used to obtain the finiteness of the measure ν, which is required for Egorov's Thm. 13.8.

Exercise 13.35. Give an example of $f_n \stackrel{a.e.}{\to} f$ such that $\int_X f_n\,d\mu \neq \int_X f\,d\mu$. For such an example, try to find a function ϕ such that $|f_n| \leqslant \phi$ for all n and check either ϕ is summable.

Exercise 13.36 (Chebyshev's inequality)**.** Show that: if f is non-negative and summable, then

$$\mu\{x \in X : f(x) > c\} < \frac{1}{c}\int_X f\,d\mu. \tag{13.7}$$

Theorem 13.37 (B. Levi's, on monotone convergence)**.** *Let* (f_n) *be monotonically increasing sequence of* μ*-summable functions on* X*. Define* $f(x) = \lim_{n\to\infty} f_n(x)$ *(allowing the value* $+\infty$*).*

(i) *If all integrals* $\int_X f_n\, d\mu$ *are bounded by the same value* $C < \infty$ *then* f *is summable and* $\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$*.*
(ii) *If* $\lim_{n\to\infty} \int_X f_n\, d\mu = +\infty$ *then function* f *is not summable.*

Proof. Replacing f_n by $f_n - f_1$ and f by $f - f_1$ we can assume $f_n \geqslant 0$ and $f \geqslant 0$. Let E be the set where f is infinite, then

$$E = \cap_N \cup_n E_{Nn}, \qquad \text{where} \quad E_{Nn} = \{x \in X : f_n(x) \geqslant N\}.$$

By Chebyshev's inequality (13.7) we have

$$N\mu(E_{Nn}) < \int_{E_{Nn}} f_n\, d\mu \leqslant \int_X f_n\, d\mu \leqslant C,$$

then $\mu(E_{Nn}) \leqslant C/N$. Thus $\mu(E) = \lim_{N\to\infty} \lim_{n\to\infty} \mu(E_{Nn}) = 0$.
Thus f is finite a.e.

Lemma 13.38. *Let* f *be a measurable non-negative function attaining only finite values.* f *is summable if and only if* $\sup \int_A f\, d\mu < \infty$*, where the supremum is taken over all finite-measure set* A *such that* f *is bounded on* A*.*

Proof of the Lemma. Necessity: if f is summable then for any set $A \subset X$ we have $\int_A f\, d\mu \leqslant \int_X f\, d\mu < \infty$, thus the supremum is finite.
Sufficiency: let $\sup \int_A f\, d\mu = M < \infty$, define $B = \{x \in X : f(x) = 0\}$ and $A_k = \{x \in X : 2^k \leqslant f(x) < 2^{k+1},\ k \in \mathbb{Z}\}$, by (13.7) we have $\mu(A_k) < M/2^k$ and $X = B \sqcup (\sqcup_{k=0}^{\infty} A_k)$. Define

$$g(x) = \begin{cases} 2^k, & \text{if } x \in A_k, \\ 0, & \text{if } x \in B, \end{cases}$$

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in \sqcup_{-n}^{n} A_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $g(x) \leqslant f(x) < 2g(x)$. Function g is a simple function, its summability follows from the estimate $\int_{\sqcup_{-n}^{n} A_k} g\, d\mu \leqslant \int_{\sqcup_{-n}^{n} A_k} f\, d\mu \leqslant M$ which is valid for any n, taking $n \to \infty$ we get summability of g. Furthermore, $f_n \stackrel{a.e.}{\to} f$ and $f_n(x) \leqslant f(x) < 2g(x)$, so we use the Lebesgue Thm. 13.33 on dominated convergence to obtain the conclusion. □

Let A be a finite measure set such that f is bounded on A, then

$$\int_A f\,d\mu \overset{\text{Cor. 13.27}}{=} \lim_{n\to\infty} \int_A f_n\,d\mu \leqslant \lim_{n\to\infty} \int_X f_n\,d\mu \leqslant C.$$

This show summability of f by the previous Lemma. The rest of statement and (contrapositive to) the second part follows from the Lebesgue Thm. 13.33 on dominated convergence. □

Now we can extend this result dropping the monotonicity assumption.

Lemma 13.39 (Fatou). *If a sequence* (f_n) *of* μ*-summable non-negative functions is such that:*

- $\int_X f_n\,d\mu \leqslant C$ *for all* n*;*
- $f_n \overset{a.e.}{\to} f$,

then f *is* μ*-summable and* $\int_X f\,d\mu \leqslant C$.

Proof. Let us replace the limit $f_n \to f$ by two monotonic limits. Define:

$$\begin{aligned} g_{kn}(x) &= \min(f_n(x), \ldots, f_{n+k}(x)),\\ g_n(x) &= \lim_{k\to\infty} g_{kn}(x). \end{aligned}$$

Then g_n is a non-decreasing sequence of functions and $\lim_{n\to\infty} g_n(x) = f(x)$ a.e. Since $g_n \leqslant f_n$, from monotonicity of integral we get $\int_X g_n\,d\mu \leqslant C$ for all n. Then Levi's Thm. 13.37 implies that f is summable and $\int_X f\,d\mu \leqslant C$. □

Remark 13.40. Note that the price for dropping monotonicity from Thm. 13.37 to Lem. 13.39 is that the limit $\int_X f_n\,d\mu \to \int_X f\,d\mu$ may not hold any more.

Exercise 13.41. Give an example such that under the Fatou's lemma condition we get $\lim_{n\to\infty} \int_X f_n\,d\mu \neq \int_X f\,d\mu$.

Now we can show that $L_1(X)$ is complete:

Theorem 13.42. $L_1(X)$ *is a Banach space.*

Proof. It is clear that the distance function d_1 indeed define a norm $\|f\|_1 = d_1(f,0)$. We only need to demonstrate the completeness. We again utilise the three-step procedure from Rem. 11.7.

Take a Cauchy sequence (f_n) and building a subsequence if necessary, assume that its *quickly convergent* that is $d_1(f_n, f_{n+1}) \leqslant 1/2^k$. Put

$$\phi_1 = f_1 \text{ and } \phi_n = f_n - f_{n-1} \text{ for } n > 1. \text{ Then } f_n = \sum_{k=1}^{n} \phi_k .$$

The sequence $\psi_n(x) = \sum_1^n |\phi_k(x)|$ is monotonic, integrals $\int_X \psi_n \, d\mu$ are bounded by the same constant $\|f_1\|_1 + 1$. Thus, by the B. Levi's Thm. 13.37 and its proof, $\psi_n \to \psi$ for a summable essentially bounded function ψ. Therefore, the series $\sum \phi_k(x)$ converges as well to a value $f(x)$ of a function f. But, this means that $f_n \overset{a.e.}{\to} f$ (the first step is completed).
We also notice $|f_n(x)| \leqslant |\psi(x)|$. Thus by the Lebesgue Thm. 13.33 on dominated convergence $f \in L_1(X)$ (the second step is completed).
Furthermore,

$$0 \leqslant \lim_{n\to\infty} \int_X |f_n - f| \, d\mu \leqslant \lim_{n\to\infty} \sum_{k=n}^{\infty} \|\phi_k\| = 0.$$

That is, $f_n \to f$ in the norm of $L_1(X)$. (That completes the third step and the whole proof). □

The next important property of the Lebesgue integral is its *absolute continuity*.

Theorem 13.43 (Absolute continuity of Lebesgue integral)**.** *Let* $f \in L_1(X)$. *Then for any* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $\left|\int_A f \, d\mu\right| < \varepsilon$ *if* $\mu(A) < \delta$.

Proof. If f is essentially bounded by M, then it is enough to set $\delta = \varepsilon/M$. In general let:

$$\begin{aligned} A_n &= \{x \in X : n \leqslant |f(x)| < n+1\}, \\ B_n &= \sqcup_0^n A_k, \\ C_n &= X \setminus B_n. \end{aligned}$$

Then $\int_X |f| \, d\mu = \sum_0^\infty \int_{A_k} |f| \, d\mu$, thus there is an N such that $\sum_N^\infty \int_{A_k} |f| \, d\mu = \int_{C_N} |f| \, d\mu < \varepsilon/2$. Now put $\delta = \frac{\varepsilon}{2N+2}$, then for any $A \subset X$ with $\mu(A) < \delta$:

$$\left|\int_A f \, d\mu\right| \leqslant \int_A |f| \, d\mu = \int_{A\cap B_N} |f| \, d\mu + \int_{A \cap C_N} |f| \, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

13.4. **Integration on Product Measures.** It is well-known geometrical interpretation of an integral in calculus as the "area under the graph". If we advance from "area" to a "measure" then the Lebesgue integral can be treated as theory of measures of very special shapes created by graphs of functions. This shapes belong to the product spaces of the function domain and its range. We introduced product measures in Defn. 12.40, now we will study them in same details using the Lebesgue integral. We start from the following

Theorem 13.44. *Let X and Y be spaces, and let S and T be semirings on X and Y respectively and μ and ν be measures on S and T respectively. If μ and ν are σ-additive, then the product measure $\nu \times \mu$ from Defn. 12.40 is σ-additive as well.*

Proof. For any $C = A \times B \in S \times T$ let us define $f_C(x) = \chi_A(x)\nu(B)$. Then

$$(\mu \times \nu)(C) = \mu(A)\nu(B) = \int_X f_C \, d\mu.$$

If the same set C has a representation $C = \sqcup_k C_k$ for $C_k \in S \times T$, then σ-additivity of ν implies $f_C = \sum_k f_{C_k}$. By the Lebesgue theorem 13.33 on dominated convergence:

$$\int_X f_C \, d\mu = \sum_k \int_X f_{C_k} \, d\mu.$$

Thus

$$(\mu \times \nu)(C) = \sum_k (\mu \times \nu)(C_k).$$

□

The above correspondence $C \mapsto f_C$ can be extended to the ring $R(S \times T)$ generated by $S \times T$ by the formula:

$$f_C = \sum_k f_{C_k}, \qquad \text{for } C = \sqcup_k C_k \in R(S \times T).$$

We have the uniform continuity of this correspondence:

$$\|f_{C_1} - f_{C_2}\|_1 \leqslant (\mu \times \nu)(C_1 \triangle C_2) = d_1(C_1, C_2)$$

because from the representation $C_1 = A_1 \sqcup B$ and $C_2 = A_2 \sqcup B$, where $B = C_1 \cap C_2$ one can see that $f_{C_1} - f_{C_2} = f_{A_1} - f_{A_2}$, $f_{C_1 \triangle C_2} = f_{A_1} + f_{A_2}$ together with $|f_{A_1} - f_{A_2}| \leqslant f_{A_1} + f_{A_2}$ for non-negative functions.

Thus the map $C \mapsto f_C$ can be extended to the map of σ-algebra $L(X \times Y)$ of $\mu \times \nu$-measurable set to $L_1(X)$ by the formula $f_{\lim_n C_n} = \lim_n f_{C_n}$.

Exercise 13.45. Describe topologies where two limits from the last formula are taken.

The following lemma provides the geometric interpretation of the function f_C as the size of the slice of the set C along $x = \text{const}$.

Lemma 13.46. *Let* $C \in L(X \times Y)$. *For almost every* $x \in X$ *the set* $C_x = \{y \in Y : (x, y) \in C\}$ *is* ν*-measurable and* $\nu(C_x) = f_C(x)$.

Proof. For sets from the ring $R(S \times T)$ it is true by the definition. If $C^{(n)}$ is a monotonic sequence of sets, then $\nu(\lim_n C_x^{(n)}) = \lim_n \nu(C_x^{(n)})$ by σ-additivity of measures. Thus the property $\nu(C_x) = f_x(C)$ is preserved by monotonic limits. The following result of the separate interest:

Lemma 13.47. *Any measurable set can be received (up to a set of zero measure) from elementary sets by two monotonic limits.*

Proof of Lem. 13.47. Let C be a measurable set, put $C_n \in R(S \times T)$ to approximate C up to 2^{-n} in $\mu \times \nu$. Let $\tilde{C} = \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} C_{n+k}$, then

$$(\mu \times \nu)\left(C \setminus \cup_{k=1}^{\infty} C_{n+k}\right) = 0 \quad \text{and} \quad (\mu \times \nu)\left(\cup_{k=1}^{\infty} C_{n+k} \setminus C\right) = 2^{1-n}.$$

Then $(\mu \times \nu)(\tilde{C} \bigtriangleup C) \leqslant 2^{1-n}$ for any $n \in \mathbb{N}$. □

Coming back to Lem. 13.46 we notice that (in the above notations) $f_C = f_{\tilde{C}}$ almost everywhere. Then:

$$f_C(x) \stackrel{\text{a.e}}{=} f_{\tilde{C}}(x) = \nu(\tilde{C}_x) = \nu(C_x).$$

□

The following theorem generalizes the meaning of the integral as "area under the graph".

Theorem 13.48. *Let* μ *and* ν *are* σ*-finite measures and* C *be a* $\mu \times \nu$ *measurable set* $X \times Y$. *We define* $C_x = \{y \in Y : (x, y) \in C\}$. *Then for* μ*-almost every* $x \in X$ *the set* C_x *is* ν*-measurable, function* $f_C(x) = \nu(C_x)$ *is* μ*-measurable and*

$$(\mu \times \nu)(C) = \int_X f_C \, d\mu, \tag{13.8}$$

where both parts may have the value $+\infty$.

Proof. If C has a finite measure, then the statement is reduced to Lem. 13.46 and a passage to limit in (13.8).

If C has an infinite measure, then there exists a sequence of $C_n \subset C$, such that $\cup_n C_n = C$ and $(\mu \times \nu)(C_n) \to \infty$. Then $f_C(x) = \lim_n f_{C_n}(x)$ and

$$\int_X f_{C_n} \, d\mu = (\mu \times \nu)(C_n) \to +\infty.$$

Thus f_C is measurable and non-summable. □

This theorem justify the well-known technique to calculation of areas (volumes) as integrals of length (areas) of the sections.

Remark 13.49. (i) The role of spaces X and Y in Theorem 13.48 is symmetric, thus we can swap them in the conclusion.
(ii) The Theorem 13.48 can be extended to any finite number of measure spaces. For the case of three spaces (X,μ), (Y,ν), (Z,λ) we have:

$$(\mu\times\nu\times\lambda)(C)=\int_{X\times Y}\lambda(C_{xy})\,d(\mu\times\nu)(x,y)=\int_Z(\mu\times\nu)(C_z)\,d\lambda(z), \tag{13.9}$$

where

$$\begin{aligned} C_{xy} &= \{z\in Z : (x,y,z)\in C\},\\ C_z &= \{(x,y)\in X\times Y : (x,y,z)\in C\}. \end{aligned}$$

Theorem 13.50 (Fubini). *Let $f(x,y)$ be a summable function on the product of spaces (X,μ) and (Y,ν). Then:*

(i) *For μ-almost every $x\in X$ the function $f(x,y)$ is summable on Y and $f_Y(x)=\int_Y f(x,y)\,d\nu(y)$ is a μ-summable on X.*
(ii) *For ν-almost every $y\in Y$ the function $f(x,y)$ is summable on X and $f_X(y)=\int_X f(x,y)\,d\mu(x)$ is a ν-summable on Y.*
(iii) *There are the identities:*

$$\begin{aligned}\int_{X\times Y} f(x,y)\,d(\mu\times\nu)(x,y) &= \int_X\left(\int_Y f(x,y)\,d\nu(y)\right)d\mu(x)\\ &= \int_Y\left(\int_X f(x,y)\,d\mu(x)\right)d\nu(y).\end{aligned}\tag{13.10}$$

(iv) *For a non-negative functions the existence of any repeated integral in (13.10) implies summability of f on $X\times Y$.*

Proof. From the decomposition $f=f_+-f_-$ we can reduce our consideration to non-negative functions. Let us consider the product of three spaces (X,μ), (Y,ν), $(\mathbb{R},\lambda)$, with $\lambda=dz$ being the Lebesgue measure on $\mathbb{R}$. Define

$$C=\{(x,y,z)\in X\times Y\times\mathbb{R} : 0\leqslant z\leqslant f(x,y)\}.$$

Using the relation (13.9) we get:

$$\begin{aligned} C_{xy} &= \{z\in\mathbb{R} : 0\leqslant z\leqslant f(x,y)\}, && \lambda(C_{xy})=f(x,y)\\ C_x &= \{(y,z)\in Y\times\mathbb{R} : 0\leqslant z\leqslant f(x,y)\}, && (\nu\times\lambda)(C_x)=\int_Y f(x,y)\,d\nu(y). \end{aligned}$$

the theorem follows from those relations. □

Exercise 13.51.
- Show that the first three conclusions of the Fubini Theorem may fail if f is not summable.
- Show that the fourth conclusion of the Fubini Theorem may fail if f has values of different signs.

13.5. **Absolute Continuity of Measures.** Here, we consider another topic in the measure theory which benefits from the integration theory.

Definition 13.52. Let X be a set with σ-algebra R and σ-finite measure μ and finite charge ν on R. The charge ν is *absolutely continuous* with respect to μ if $\mu(A) = 0$ for $A \in R$ implies $\nu(A) = 0$. Two charges ν_1 and ν_2 are *equivalent* if two conditions $|\nu_1|(A) = 0$ and $|\nu_2|(A) = 0$ are equivalent.

The above definition seems to be not justifying "absolute continuity" name, but this will become clear from the following important theorem.

Theorem 13.53 (Radon–Nikodym)**.** *Any charge ν which absolutely continuous with respect to a measure μ has the form*

$$\nu(A) = \int_A f\,d\mu,$$

where f is a function from L_1. The function $f \in L_1$ is uniquely defined by the charge ν.

Sketch of the proof. First we will assume that ν is a measure. Let D be the collection of measurable functions $g : X \to [0, \infty)$ such that

$$\int_E g\,d\mu \leqslant \nu(E) \qquad (E \in L).$$

Let $\alpha = \sup_{g \in D} \int_X g\,d\mu \leqslant \nu(X) < \infty$. So we can find a sequence (g_n) in D with $\int_X g_n\,d\mu \to \alpha$.
We define $f_0(x) = \sup_n g_n(x)$. We can show that $f_0 = \infty$ only on a set of μ-measure zero, so if we adjust f_0 on this set, we get a measurable function $f : X \to [0, \infty)$. There is now a long argument to show that f is as required.
If ν is a charge, we can find f by applying the previous operation to the measures ν_+ and ν_- (as it is easy to verify that $\nu_+, \nu_- \lll \mu$).
We show that f is essentially unique. If g is another function inducing ν, then

$$\int_E f - g\,d\mu = \nu(E) - \nu(E) = 0 \qquad (E \in L).$$

Let $E = \{x \in X : f(x) - g(x) \geqslant 0\}$, so as $f - g$ is measurable, $E \in L$. Then $\int_E f - g\,d\mu = 0$ and $f - g \geqslant 0$ on E, so by our result from integration theory, we have that $f - g = 0$ almost everywhere on E. Similarly, if $F = \{x \in X : f(x) - g(x) \leqslant 0\}$, then $F \in L$ and $f - g = 0$ almost everywhere on F. As $E \cup F = X$, we conclude that $f = g$ almost everywhere. □

Corollary 13.54. *Let μ be a measure on X, ν be a finite charge, which is absolutely continuous with respect to μ. For any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $|\nu|(A) < \varepsilon$.*

Proof. By the Radon–Nikodym theorem there is a function $f \in L_1(X, \mu)$ such that $\nu(A) = \int_A f\, d\mu$. Then $|\nu|(A) = \int_A |f|\, d\mu$ ad we get the statement from Theorem 13.43 on absolute continuity of the Lebesgue integral. □

14. Functional Spaces

In this section we describe various Banach spaces of functions on sets with measure.

14.1. **Integrable Functions.** Let (X, L, μ) be a measure space. For $1 \leqslant p < \infty$, we define $\mathcal{L}_p(\mu)$ to be the space of measurable functions $f : X \to \mathbb{K}$ such that

$$\int_X |f|^p \, d\mu < \infty.$$

We define $\|\cdot\|_p : \mathcal{L}_p(\mu) \to [0, \infty)$ by

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p} \qquad (f \in \mathcal{L}_p(\mu)).$$

Notice that if $f = 0$ almost everywhere, then $|f|^p = 0$ almost everywhere, and so $\|f\|_p = 0$. However, there can be non-zero functions such that $f = 0$ almost everywhere. So $\|\cdot\|_p$ is *not* a norm on $\mathcal{L}_p(\mu)$.

Exercise 14.1. Find a measure space (X, μ) such that $\ell_p = \mathcal{L}_p(\mu)$, that is the space of sequences ℓ_p is a particular case of function spaces considered in this section. It also explains why the following proofs are referencing to Section 11 so often.

Lemma 14.2 (Integral Hölder inequality)**.** *Let $1 < p < \infty$, let $q \in (1, \infty)$ be such that $1/p + 1/q = 1$. For $f \in \mathcal{L}_p(\mu)$ and $g \in \mathcal{L}_q(\mu)$, we have that fg is summable, and*

$$\int_X |fg| \, d\mu \leqslant \|f\|_p \, \|g\|_q \,. \tag{14.1}$$

Proof. Recall that we know from Lem. 11.2 that

$$|ab| \leqslant \frac{|a|^p}{p} + \frac{|b|^q}{q} \qquad (a, b \in \mathbb{K}).$$

Now we follow the steps in proof of Prop. 11.4. Define measurable functions $a, b : X \to \mathbb{K}$ by setting

$$a(x) = \frac{f(x)}{\|f\|_p}, \quad b(x) = \frac{g(x)}{\|g\|_q} \qquad (x \in X).$$

So we have that

$$|a(x)b(x)| \leqslant \frac{|f(x)|^p}{p\,\|f\|_p^p} + \frac{|g(x)|^q}{q\,\|g\|_q^q} \qquad (x \in X).$$

By integrating, we see that

$$\int_X |ab| \; d\mu \leqslant \frac{1}{p\,\|f\|_p^p} \int_X |f|^p \; d\mu + \frac{1}{q\,\|g\|_q^q} \int_X |g|^q \; d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

Hence, by the definition of a and b,

$$\int_X |fg| \leqslant \|f\|_p \, \|g\|_q \, ,$$

as required. □

Lemma 14.3. *Let* $f, g \in \mathcal{L}_p(\mu)$ *and let* $a \in \mathbb{K}$. *Then:*

(i) $\|af\|_p = |a| \, \|f\|_p$;
(ii) $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$.

In particular, $\mathcal{L}_p$ *is a vector space.*

Proof. Part 14.3(i) is easy. For 14.3(ii), we need a version of Minkowski's Inequality, which will follow from the previous lemma. We essentially repeat the proof of Prop. 11.5.

Notice that the $p = 1$ case is easy, so suppose that $1 < p < \infty$. We have that

$$\begin{aligned}
\int_X |f+g|^p \; d\mu &= \int_X |f+g|^{p-1} \, |f+g| \; d\mu \\
&\leqslant \int_X |f+g|^{p-1} \, (|f| + |g|) \; d\mu \\
&= \int_X |f+g|^{p-1} \, |f| \; d\mu + \int_X |f+g|^{p-1} \, |g| \; d\mu.
\end{aligned}$$

Applying the lemma, this is

$$\leqslant \|f\|_p \left(\int_X |f+g|^{q(p-1)} \; d\mu \right)^{1/q} + \|g\|_p \left(\int_X |f+g|^{q(p-1)} \; d\mu \right)^{1/q}.$$

As $q(p-1) = p$, we see that

$$\|f+g\|_p^p \leqslant \left(\|f\|_p + \|g\|_p\right) \|f+g\|_p^{p/q}.$$

As $p - p/q = 1$, we conclude that

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p,$$

as required.

In particular, if $f, g \in \mathcal{L}_p(\mu)$ then $\alpha f + g \in \mathcal{L}_p(\mu)$, showing that $\mathcal{L}_p(\mu)$ is a vector space. □

We define an equivalence relation $\sim$ on the space of measurable functions by setting $f \sim g$ if and only if $f = g$ almost everywhere. We can check that $\sim$ is an equivalence relation (the slightly non-trivial part is that $\sim$ is transitive).

Proposition 14.4. *For $1 \leqslant p < \infty$, the collection of equivalence classes $\mathcal{L}_p(\mu)/\sim$ is a vector space, and $\|\cdot\|_p$ is a well-defined norm on $\mathcal{L}_p(\mu)/\sim$.*

Proof. We need to show that addition, and scalar multiplication, are well-defined on $\mathcal{L}_p(\mu)/\sim$. Let $a \in \mathbb{K}$ and $f_1, f_2, g_1, g_2 \in \mathcal{L}_p(\mu)$ with $f_1 \sim f_2$ and $g_1 \sim g_2$. Then it's easy to see that $af_1 + g_1 \sim af_2 + g_2$; but this is all that's required!

If $f \sim g$ then $|f|^p = |g|^p$ almost everywhere, and so $\|f\|_p = \|g\|_p$. So $\|\cdot\|_p$ is well-defined on equivalence classes. In particular, if $f \sim 0$ then $\|f\|_p = 0$. Conversely, if $\|f\|_p = 0$ then $\int_X |f|^p \, d\mu = 0$, so as $|f|^p$ is a positive function, we must have that $|f|^p = 0$ almost everywhere. Hence $f = 0$ almost everywhere, so $f \sim 0$. That is,

$$\{f \in \mathcal{L}_p(\mu) : f \sim 0\} = \left\{f \in \mathcal{L}_p(\mu) : \|f\|_p = 0\right\}.$$

It follows from the above lemma that this is a subspace of $\mathcal{L}_p(\mu)$.

The above lemma now immediately shows that $\|\cdot\|_p$ is a norm on $\mathcal{L}_p(\mu)/\sim$. □

Definition 14.5. We write $L_p(\mu)$ for the normed space $(\mathcal{L}_p(\mu)/\sim, \|\cdot\|_p)$.

We will abuse notation and continue to write members of $L_p(\mu)$ as functions. Really they are equivalence classes, and so care must be taken when dealing with $L_p(\mu)$. For example, if $f \in L_p(\mu)$, it does *not* make sense to talk about the value of f at a point.

Theorem 14.6. *Let (f_n) be a Cauchy sequence in $L_p(\mu)$. There exists $f \in L_p(\mu)$ with $\|f_n - f\|_p \to 0$. In fact, we can find a subsequence (n_k) such that $f_{n_k} \to f$ pointwise, almost everywhere.*

Proof. Consider first the case of a finite measure space X. We again follow the three steps scheme from Rem. 11.7. Let f_n be a Cauchy sequence in $L_p(\mu)$. From the Hölder inequality (14.1) we see that $\|f_n - f_m\|_1 \leqslant \|f_n - f_m\|_p (\mu(X))^{1/q}$. Thus, f_n is also a Cauchy sequence in $L_1(\mu)$. Thus by the Theorem 13.42 there is the limit function $f \in L_1(\mu)$. Moreover, from the proof of that theorem we know that there is a subsequence f_{n_k} of f_n convergent to f almost everywhere. Thus in the Cauchy sequence inequality

$$\int_X |f_{n_k} - f_{n_m}|^p \, d\mu < \varepsilon$$

we can pass to the limit $m \to \infty$ by the Fatou Lemma 13.39 and conclude:

$$\int_X |f_{n_k} - f|^p \, d\mu < \varepsilon.$$

So, f_{n_k} converges to f in $L_p(\mu)$, then f_n converges to f in $L_p(\mu)$ as well.
For a σ-finite measure μ we represent $X = \sqcup_k X_k$ with $\mu(X_k) < +\infty$ for all k. The restriction $(f_n^{(k)})$ of a Cauchy sequence $(f_n) \subset L_p(X, \mu)$ to every X_k is a Cauchy sequence in $L_p(X_k, \mu)$. By the previous paragraph there is the limit $f^{(k)} \in L_p(X_k, \mu)$. Define a function $f \in L_p(X, \mu)$ by the identities $f(x) = f^{(k)}$ if $x \in X_k$. By the additivity of integral, the Cauchy condition on (f_n) can be written as:

$$\int_X |f_n - f_m|^p \, d\mu = \sum_{k=1}^{\infty} \int_{X_k} \left| f_n^{(k)} - f_m^{(k)} \right|^p \, d\mu < \varepsilon.$$

It implies for any M:

$$\sum_{k=1}^{M} \int_{X_k} \left| f_n^{(k)} - f_m^{(k)} \right|^p \, d\mu < \varepsilon.$$

In the last inequality we can pass to the limit $m \to \infty$:

$$\sum_{k=1}^{M} \int_{X_k} \left| f_n^{(k)} - f^{(k)} \right|^p \, d\mu < \varepsilon.$$

Since the last inequality is independent of M we conclude:

$$\int_X |f_n - f|^p \, d\mu = \sum_{k=1}^{\infty} \int_{X_k} \left| f_n^{(k)} - f^{(k)} \right|^p \, d\mu < \varepsilon.$$

Thus we conclude that $f_n \to f$ in $L_p(X, \mu)$. □

Corollary 14.7. $L_p(\mu)$ *is a Banach space.*

Example 14.8. If $p = 2$ then $L_p(\mu) = L_2(\mu)$ can be equipped with the inner product:

$$\langle f, g \rangle = \int_X f\bar{g}\, d\mu. \tag{14.2}$$

The previous Corollary implies that $L_2(\mu)$ is a Hilbert space, see a preliminary discussion in Defn. 2.22.

Proposition 14.9. *Let* $(X, \mathfrak{L}, \mu)$ *be a measure space, and let* $1 \leqslant p < \infty$. *We can define a map* $\Phi : L_q(\mu) \to L_p(\mu)^*$ *by setting* $\Phi(f) = F$, *for* $f \in L_q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$, *where*

$$F : L_p(\mu) \to \mathbb{K}, \quad g \mapsto \int_X fg\, d\mu \qquad (g \in L_p(\mu)).$$

Proof. This proof very similar to proof of Thm. 11.13. For $f \in L_q(\mu)$ and $g \in L_p(\mu)$, it follows by the Hölder's Inequality (14.1), that fg is summable, and

$$\left| \int_X fg\, d\mu \right| \leqslant \int_X |fg|\, d\mu \leqslant \|f\|_q \|g\|_p .$$

Let $f_1, f_2 \in L_q(\mu)$ and $g_1, g_2 \in L_p(\mu)$ with $f_1 \sim f_2$ and $g_1 \sim g_2$. Then $f_1 g_1 = f_2 g_1$ almost everywhere and $f_2 g_1 = f_2 g_2$ almost everywhere, so $f_1 g_1 = f_2 g_2$ almost everywhere, and hence

$$\int_X f_1 g_1\, d\mu = \int_X f_2 g_2\, d\mu.$$

So Φ is well-defined.

Clearly Φ is linear, and we have shown that $\|\Phi(f)\| \leqslant \|f\|_q$.

Let $f \in L_q(\mu)$ and define $g : X \to \mathbb{K}$ by

$$g(x) = \begin{cases} \overline{f(x)}\, |f(x)|^{q-2} & : f(x) \neq 0, \\ 0 & : f(x) = 0. \end{cases}$$

Then $|g(x)| = |f(x)|^{q-1}$ for all $x \in X$, and so

$$\int_X |g|^p\, d\mu = \int_X |f|^{p(q-1)}\, d\mu = \int_X |f|^q\, d\mu,$$

so $\|g\|_p = \|f\|_q^{q/p}$, and so, in particular, $g \in L_p(\mu)$. Let $F = \Phi(f)$, so that

$$F(g) = \int_X fg\, d\mu = \int_X |f|^q\, d\mu = \|f\|_q^q .$$

Thus $\|F\| \geqslant \|f\|_q^q / \|g\|_p = \|f\|_q$. So we conclude that $\|F\| = \|f\|_q$, showing that Φ is an isometry. □

Proposition 14.10. *Let* (X, L, μ) *be a finite measure space, let* $1 \leqslant p < \infty$, *and let* $F \in L_p(\mu)^*$. *Then there exists* $f \in L_q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$ *such that*

$$F(g) = \int_X fg \, d\mu \qquad (g \in L_p(\mu)).$$

Sketch of the proof. As $\mu(X) < \infty$, for $E \in L$, we have that $\|\chi_E\|_p = \mu(E)^{1/p} < \infty$. So $\chi_E \in L_p(\mu)$, and hence we can define

$$\nu(E) = F(\chi_E) \qquad (E \in L).$$

We proceed to show that ν is a signed (or complex) measure. Then we can apply the Radon-Nikodym Theorem 13.53 to find a function $f : X \to \mathbb{K}$ such that

$$F(\chi_E) = \nu(E) = \int_E f \, d\mu \qquad (E \in L).$$

There is then a long argument to show that $f \in L_q(\mu)$, which we skip here. Finally, we need to show that

$$\int_X fg \, d\mu = F(g)$$

for all $g \in L_p(\mu)$, and not just for $g = \chi_E$. That follows for simple functions with a finite set of values by linearity of the Lebesgue integral and F. Then, it can be extended by continuity to the entire space $L_p(\mu)$ in view in the following Prop. 14.14. □

Proposition 14.11. *For* $1 < p < \infty$, *we have that* $L_p(\mu)^* = L_q(\mu)$ *isometrically, under the identification of the above results.*

Remark 14.12. (i) For $p = q = 2$ we obtain a special case of the Riesz–Frechét theorem 4.11 about self-duality of the Hilbert space $L_2(\mu)$.
(ii) Note that L_∞^* is not isomorphic to L_1, except finite-dimensional situation. Moreover if μ is not a point measure L_1 is not a dual to any Banach space.

Exercise 14.13. Let μ be a measure on the real line.
(i) Show that the space $L_\infty(\mathbb{R}, \mu)$ is either finite-dimensional or non-separable.
(ii) Show that for $p \neq q$ neither $L_p(\mathbb{R}, \mu)$ nor $L_q(\mathbb{R}, \mu)$ contains the other space.

14.2. **Dense Subspaces in L_p.** We note that $f \in L_p(X)$ if and only if $|f|^p$ is summable, thus we can use all results from Section 13 to investigate $L_p(X)$.

Proposition 14.14. *Let (X, L, μ) be a finite measure space, and let $1 \leqslant p < \infty$. Then the collection of simple bounded functions attained only a finite number of values is dense in $L_p(\mu)$.*

Proof. Let $f \in L_p(\mu)$, and suppose for now that $f \geqslant 0$. For each $n \in \mathbb{N}$, let

$$f_n = \min(n, \tfrac{1}{n}\lfloor nf \rfloor).$$

Then each f_n is simple, $f_n \uparrow f$, and $|f_n - f|^p \to 0$ pointwise. For each n, we have that

$$0 \leqslant f_n \leqslant f \implies 0 \leqslant f - f_n \leqslant f,$$

so that $|f - f_n|^p \leqslant |f|^p$ for all n. As $\int |f|^p \, d\mu < \infty$, we can apply the Dominated Convergence Theorem to see that

$$\lim_n \int_X |f_n - f|^p \, d\mu = 0,$$

that is, $\|f_n - f\|_p \to 0$.

The general case follows by taking positive and negative parts, and if $\mathbb{K} = \mathbb{C}$, by taking real and imaginary parts first. □

Corollary 14.15. *Let μ be the Lebesgue measure on the real line. The collection of simple bounded functions with compact supports attained only a finite number of values is dense in $L_p(\mathbb{R}, \mu)$.*

Proof. Let $f \in L_p(\mathbb{R}, \mu)$, since $\int_{\mathbb{R}} |f|^p \, d\mu = \sum_{k=-\infty}^{\infty} \int_{[k,k+1)} |f|^p \, d\mu$ there exists N such that $\sum_{k=-\infty}^{-N} + \sum_{N}^{k=\infty} \int_{[k,k+1)} |f|^p \, d\mu < \varepsilon$. By the previous Proposition, the restriction of f to $[-N, N]$ can be ε-approximated by a simple bounded function f_1 with support in $[-N, N]$ attained only a finite number of values. Therefore f_1 will be also (2ε)-approximation to f as well. □

Definition 14.16. A function $f : \mathbb{R} \to \mathbb{C}$ is called *step function* if it a linear combination of a finite number of indicator functions of half-open disjoint intervals: $f = \sum_k c_k \chi_{[a_k, b_k)}$.

The regularity of the Lebesgue measure allows to make a stronger version of Prop. 14.14.

Lemma 14.17. *The space of step functions is dense in* $L_p(\mathbb{R})$.

Proof. By Prop. 14.14, for a given $f \in L_p(\mathbb{R})$ and $\varepsilon > 0$ there exists a simple function $f_0 = \sum_{k=1}^{n} c_k \chi_{A_k}$ such that $\|f - f_0\|_p < \frac{\varepsilon}{2}$. Let $M = \|f_0\|_\infty < \infty$. By measurability of the set A_k there is $C_k = \sqcup_j^{m_k} [a_{j_k}, b_{j_k})$ a disjoint finite union of half-open intervals such that $\mu(C_k \vartriangle A_k) < \frac{\varepsilon}{2n^3 M}$. Since A_k and A_j are disjoint for $k \neq j$ we also obtain by the triangle inequality: $\mu(C_j \cap A_k) < \frac{\varepsilon}{2n^3 M}$ and $\mu(C_j \cap C_k) < \frac{2\varepsilon}{2n^3 M}$. We define a step function

$$f_1 = \sum_{k=1}^{n} c_k \chi_{C_k} = \sum_{k=1}^{n} \sum_{j}^{m_k} c_k \chi_{[a_{j_k}, b_{jk})}.$$

Clearly

$$f_1(x) = c_k \qquad \text{for all } x \in A_k \setminus ((C_k \vartriangle A_k) \cup (\cup_{j \neq k} C_j)).$$

Thus:

$$\mu(\{x \in \mathbb{R} \mid f_0(x) \neq f_1(x)\}) \leqslant n \cdot n \cdot \frac{\varepsilon}{2n^3 M} = \frac{\varepsilon}{2nM}.$$

Then $\|f_0 - f_1\|_p \leqslant nM \cdot \frac{\varepsilon}{2nM} = \frac{\varepsilon}{2}$ because $\|f_1\|_\infty < nM$. Thus $\|f - f_1\|_p < \varepsilon$. □

Corollary 14.18. *The collection of continuous function belonging to* $L_p(\mathbb{R})$ *is dense in* $L_p(\mathbb{R})$.

Proof. In view of Rem. 2.29 and the previous Lemma it is enough to show that the characteristic function of an interval $[a, b]$ can be approximated by a continuous function in $L_p(\mathbb{R})$. The idea of such approximation is illustrated by Fig. 4 and we skip the technical details. □

We will establish denseness of the subspace of smooth function in § 15.4.

Exercise 14.19. Show that every $f \in L_1(\mathbb{R})$ is *continuous on average*, that is for any $\varepsilon > 0$ there is $\delta > 0$ such that for all t such that $|t| < \delta$ we have:

$$\int_{\mathbb{R}} |f(x) - f(x+t)| \, dx < \varepsilon \,. \tag{14.3}$$

Here is an alternative demonstration of a similar result, it essentially encapsulate all the above separate statements. Let $([0,1], L, \mu)$ be the restriction of Lebesgue measure to $[0, 1]$. We often write $L_p([0, 1])$ instead of $L_p(\mu)$.

Proposition 14.20. *For* $1 \leqslant p < \infty$, *we have that* $C_{\mathbb{K}}([0,1])$ *is dense in* $L_p([0,1])$.

Proof. As $[0,1]$ is a finite measure space, and each member of $C_{\mathbb{K}}([0,1])$ is bounded, it is easy to see that each $f \in C_{\mathbb{K}}([0,1])$ is such that $\|f\|_p < \infty$. So it makes sense to regard $C_{\mathbb{K}}([0,1])$ as a subspace of $L_p(\mu)$. If $C_{\mathbb{K}}([0,1])$ is not dense in $L_p(\mu)$, then we can find a non-zero $F \in L_p([0,1])^*$ with $F(f) = 0$ for each $f \in C_{\mathbb{K}}([0,1])$. This was a corollary of the Hahn-Banach theorem 11.15.
So there exists a non-zero $g \in L_q([0,1])$ with

$$\int_{[0,1]} fg \, d\mu = 0 \qquad (f \in C_{\mathbb{K}}([0,1])).$$

Let $a < b$ in $[0,1]$. By approximating $\chi_{(a,b)}$ by a continuous function, we can show that $\int_{(a,b)} g \, d\mu = \int g\chi_{(a,b)} \, d\mu = 0$.
Suppose for now that $\mathbb{K} = \mathbb{R}$. Let $A = \{x \in [0,1] : g(x) \geqslant 0\} \in L$. By the definition of the Lebesgue (outer) measure, for $\epsilon > 0$, there exist sequences (a_n) and (b_n) with $A \subseteq \cup_n (a_n, b_n)$, and $\sum_n (b_n - a_n) \leqslant \mu(A) + \epsilon$.
For each N, consider $\cup_{n=1}^N (a_n, b_n)$. If some (a_i, b_i) overlaps (a_j, b_j), then we could just consider the larger interval $(\min(a_i, a_j), \max(b_i, b_j))$. Formally by an induction argument, we see that we can write $\cup_{n=1}^N (a_n, b_n)$ as a finite union of some *disjoint* open intervals, which we abusing notations still denote by (a_n, b_n). By linearity, it hence follows that for $N \in \mathbb{N}$, if we set $B_N = \sqcup_{n=1}^N (a_n, b_n)$, then

$$\int g\chi_{B_N} \, d\mu = \int g\chi_{(a_1,b_1)\sqcup\cdots\sqcup(a_N,b_N)} \, d\mu = 0.$$

Let $B = \cup_n (a_n, b_n)$, so $A \subseteq B$ and $\mu(B) \leqslant \sum_n (b_n - a_n) \leqslant \mu(A) + \epsilon$. We then have that

$$\left| \int g\chi_{B_N} \, d\mu - \int g\chi_B \, d\mu \right| = \left| \int g\chi_{B\setminus(a_1,b_1)\sqcup\cdots\sqcup(a_N,b_N)} \, d\mu \right|.$$

We now apply Hölder's inequality to get

$$\begin{aligned}\left(\int \chi_{B\setminus(a_1,b_1)\cup\cdots\cup(a_N,b_N)} \, d\mu \right)^{1/p} \|g\|_q &= \mu(B \setminus (a_1,b_1) \sqcup \cdots \sqcup (a_N,b_N))^{1/p} \|g\|_q \\ &\leqslant \left(\sum_{n=N+1}^{\infty} (b_n - a_n) \right)^{1/p} \|g\|_q .\end{aligned}$$

We can make this arbitrarily small by making N large. Hence we conclude that

$$\int g\chi_B \, d\mu = 0.$$

Then we apply Hölder's inequality again to see that

$$\left| \int g\chi_A \, d\mu \right| = \left| \int g\chi_A \, d\mu - \int g\chi_B \, d\mu \right| = \left| \int g\chi_{B\setminus A} \, d\mu \right| \leqslant \|g\|_q \, \mu(B\setminus A)^{1/p} \leqslant \|g\|_q \, \epsilon^{1/p}.$$

As $\epsilon > 0$ was arbitrary, we see that $\int_A g \, d\mu = 0$. As g is positive on A, we conclude that $g = 0$ almost everywhere on A.
A similar argument applied to the set $\{x \in [0,1] : g(x) \leqslant 0\}$ allows us to conclude that $g = 0$ almost everywhere. If $\mathbb{K} = \mathbb{C}$, then take real and imaginary parts. □

14.3. **Continuous functions.** Let K be a compact (always assumed Hausdorff) topological space.

Definition 14.21. The *Borel σ-algebra*, $\mathcal{B}(K)$, on K, is the σ-algebra generated by the open sets in K (recall what this means from Section 11.5). A member of $\mathcal{B}(K)$ is a *Borel* set.

Notice that if $f : K \to \mathbb{K}$ is a continuous function, then clearly f is $\mathcal{B}(K)$-measurable (the inverse image of an open set will be open, and hence certainly Borel). So if $\mu : \mathcal{B}(K) \to \mathbb{K}$ is a finite real or complex charge (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ respectively), then f will be μ-summable (as f is bounded) and so we can define

$$\phi_\mu : C_{\mathbb{K}}(K) \to \mathbb{K}, \quad \phi_\mu(f) = \int_K f \, d\mu \qquad (f \in C_{\mathbb{K}}(K)).$$

Clearly ϕ_μ is linear. Suppose for now that μ is positive, so that

$$|\phi_\mu(f)| \leqslant \int_K |f| \, d\mu \leqslant \|f\|_\infty \, \mu(K) \qquad (f \in C_{\mathbb{K}}(K)).$$

So $\phi_\mu \in C_{\mathbb{K}}(K)^*$ with $\|\phi_\mu\| \leqslant \mu(K)$.

The aim of this section is to show that all of $C_{\mathbb{K}}(K)^*$ arises in this way. First we need to define a class of measures which are in a good agreement with the topological structure.

Definition 14.22. A measure $\mu : B(K) \to [0, \infty)$ is *regular* if for each $A \in B(K)$, we have

$$\begin{aligned} \mu(A) &= \sup\{\mu(E) : E \subseteq A \text{ and } E \text{ is compact}\} \\ &= \inf\{\mu(U) : A \subseteq U \text{ and } U \text{ is open}\}. \end{aligned}$$

A charge $\nu = \nu_+ - \nu_-$ is *regular* if ν_+ and ν_- are regular measures. A complex measure is *regular* if its real and imaginary parts are regular.

Note the similarity between this notion and definition of outer measure.

Example 14.23. (i) Many common measures on the real line, e.g. the Lebesgue measure, point measures, etc., are regular.

(ii) An example of the measure μ on $[0,1]$ which is *not* regular:

$$\mu(\varnothing)=0, \qquad \mu(\{\tfrac{1}{2}\})=1, \qquad \mu(A)=+\infty,$$

for any other subset $A\subset[0,1]$.

(iii) Another example of a σ-additive measure μ on $[0,1]$ which is *not* regular:

$$\mu(A)=\begin{cases} 0, & \text{if } A \text{ is at most countable;}\\ +\infty & \text{otherwise.}\end{cases}$$

The following subspace of the space of all simple functions is helpful.

As we are working only with compact spaces, for us, "compact" is the same as "closed". Regular measures somehow interact "well" with the underlying topology on K.

We let $M_{\mathbb{R}}(K)$ and $M_{\mathbb{C}}(K)$ be the collection of all finite, regular real or complex charges (that is, signed or complex measures) on $B(K)$.

Exercise 14.24. Check that, $M_{\mathbb{R}}(K)$ and $M_{\mathbb{C}}(K)$ are real or complex, respectively, vector spaces for the obvious definition of addition and scalar multiplication.

Recall, Defn. 12.31, that for $\mu\in M_{\mathbb{K}}(K)$ we define the *variation* of μ

$$\|\mu\|=\sup\left\{\sum_{n=1}^{\infty}|\mu(A_n)|\right\},$$

where the supremum is taken over all sequences (A_n) of pairwise disjoint members of $B(K)$, with $\sqcup_n A_n=K$. Such (A_n) are called *partitions*.

Proposition 14.25. *The variation $\|\cdot\|$ is a norm on $M_{\mathbb{K}}(K)$.*

Proof. If $\mu=0$ then clearly $\|\mu\|=0$. If $\|\mu\|=0$, then for $A\in B(K)$, let $A_1=A$, $A_2=K\setminus A$ and $A_3=A_4=\cdots=\varnothing$. Then (A_n) is a partition, and so

$$0=\sum_{n=1}^{\infty}|\mu(A_n)|=|\mu(A)|+|\mu(K\setminus A)|.$$

Hence $\mu(A)=0$, and so as A was arbitrary, we have that $\mu=0$.
Clearly $\|a\mu\|=|a|\,\|\mu\|$ for $a\in\mathbb{K}$ and $\mu\in M_{\mathbb{K}}(K)$.
For $\mu,\lambda\in M_{\mathbb{K}}(K)$ and a partition (A_n), we have that

$$\sum_n|(\mu+\lambda)(A_n)|=\sum_n|\mu(A_n)+\lambda(A_n)|\leqslant\sum_n|\mu(A_n)|+\sum_n|\lambda(A_n)|\leqslant\|\mu\|+\|\lambda\|.$$

As (A_n) was arbitrary, we see that $\|\mu+\lambda\|\leqslant\|\mu\|+\|\lambda\|$. □

To get a handle on the "regular" condition, we need to know a little more about $C_{\mathbb{K}}(K)$.

Theorem 14.26 (Urysohn's Lemma)**.** *Let K be a compact space, and let E, F be closed subsets of K with $E \cap F = \varnothing$. There exists $f : K \to [0,1]$ continuous with $f(x) = 1$ for $x \in E$ and $f(x) = 0$ for $x \in F$ (written $f(E) = \{1\}$ and $f(F) = \{0\}$).*

Proof. See a book on (point set) topology. □

Lemma 14.27. *Let $\mu : B(K) \to [0, \infty)$ be a regular measure. Then for $U \subseteq K$ open, we have*

$$\mu(U) = \sup\left\{\int_K f \, d\mu \;:\; f \in C_{\mathbb{R}}(K),\; 0 \leqslant f \leqslant \chi_U\right\}.$$

Proof. If $0 \leqslant f \leqslant \chi_U$, then

$$0 = \int_K 0 \, d\mu \leqslant \int_K f \, d\mu \leqslant \int_K \chi_U \, d\mu = \mu(U).$$

Conversely, let $F = K \setminus U$, a closed set. Let $E \subseteq U$ be closed. By Urysohn Lemma 14.26, there exists $f : K \to [0,1]$ continuous with $f(E) = \{1\}$ and $f(F) = \{0\}$. So $\chi_E \leqslant f \leqslant \chi_U$, and hence

$$\mu(E) \leqslant \int_K f \, d\mu \leqslant \mu(U).$$

As μ is regular,

$$\mu(U) = \sup\{\mu(E) : E \subseteq U \text{ closed}\} \leqslant \sup\left\{\int_K f \, d\mu : 0 \leqslant f \leqslant \chi_U\right\} \leqslant \mu(U).$$

Hence we have equality throughout. □

The next result tells that the variation coincides with the norm on real charges viewed as linear functionals on $C_{\mathbb{R}}(K)$.

Lemma 14.28. *Let $\mu \in M_{\mathbb{R}}(K)$. Then*

$$\|\mu\| = \|\phi_\mu\| := \sup\left\{\left|\int_K f \, d\mu\right| : f \in C_{\mathbb{R}}(K), \|f\|_\infty \leqslant 1\right\}.$$

Proof. Let (A, B) be a Hahn decomposition (Thm. 12.36) for μ. For $f \in C_{\mathbb{R}}(K)$ with $\|f\|_\infty \leqslant 1$, we have that

$$\left|\int_K f\,d\mu\right| \leqslant \left|\int_A f\,d\mu\right| + \left|\int_B f\,d\mu\right| = \left|\int_A f\,d\mu_+\right| + \left|\int_B f\,d\mu_-\right|$$
$$\leqslant \int_A |f|\,d\mu_+ + \int_B |f|\,d\mu_- \leqslant \|f\|_\infty (\mu(A) - \mu(B)) \leqslant \|f\|_\infty \|\mu\|,$$

using the fact that $\mu(B) \leqslant 0$ and that (A, B) is a partition of K.
Conversely, as μ is regular, for $\epsilon > 0$, there exist closed sets E and F with $E \subseteq A$, $F \subseteq B$, and with $\mu_+(E) > \mu_+(A) - \epsilon$ and $\mu_-(F) > \mu_-(B) - \epsilon$. By Urysohn Lemma 14.26, there exists $f : K \to [0, 1]$ continuous with $f(E) = \{1\}$ and $f(F) = \{0\}$. Let $g = 2f - 1$, so g is continuous, g takes values in $[-1, 1]$, and $g(E) = \{1\}$, $g(F) = \{-1\}$. Then

$$\int_K g\,d\mu = \int_E 1\,d\mu + \int_F -1\,d\mu + \int_{K\setminus(E\cup F)} g\,d\mu$$
$$= \mu(E) - \mu(F) + \int_{A\setminus E} g\,d\mu + \int_{B\setminus F} g\,d\mu$$

As $E \subseteq A$, we have $\mu(E) = \mu_+(E)$, and as $F \subseteq B$, we have $-\mu(F) = \mu_-(F)$. So

$$\int_K g\,d\mu > \mu_+(A) - \epsilon + \mu_-(B) - \epsilon + \int_{A\setminus E} g\,d\mu + \int_{B\setminus F} g\,d\mu$$
$$\geqslant |\mu(A)| + |\mu(B)| - 2\epsilon - |\mu(A \setminus E)| - |\mu(B \setminus F)|$$
$$\geqslant |\mu(A)| + |\mu(B)| - 4\epsilon.$$

As $\epsilon > 0$ was arbitrary, we see that $\|\phi_\mu\| \geqslant |\mu(A)| + |\mu(B)| = \|\mu\|$. □

Thus, we know that $M_{\mathbb{R}}(K)$ is isometrically embedded in $C_{\mathbb{R}}(K)^*$.

14.4. Riesz Representation Theorem.

To facilitate an approach to the key point of this Subsection we will require some more definitions.

Definition 14.29. A functional F on $C_(K)$ is *positive* if for any non-negative function $f \geqslant 0$ we have $F(f) \geqslant 0$.

Lemma 14.30. *Any positive linear functional F on $C(X)$ is continuous and $\|F\| = F(\mathbf{1})$, where $\mathbf{1}$ is the function identically equal to 1 on X.*

Proof. For any function f such that $\|f\|_\infty \leqslant 1$ the function $1 - f$ is non negative thus: $F(\mathbf{1}) - F(f) = F(1 - f) > 0$, Thus $F(\mathbf{1}) > F(f)$, that is F is bounded and its norm is $F(\mathbf{1})$. □

So for a positive functional you know the exact place where to spot its norm, while a linear functional can attain its norm in an generic point (if any) of the unit ball in $C(X)$. It is also remarkable that any bounded linear functional can be represented by a pair of positive ones.

Lemma 14.31. *Let* λ *be a continuous linear functional on* $C(X)$*. Then there are positive functionals* λ_+ *and* λ_- *on* $C(X)$*, such that* $\lambda = \lambda_+ - \lambda_-$*.*

Proof. First, for $f \in C_{\mathbb{R}}(K)$ with $f \geqslant 0$, we define

$$\begin{aligned}
\lambda_+(f) &= \sup\left\{\lambda(g) : g \in C_{\mathbb{R}}(K), 0 \leqslant g \leqslant f\right\} \geqslant 0,\\
\lambda_-(f) &= \lambda_+(f) - \lambda(f) = \sup\left\{\lambda(g) - \lambda(f) :\ g \in C_{\mathbb{R}}(K),\ 0 \leqslant g \leqslant f\right\}\\
&= \sup\left\{\lambda(h) :\ h \in C_{\mathbb{R}}(K),\ 0 \leqslant h + f \leqslant f\right\}\\
&= \sup\left\{\lambda(h) :\ h \in C_{\mathbb{R}}(K),\ -f \leqslant h \leqslant 0\right\} \geqslant 0.
\end{aligned}$$

In a sense, this is similar to the Hahn decomposition (Thm. 12.36).
We can check that

$$\lambda_+(tf) = t\lambda_+(f), \quad \lambda_-(tf) = t\lambda_-(f) \qquad (t \geqslant 0, f \geqslant 0).$$

For $f_1, f_2 \geqslant 0$, we have that

$$\begin{aligned}
\lambda_+(f_1 + f_2) &= \sup\{\lambda(g) :\ 0 \leqslant g \leqslant f_1 + f_2\}\\
&= \sup\{\lambda(g_1 + g_2) :\ 0 \leqslant g_1 + g_2 \leqslant f_1 + f_2\}\\
&\geqslant \sup\{\lambda(g_1) + \lambda(g_2) :\ 0 \leqslant g_1 \leqslant f_1,\ 0 \leqslant g_2 \leqslant f_2\}\\
&= \lambda_+(f_1) + \lambda_+(f_2).
\end{aligned}$$

Conversely, if $0 \leqslant g \leqslant f_1 + f_2$, then set $g_1 = \min(g, f_1)$, so $0 \leqslant g_1 \leqslant f_1$. Let $g_2 = g - g_1$ so $g_1 \leqslant g$ implies that $0 \leqslant g_2$. For $x \in K$, if $g_1(x) = g(x)$ then $g_2(x) = 0 \leqslant f_2(x)$; if $g_1(x) = f_1(x)$ then $f_1(x) \leqslant g(x)$ and so $g_2(x) = g(x) - f_1(x) \leqslant f_2(x)$. So $0 \leqslant g_2 \leqslant f_2$, and $g = g_1 + g_2$. So in the above displayed equation, we really have equality throughout, and so $\lambda_+(f_1 + f_2) = \lambda_+(f_1) + \lambda_+(f_2)$. As λ is additive, it is now immediate that $\lambda_-(f_1 + f_2) = \lambda_-(f_1) + \lambda_-(f_2)$
For $f \in C_{\mathbb{R}}(K)$ we put $f_+(x) = \max(f(x), 0)$ and $f_-(x) = -\min(f(x), 0)$. Then $f_\pm \geqslant 0$ and $f = f_+ - f_-$. We define:

$$\lambda_+(f) = \lambda_+(f_+) - \lambda_+(f_-), \quad \lambda_-(f) = \lambda_-(f_+) - \lambda_-(f_-).$$

As when we were dealing with integration, we can check that λ_+ and λ_- become linear functionals; by the previous Lemma they are bounded. □

Finally, we need a technical definition.

Definition 14.32. For $f \in C_{\mathbb{R}}(K)$, we define the *support* of f, written $\operatorname{supp}(f)$, to be the closure of the set $\{x \in K : f(x) \neq 0\}$.

Theorem 14.33 (Riesz Representation)**.** *Let K be a compact (Hausdorff) space, and let $\lambda \in C_{\mathbb{K}}(K)^*$. There exists a unique $\mu \in M_{\mathbb{K}}(K)$ such that*

$$\lambda(f) = \int_K f \, d\mu \qquad (f \in C_{\mathbb{K}}(K)).$$

Furthermore, $\|\lambda\| = \|\mu\|$.

Proof. Let us show *uniqueness*. If $\mu_1, \mu_2 \in M_{\mathbb{K}}(K)$ both induce λ then $\mu = \mu_1 - \mu_2$ induces the zero functional on $C_{\mathbb{K}}(K)$. So for $f \in C_{\mathbb{R}}(K)$,

$$\begin{aligned} 0 &= \Re \int_K f \, d\mu = \int_K f \, d\mu_r \\ &= \Im \int_K f \, d\mu = \int_K f \, d\mu_i. \end{aligned}$$

So μ_r and μ_i both induce the zero functional on $C_{\mathbb{R}}(K)$. By Lemma 14.28, this means that $\|\mu_r\| = \|\mu_i\| = 0$, showing that $\mu = \mu_r + i\mu_i = 0$, as required.
Existence is harder, and we shall only sketch it here. Firstly, we shall suppose that $\mathbb{K} = \mathbb{R}$ and that λ is *positive*.
Motivated by the above Lemmas 14.27 and 14.28, for $U \subseteq K$ open, we define

$$\mu^*(U) = \sup\left\{\lambda(f) : \ f \in C_{\mathbb{R}}(K), \ 0 \leqslant f \leqslant \chi_U, \ \operatorname{supp}(f) \subseteq U\right\}.$$

For $A \subseteq K$ general, we define

$$\mu^*(A) = \inf\{\mu^*(U) : \ U \subseteq K \text{ is open}, \ A \subseteq U\}.$$

We then proceed to show that

- μ^* is an outer measure: this requires a technical topological lemma, where we make use of the support condition in the definition.
- We then check that every open set in μ^*-measurable.
- As $B(K)$ is generated by open sets, and the collection of μ^*-measurable sets is a σ-algebra, it follows that every member of $B(K)$ is μ^*-measurable.

- By using results from Section 12, it follows that if we let μ be the restriction of μ^* to $B(K)$, then μ is a measure on $B(K)$.
- We then check that this measure is regular.
- Finally, we show that μ does induce the functional λ. Arguably, it is this last step which is the hardest (or least natural to prove).

If λ is not positive, then by Lemma 14.31 represent it as $\lambda = \lambda_+ - \lambda_-$ for positive $\lambda_\pm$. As λ_+ and λ_- are positive functionals, we can find μ_+ and μ_- positive measures in $M_{\mathbb{R}}(K)$ such that

$$\lambda_+(f) = \int_K f\,d\mu_+, \quad \lambda_-(f) = \int_K f\,d\mu_- \qquad (f \in C_{\mathbb{R}}(K)).$$

Then if $\mu = \mu_+ - \mu_-$, we see that

$$\lambda(f) = \lambda_+(f) - \lambda_-(f) = \int_K f\,d\mu \qquad (f \in C_{\mathbb{R}}(K)).$$

Finally, if $\mathbb{K} = \mathbb{C}$, then we use the same "complexification" trick from the proof of the Hahn-Banach Theorem 11.15. Namely, let $\lambda \in C_{\mathbb{C}}(K)^*$, and define $\lambda_r, \lambda_i \in C_{\mathbb{R}}(K)^*$ by

$$\lambda_r(f) = \Re\lambda(f), \quad \lambda_i(f) = \Im\lambda(f) \qquad (f \in C_{\mathbb{R}}(K)).$$

These are both clearly $\mathbb{R}$-linear. Notice also that $|\lambda_r(f)| = |\Re\lambda(f)| \leqslant |\lambda(f)| \leqslant \|\lambda\|\,\|f\|_\infty$, so λ_r is bounded; similarly λ_i.
By the real version of the Riesz Representation Theorem, there exist charges μ_r and μ_i such that

$$\Re\lambda(f) = \lambda_r(f) = \int_K f\,d\mu_r, \quad \Im\lambda(f) = \lambda_i(f) = \int_K f\,d\mu_i \qquad (f \in C_{\mathbb{R}}(K)).$$

Then let $\mu = \mu_r + i\mu_i$, so for $f \in C_{\mathbb{C}}(K)$,

$$\begin{aligned}
\int_K f\,d\mu &= \int_K f\,d\mu_r + i\int_K f\,d\mu_i \\
&= \int_K \Re(f)\,d\mu_r + i\int_K \Im(f)\,d\mu_r + i\int_K \Re(f)\,d\mu_i - \int_K \Im(f)\,d\mu_i \\
&= \lambda_r(\Re(f)) + i\lambda_r(\Im(f)) + i\lambda_i(\Re(f)) - \lambda_i(\Im(f)) \\
&= \Re\lambda(\Re(f)) + i\Re\lambda(\Im(f)) + i\Im\lambda(\Re(f)) - \Im\lambda(\Im(f)) \\
&= \lambda(\Re(f) + i\Im(f)) = \lambda(f),
\end{aligned}$$

as required. □

Notice that we have not currently proved that $\|\mu\| = \|\lambda\|$ in the case $\mathbb{K} = \mathbb{C}$. See a textbook for this.

15. Fourier Transform

In this section we will briefly present a theory of Fourier transform focusing on commutative group approach. We mainly follow footsteps of [3, Ch. IV].

15.1. Convolutions on Commutative Groups.

Let G be a commutative group, we will use $+$ sign to denote group operation, respectively the inverse elements of $g \in G$ will be denoted $-g$. We assume that G has a Hausdorff topology such that operations $(g_1, g_2) \mapsto g_1 + g_2$ and $g \mapsto -g$ are continuous maps. We also assume that the topology is *locally compact*, that is the group neutral element has a neighbourhood with a compact closure.

Example 15.1. Our main examples will be as follows:

(i) $G = \mathbb{Z}$ the group of integers with operation of addition and the discrete topology (each point is an open set).
(ii) $G = \mathbb{R}$ the group of real numbers with addition and the topology defined by open intervals.
(iii) $G = \mathbb{T}$ the group of Euclidean rotations the unit circle in $\mathbb{R}^2$ with the natural topology. Another realisations of the same group:
  - Unimodular complex numbers under multiplication.
  - Factor group $\mathbb{R}/\mathbb{Z}$, that is addition of real numbers modulo 1.

  There is a homomorphism between two realisations given by $z = e^{2\pi i t}$, $t \in [0, 1)$, $|z| = 1$.

We assume that G has a regular Borel measure which is invariant in the following sense.

Definition 15.2. Let μ be a measure on a commutative group G, μ is called *invariant* (or *Haar measure*) if for any measurable X and any $g \in G$ the sets $g + X$ and $-X$ are also measurable and $\mu(X) = \mu(g + X) = \mu(-X)$.

Such an invariant measure exists if and only if the group is locally compact, in this case the measure is uniquely defined up to the constant factor.

Exercise 15.3. Check that in the above three cases invariant measures are:

- $G = \mathbb{Z}$, the invariant measure of X is equal to number of elements in X.
- $G = \mathbb{R}$ the invariant measure is the Lebesgue measure.
- $G = \mathbb{T}$ the invariant measure coincides with the Lebesgue measure.

Definition 15.4. A *convolution* of two functions on a commutative group G with an invariant measure μ is defined by:

$$(f_1 * f_2)(x) = \int_G f_1(x - y)\, f_2(y)\, d\mu(y) = \int_G f_1(y)\, f_2(x - y)\, d\mu(y). \tag{15.1}$$

Theorem 15.5. *If f_1, $f_2 \in L_1(G, \mu)$, then the integrals in (15.1) exist for almost every $x \in G$, the function $f_1 * f_2$ is in $L_1(G, \mu)$ and $\|f_1 * f_2\| \leqslant \|f_1\| \cdot \|f_2\|$.*

Proof. If f_1, $f_2 \in L_1(G, \mu)$ then by Fubini's Thm. 13.50 the function $\phi(x, y) = f_1(x) * f_2(y)$ is in $L_1(G \times G, \mu \times \mu)$ and $\|\phi\| = \|f_1\| \cdot \|f_2\|$.
Let us define a map $\tau : G \times G \to G \times G$ such that $\tau(x, y) = (x + y, y)$. It is measurable (send Borel sets to Borel sets) and preserves the measure $\mu \times \mu$. Indeed, for an elementary set $C = A \times B \subset G \times G$ we have:

$$\begin{aligned}
(\mu \times \mu)(\tau(C)) &= \int_{G\times G} \chi_{\tau(C)}(x, y)\, d\mu(x)\, d\mu(y)\\
&= \int_{G\times G} \chi_C(x - y, y)\, d\mu(x)\, d\mu(y)\\
&= \int_G \left(\int_G \chi_C(x - y, y)\, d\mu(x)\right) d\mu(y)\\
&= \int_B \mu(A + y)\, d\mu(y) = \mu(A) \times \mu(B) = (\mu \times \mu)(C).
\end{aligned}$$

We used invariance of μ and Fubini's Thm. 13.50. Therefore we have an isometric isomorphism of $L_1(G \times G, \mu \times \mu)$ into itself by the formula:

$$T\phi(x, y) = \phi(\tau(x, y)) = \phi(x - y, y).$$

If we apply this isomorphism to the above function $\phi(x, y) = f_1(x) * f_2(y)$ we shall obtain the statement. □

Definition 15.6. Denote by $S(k)$ the map $S(k) : f \mapsto k * f$ which we will call *convolution operator* with the *kernel* k.

Corollary 15.7. *If $k \in L_1(G)$ then the convolution $S(k)$ is a bounded linear operator on $L_1(G)$.*

Theorem 15.8. *Convolution is a commutative, associative and distributive operation. In particular* $S(f_1)S(f_2) = S(f_2)S(f_1) = S(f_1 * f_2)$.

Proof. Direct calculation using change of variables. □

It follows from Thm. 15.5 that convolution is a closed operation on $L_1(G)$ and has nice properties due to Thm. 15.8. We fix this in the following definition.

Definition 15.9. $L_1(G)$ equipped with the operation of convolution is called *convolution algebra* $L_1(G)$.

The following operators of special interest.

Definition 15.10. An operator of *shift* $T(a)$ acts on functions by $T(a) : f(x) \mapsto f(x+a)$.

Lemma 15.11. *An operator of shift is an isometry of* $L_p(G)$, $1 \leqslant p \leqslant \infty$.

Theorem 15.12. *Operators of shifts and convolutions commute:*

$$T(a)(f_1 * f_2) = T(a)f_1 * f_2 = f_1 * T(a)f_2,$$

or

$$T(a)S(f) = S(f)T(a) = S(T(a)f).$$

Proof. Just another calculation with a change of variables. □

Remark 15.13. Note that operator of shifts $T(a)$ provide a *representation* of the group G by linear isometric operators in $L_p(G)$, $1 \leqslant p \leqslant \infty$. A map $f \mapsto S(f)$ is a *representation* of the convolution algebra

There is a useful relation between support of functions and their convolutions.

Lemma 15.14. *For any* $f_1, f_2 \in L_1(G)$ *we have:*

$$\operatorname{supp}(f_1 * f_2) \subset \operatorname{supp}(f_1) + \operatorname{supp}(f_2).$$

Proof. If $x \notin \operatorname{supp}(f_1) + \operatorname{supp}(f_2)$ then for any $y \in \operatorname{supp}(f_2)$ we have $x - y \notin \operatorname{supp}(f_1)$. Thus for such x convolution is the integral of the identical zero. □

Exercise 15.15. Suppose that the function f_1 is compactly supported and k times continuously differentiate in $\mathbb{R}$, and that the function f_2 belongs to $L_1(\mathbb{R})$. Prove that the convolution $f_1 * f_2$ has continuous derivatives up to order k.
[*Hint:* Express the derivative $\frac{d}{dx}$ as the limit of operators $(T(h) - I)/h$ when $h \to 0$ and use Thm. 15.12.]

15.2. **Characters of Commutative Groups.** Our purpose is to map the commutative algebra of convolutions to a commutative algebra of functions with point-wise multiplication. To this end we first represent elements of the group as operators of multiplication.

Definition 15.16. A *character* $\chi : G \to \mathbb{T}$ is a continuous homomorphism of an abelian topological group G to the group $\mathbb{T}$ of unimodular complex numbers under multiplications:

$$\chi(x + y) = \chi(x)\chi(y).$$

Note, that a character is an eigenfunction for a shift operator $T(a)$ with the eigenvalue $\chi(a)$. Furthermore, if a function f on G is an eigenfunction for all shift operators $T(a)$, $a \in G$ then the collection of respective eigenvalues $\lambda(a)$ is a homomorphism of G to $\mathbb{C}$ and $f(a) = \alpha\lambda(a)$ for some $\alpha \in \mathbb{C}$. Moreover, if $T(a)$ act by isometries on the space containing $f(a)$ then $\lambda(a)$ is a homomorphism to $\mathbb{T}$.

Lemma 15.17. *The product of two characters of a group is again a character of the group. If χ is a character of G then $\chi^{-1} = \bar{\chi}$ is a character as well.*

Proof. Let χ_1 and χ_2 be characters of G. Then:

$$\begin{aligned} \chi_1(gh)\chi_2(gh) &= \chi_1(g)\chi_1(h)\chi_2(g)\chi_2(h) \\ &= (\chi_1(g)\chi_2(g))(\chi_1(h)\chi_2(h)) \in \mathbb{T}. \end{aligned}$$

□

Definition 15.18. The *dual group* $\hat{G}$ is collection of all characters of G with operation of multiplication.

The dual group becomes a topological group with the *uniform convergence on compacts*: for any compact subset $K \subset G$ and any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $|\chi_n(x) - \chi(x)| < \varepsilon$ for all $x \in K$ and $n > N$.

Exercise 15.19. Check that

(i) The sequence $f_n(x) = x^n$ does not converge uniformly on compacts if considered on $[0, 1]$. However it does converges uniformly on compacts if considered on $(0, 1)$.

(ii) If X is a compact set then the topology of uniform convergence on compacts and the topology uniform convergence on X coincide.

Example 15.20. If $G = \mathbb{Z}$ then any character χ is defined by its values $\chi(1)$ since

$$\chi(n) = [\chi(1)]^n. \tag{15.2}$$

Since $\chi(1)$ can be any number on $\mathbb{T}$ we see that $\hat{\mathbb{Z}}$ is parametrised by $\mathbb{T}$.

Theorem 15.21. *The group $\hat{\mathbb{Z}}$ is isomorphic to $\mathbb{T}$.*

Proof. The correspondence from the above example is a group homomorphism. Indeed if χ_z is the character with $\chi_z(1) = z$, then $\chi_{z_1}\chi_{z_2} = \chi_{z_1 z_2}$. Since $\mathbb{Z}$ is discrete, every compact consists of a finite number of points, thus uniform convergence on compacts means point-wise convergence. The equation (15.2) shows that $\chi_{z_n} \to \chi_z$ if and only if $\chi_{z_n}(1) \to \chi_z(1)$, that is $z_n \to z$. □

Theorem 15.22. *The group $\hat{\mathbb{T}}$ is isomorphic to $\mathbb{Z}$.*

Proof. For every $n \in \mathbb{Z}$ define a character of $\mathbb{T}$ by the identity

$$\chi_n(z) = z^n, \qquad z \in \mathbb{T}. \tag{15.3}$$

We will show that these are the only characters in Cor. 15.26. The isomorphism property is easy to establish. The topological isomorphism follows from discreteness of $\hat{\mathbb{T}}$. Indeed due to compactness of $\mathbb{T}$ for $n \neq m$:

$$\max_{z\in\mathbb{T}} |\chi_n(z) - \chi_m(z)|^2 = \max_{z\in\mathbb{T}} \left|1 - \Re z^{m-n}\right|^2 = 2^2 = 4.$$

Thus, any convergent sequence (n_k) have to be constant for sufficiently large k, that corresponds to a discrete topology on $\mathbb{Z}$. □

The two last Theorem are an illustration to the following general statement.

Principle 15.23 (Pontryagin's duality). For any locally compact commutative topological group G the natural map $G \to \hat{\hat{G}}$, such that it maps $g \in G$ to a character f_g on $\hat{G}$ by the formula:

$$f_g(\chi) = \chi(g), \qquad \chi \in \hat{G}, \tag{15.4}$$

is an isomorphism of topological groups.

Remark 15.24. (i) The principle is not true for commutative group which are not locally compact.
(ii) Note the similarity with an embedding of a vector space into the second dual.

In particular, the Pontryagin's duality tells that the collection of all characters contains enough information to rebuild the initial group.

Theorem 15.25. *The group $\hat{\mathbb{R}}$ is isomorphic to $\mathbb{R}$.*

Proof. For $\lambda \in \mathbb{R}$ define a character $\chi_\lambda \in \hat{\mathbb{R}}$ by the identity

$$\chi_\lambda(x) = e^{2\pi i \lambda x}, \qquad x \in \mathbb{R}. \tag{15.5}$$

Moreover any smooth character of the group $G = (\mathbb{R}, +)$ has the form (15.5). Indeed, let χ be a smooth character of $\mathbb{R}$. Put $c = \chi'(t)|_{t=0} \in \mathbb{C}$. Then $\chi'(t) = c\chi(t)$ and $\chi(t) = e^{ct}$. We also get $c \in i\mathbb{R}$ and any such c defines a character. Then the multiplication of characters is: $\chi_1(t)\chi_2(t) = e^{c_1 t}e^{c_2 t} = e^{(c_2+c_1)t}$. So we have a group isomorphism.
For a generic character we can apply first the *smoothing technique* and reduce to the above case.
Let us show topological homeomorphism. If $\lambda_n \to \lambda$ then $\chi_{\lambda_n} \to \chi_\lambda$ uniformly on any compact in $\mathbb{R}$ from the explicit formula of the character. Reverse, let $\chi_{\lambda_n} \to \chi_\lambda$ uniformly on any interval. Then $\chi_{\lambda_n - \lambda}(x) \to 1$ uniformly on any compact, in particular, on $[0, 1]$. But

$$\begin{aligned}\sup_{[0,1]} |\chi_{\lambda_n-\lambda}(x) - 1| &= \sup_{[0,1]} |\sin \pi(\lambda_n - \lambda)x| \\ &= \begin{cases} 1, & \text{if } |\lambda_n - \lambda| \geqslant 1/2, \\ \sin \pi |\lambda_n - \lambda|, & \text{if } |\lambda_n - \lambda| \leqslant 1/2. \end{cases}\end{aligned}$$

Thus $\lambda_n \to \lambda$. □

Corollary 15.26. *Any character of the group $\mathbb{T}$ has the form* (15.3).

Proof. Let $\chi \in \hat{\mathbb{T}}$, consider $\chi_1(t) = \chi(e^{2\pi i t})$ which is a character of $\mathbb{R}$. Thus $\chi_1(t) = e^{2\pi i \lambda t}$ for some $\lambda \in \mathbb{R}$. Since $\chi_1(1) = 1$ then $\lambda = n \in \mathbb{Z}$. Thus $\chi_1(t) = e^{2\pi i n t}$, that is $\chi(z) = z^n$ for $z = e^{2\pi i t}$. □

Remark 15.27. Although $\hat{\mathbb{R}}$ is isomorphic to $\mathbb{R}$ there is no a canonical form for this isomorphism (unlike for $\mathbb{R} \to \hat{\hat{\mathbb{R}}}$). Our choice is convenient for the Poisson formula below, however some other popular definitions are $\lambda \to e^{i\lambda x}$ or $\lambda \to e^{-i\lambda x}$.

We can unify the previous three Theorem into the following statement.

Theorem 15.28. *Let* $G = \mathbb{R}^n \times \mathbb{Z}^k \times \mathbb{T}^l$ *be the direct product of groups. Then the dual group is* $\hat{G} = \mathbb{R}^n \times \mathbb{T}^k \times \mathbb{Z}^l$.

15.3. **Fourier Transform on Commutative Groups.**

Definition 15.29. Let G be a locally compact commutative group with an invariant measure μ. For any $f \in L_1(G)$ define the *Fourier transform* $\hat{f}$ by

$$\hat{f}(\chi) = \int_G f(x)\,\bar{\chi}(x)\,d\mu(x), \qquad \chi \in \hat{G}. \tag{15.6}$$

That is the Fourier transform $\hat{f}$ is a function on the dual group $\hat{G}$.

Example 15.30. (i) If $G = \mathbb{Z}$, then $f \in L_1(Z)$ is a two-sided summable sequence $(c_n)_{n\in\mathbb{Z}}$. Its Fourier transform is the function $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ on $\mathbb{T}$. Sometimes $f(z)$ is called *generating function* of the sequence (c_n).
(ii) If $G = \mathbb{T}$, then the Fourier transform of $f \in L_1(\mathbb{T})$ is its *Fourier coefficients*, see Section 5.1.
(iii) If $G = \mathbb{R}$, the Fourier transform is also the function on $\mathbb{R}$ given by the *Fourier integral*:

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x)\,e^{-2\pi i\lambda x}\,dx. \tag{15.7}$$

The important properties of the Fourier transform are captured in the following statement.

Theorem 15.31. *Let G be a locally compact commutative group with an invariant measure μ. The Fourier transform maps functions from $L_1(G)$ to continuous bounded functions on $\hat{G}$. Moreover, a convolution is transformed to point-wise multiplication:*

$$(f_1 * f_2)\hat{}(\chi) = \hat{f}_1(\chi)\cdot\hat{f}_2(\chi), \tag{15.8}$$

a shift operator $T(a)$, $a \in G$ is transformed in multiplication by the character $f_a \in \hat{\hat{G}}$:

$$(T(a)f)\hat{}(\chi) = f_a(\chi)\cdot\hat{f}(\chi), \qquad f_a(\chi) = \chi(a) \tag{15.9}$$

and multiplication by a character $\chi \in \hat{G}$ is transformed to the shift $T(\chi^{-1})$:

$$(\chi \cdot f)\hat{}(\chi_1) = T(\chi^{-1})\hat{f}(\chi_1) = \hat{f}(\chi^{-1}\chi_1). \tag{15.10}$$

Proof. Let $f \in L_1(G)$. For any $\varepsilon > 0$ there is a compact $K \subset G$ such that $\int_{G\setminus K} |f|\, d\mu < \varepsilon$. If $\chi_n \to \chi$ in $\hat{G}$, then we have the uniform convergence of $\chi_n \to \chi$ on K, so there is $n(\varepsilon)$ such that for $k > n(\varepsilon)$ we have $|\chi_k(x) - \chi(x)| < \varepsilon$ for all $x \in K$. Then

$$\begin{aligned} \left|\hat{f}(\chi_n) - \hat{f}(\chi)\right| &\leqslant \int_K |f(x)|\,|\chi_n(x) - \chi(x)|\, d\mu(x) + \int_{G\setminus K} |f(x)|\,|\chi_n(x) - \chi(x)|\, d\mu(x) \\ &\leqslant \varepsilon \|f\| + 2\varepsilon. \end{aligned}$$

Thus $\hat{f}$ is continuous. Its boundedness follows from the integral estimations. Algebraic maps (15.8)–(15.10) can be obtained by changes of variables under integration. For example, using Fubini's Thm. 13.50 and invariance of the measure:

$$\begin{aligned} (f_1 * f_2)\hat{}(\chi) &= \int_G \int_G f_1(s)\, f_2(t-s)\, ds\, \bar{\chi}(t)\, dt \\ &= \int_G \int_G f_1(s)\, \chi(\bar{s})\, f_2(t-s)\, \bar{\chi}(t-s)\, ds\, dt \\ &= \hat{f}_1(\chi)\hat{f}_2(\chi). \end{aligned}$$

□

15.4. **The Schwartz space of smooth rapidly decreasing functions.** We say that a function f is *rapidly decreasing* if $\lim_{x\to\pm\infty} \left|x^k f(x)\right| = 0$ for any $k \in \mathbb{N}$.

Definition 15.32. The *Schwartz space* denoted by S or space of rapidly decreasing functions on Rn is the space of infinitely differentiable functions such that:

$$S = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \left|x^\alpha f^{(\beta)}(x)\right| < \infty \quad \forall \alpha, \beta \in \mathbb{N} \right\}. \tag{15.11}$$

Example 15.33. An example of a rapidly decreasing function is the Gaussian $e^{-\pi x^2}$.

It is worth to notice that $S \subset L_p(\mathbb{R})$ for any $1 < p < \infty$. Moreover, S is dense in $L_p(\mathbb{R})$, for $p = 1$ this can be shown in the following steps (other values of p can be done similarly but require some more care). First we will show that S is an ideal of the convolution algebra $L_1(\mathbb{R})$.

Exercise 15.34. For any $g \in S$ and $f \in L_1(\mathbb{R})$ with compact support their convolution $f * g$ belongs to S. [*Hint:* smoothness follows from Ex. 15.15.]

Define the family of functions $g_t(x)$ for $t > 0$ in S by scaling the Gaussian:

$$g_t(x) = \frac{1}{t} e^{-\pi(x/t)^2}.$$

Exercise 15.35. Show that $g_t(x)$ satisfies the following properties, cf. Lem 5.7:

(i) $g_t(x) > 0$ for all $x \in \mathbb{R}$ and $t > 0$.

(ii) $\int_{\mathbb{R}} g_t(x)\,dx = 1$ for all $t > 0$. [*Hint:* use the table integral $\int_{\mathbb{R}} e^{-\pi x^2}\,dx = 1$.]

(iii) For any $\varepsilon > 0$ and any $\delta > 0$ there exists $T > 0$ such that for all positive $t < T$ we have:

$$0 < \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} g_t(x)\,dx < \varepsilon.$$

It is easy to see, that the above properties 15.35(i)–15.35(iii) are not unique to the Gaussian and a wide class have them. Such a family a family of functions is known as *approximation of the identity* [6] due to the next property (15.12).

Exercise 15.36. (i) Let f be a continuous function with compact support, then

$$\lim_{t \to 0} \|f - g_t * f\|_1 = 0\,. \tag{15.12}$$

[*Hint:* use the proof of Thm. 5.8.]

(ii) The Schwartz space S is dense in $L_1(\mathbb{R})$. [*Hint:* use Prop. 14.20, Ex. 15.34 and (15.12).]

15.5. **Fourier Integral.** We recall the formula (15.7):

Definition 15.37. We define the *Fourier integral* of a function $f \in L_1(\mathbb{R})$ by

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i \lambda x}\,dx. \tag{15.13}$$

We already know that $\hat{f}$ is a bounded continuous function on $\mathbb{R}$, a further property is:

Lemma 15.38. *If a sequence of functions $(f_n) \subset L_1(\mathbb{R})$ converges in the metric $L_1(\mathbb{R})$, then the sequence $(\hat{f}_n)$ converges uniformly on the real line.*

Proof. This follows from the estimation:

$$\left|\hat{f}_n(\lambda) - \hat{f}_m(\lambda)\right| \leqslant \int_{\mathbb{R}} |f_n(x) - f_m(x)| \, dx.$$

□

Lemma 15.39. *The Fourier integral $\hat{f}$ of $f \in L_1(\mathbb{R})$ has zero limits at $-\infty$ and $+\infty$.*

Proof. Take f the indicator function of $[a, b]$. Then $\hat{f}(\lambda) = \frac{1}{-2\pi i\lambda}(e^{-2\pi i a} - e^{-2\pi i b})$, $\lambda \neq 0$. Thus $\lim_{\lambda\to\pm\infty} \hat{f}(\lambda) = 0$. By continuity from the previous Lemma this can be extended to the closure of step functions, which is the space $L_1(\mathbb{R})$ by Lem. 14.17. □

Lemma 15.40. *If f is absolutely continuous on every interval and $f' \in L_1(\mathbb{R})$, then*

$$(f')\hat{} = 2\pi i\lambda\hat{f}.$$

More generally:

$$(f^{(k)})\hat{} = (2\pi i\lambda)^k\hat{f}. \tag{15.14}$$

Proof. A direct demonstration is based on integration by parts, which is possible because assumption in the Lemma.

It may be also interesting to mention that the operation of differentiation D can be expressed through the shift operatot T_a:

$$D = \lim_{\Delta t\to 0} \frac{T_{\Delta t} - I}{\Delta t}. \tag{15.15}$$

By the formula (15.9), the Fourier integral transforms $\frac{1}{\Delta t}(T_{\Delta t} - I)$ into $\frac{1}{\Delta t}(\chi_\lambda(\Delta t) - 1)$. Providing we can justify that the Fourier integral commutes with the limit, the last operation is multiplication by $\chi'_\lambda(0) = 2\pi i\lambda$. □

Corollary 15.41. *If $f^{(k)} \in L_1(\mathbb{R})$ then*

$$\left|\hat{f}\right| = \frac{\left|(f^{(k)})\hat{}\right|}{|2\pi\lambda|^k} \to 0 \quad \text{as } \lambda \to \infty,$$

that is $\hat{f}$ decrease at infinity faster than $|\lambda|^{-k}$.

Lemma 15.42. *Let* $f(x)$ *and* $xf(x)$ *are both in* $L_1(\mathbb{R})$*, then* $\hat{f}$ *is differentiable and*

$$\hat{f}' = (-2\pi i x f)\hat{}.$$

More generally

$$\hat{f}^{(k)} = ((-2\pi i x)^k f)\hat{}. \tag{15.16}$$

Proof. There are several strategies to prove this results, all having their own merits:

(i) The most straightforward uses the differentiation under the integration sign.
(ii) We can use the intertwining property (15.10) of the Fourier integral and the connection of derivative with shifts (15.15).
(iii) Using the inverse Fourier integral (see below), we regard this Lemma as the dual to the Lemma 15.40.

□

Corollary 15.43. *The Fourier transform of a smooth rapidly decreasing function is a smooth rapidly decreasing function.*

Corollary 15.44. *The Fourier integral of the Gaussian* $e^{-\pi x^2}$ *is* $e^{-\pi\lambda^2}$.

Proof. [2] Note that the Gaussian $g(x) = e^{-\pi x^2}$ is a unique (up to a factor) solution of the equation $g' + 2\pi x g = 0$. Then, by Lemmas 15.40 and 15.42, its Fourier transform shall satisfy to the equation $2\pi i \lambda \hat{g} + i\hat{g}' = 0$. Thus, $\hat{g} = c \cdot e^{-\pi\lambda^2}$ with a constant factor c, its value 1 can be found from the classical integral $\int_{\mathbb{R}} e^{-\pi x^2}\,dx = 1$ which represents $\hat{g}(0)$. □

The relation (15.14) and (15.16) allows to reduce many partial differential equations to algebraic one, see § 0.2 and 5.4. To convert solutions of algebraic equations into required differential equations we need the inverse of the Fourier transform.

Definition 15.45. We define the *inverse Fourier transform* on $L_1(\mathbb{R})$:

$$\check{f}(\lambda) = \int_{\mathbb{R}} f(x)\, e^{2\pi i \lambda x}\,dx. \tag{15.17}$$

We can notice the formal correspondence $\check{f}(\lambda) = \hat{f}(-\lambda) = \overline{\hat{f}(\lambda)}$, which is a manifestation of the group duality $\hat{\mathbb{R}} = \mathbb{R}$ for the real line. This immediately generates analogous results from Lem. 15.38 to Cor. 15.44 for the inverse Fourier transform.

Theorem 15.46. *The Fourier integral and the inverse Fourier transform are inverse maps. That is, if $g = \hat{f}$ then $f = \check{g}$.*

Sketch of a proof. The exact meaning of the statement depends from the spaces which we consider as the domain and the range. Various variants and their proofs can be found in the literature. For example, in [3, § IV.2.3], it is proven for the Schwartz space $\mathcal{S}$ of smooth rapidly decreasing functions.

The outline of the proof is as follows. Using the intertwining relations (15.14) and (15.16), we conclude the composition of Fourier integral and the inverse Fourier transform commutes both with operator of multiplication by x and differentiation. Then we need a result, that any operator commuting with multiplication by x is an operator of multiplication by a function f. For this function, the commutation with differentiation implies $f' = 0$, that is $f = \mathrm{const}$. The value of this constant can be evaluated by a Fourier transform on a single function, say the Gaussian $e^{-\pi x^2}$ from Cor. 15.44. □

The above Theorem states that the Fourier integral is an invertible map. For the Hilbert space $L_2(\mathbb{R})$ we can show a stronger property—its unitarity.

Theorem 15.47 (Plancherel identity)**.** *The Fourier transform extends uniquely to a unitary map $L_2(\mathbb{R}) \to L_2(\mathbb{R})$:*

$$\int_{\mathbb{R}} |f|^2 \, dx = \int_{\mathbb{R}} \left|\hat{f}\right|^2 \, d\lambda. \tag{15.18}$$

Proof. The proof will be done in three steps: first we establish the identity for smooth rapidly decreasing functions, then for L_2 functions with compact support and finally for any L_2 function.

(i) Take f_1 and $f_2 \in S$ be smooth rapidly decreasing functions and g_1 and g_2 be their Fourier transform. Then (using Fubini's Thm. 13.50):

$$\begin{aligned}\int_{\mathbb{R}} f_1(t)\bar{f}_2(t)\,dt &= \int_{\mathbb{R}}\int_{\mathbb{R}} g_1(\lambda)\, e^{2\pi i\lambda t}\,d\lambda\, \bar{f}_2(t)\,dt \\ &= \int_{\mathbb{R}} g_1(\lambda) \int_{\mathbb{R}} e^{2\pi i\lambda t}\, \bar{f}_2(t)\,dt\,d\lambda \\ &= \int_{\mathbb{R}} g_1(\lambda)\,\bar{g}_2(\lambda)\,d\lambda\end{aligned}$$

Put $f_1 = f_2 = f$ (and therefore $g_1 = g_2 = \hat{f}$) we get the identity $\int |f|^2\,dx = \int \left|\hat{f}\right|^2\,d\lambda$.

The same identity (15.18) can be obtained from the property $(f_1 f_2)\hat{} = \hat{f}_1 * \hat{f}_2$, cf. (15.8), or explicitly:

$$\int_{\mathbb{R}} f_1(x)\, f_2(x)\, e^{-2\pi i\lambda x}\,dx = \int_{\mathbb{R}} \hat{f}_1(t)\,\hat{f}_2(\lambda - t)\,dt.$$

Now, substitute $\lambda = 0$ and $f_2 = \bar{f}_1$ (with its corollary $\hat{f}_2(t) = \bar{\hat{f}}_1(-t)$) and obtain (15.18).

(ii) Next let $f \in L_2(\mathbb{R})$ with a support in $(-a, a)$ then $f \in L_1(\mathbb{R})$ as well, thus the Fourier transform is well-defined. Let $f_n \in S$ be a sequence with support on $(-a, a)$ which converges to f in L_2 and thus in L_1. The Fourier transform g_n converges to g uniformly and is a Cauchy sequence in L_2 due to the above identity. Thus $g_n \to g$ in L_2 and we can extend the Plancherel identity by continuity to L_2 functions with compact support.

(iii) The final bit is done for a general $f \in L_2$ the sequence

$$f_n(x) = \begin{cases} f(x), & \text{if } |x| < n, \\ 0, & \text{otherwise;}\end{cases}$$

of truncations to the interval $(-n, n)$. For f_n the Plancherel identity is established above, and $f_n \to f$ in $L_2(\mathbb{R})$. We also build their Fourier images g_n and see that this is a Cauchy sequence in $L_2(\mathbb{R})$, so $g_n \to g$.

If $f \in L_1 \cap L_2$ then the above g coincides with the ordinary Fourier transform on L_1. □

We note that Plancherel identity and the Parseval's identity (5.7) are cousins—they both states that the Fourier transform $L_2(G) \to L_2(\hat{G})$ is an isometry for $G = \mathbb{R}$ and $G = \mathbb{T}$ respectively. They may be combined to state the unitarity of the Fourier transform on $L_2(G)$ for the group $G = \mathbb{R}^n \times \mathbb{Z}^k \times \mathbb{T}^l$ cf. Thm. 15.28.

Proofs of the following statements are not examinable Thms. 12.23, 12.36, 13.53, 14.33, 15.46, Props. 14.14, 14.20.

16. Advances of Metric Spaces

16.1. **Contraction mappings and fixed point theorems.**

16.1.1. *The Banach fixed point theorem.* An important tool in numerical Analysis, but also in constructions of solutions of differential equations are fixed point approximations. In order to understand this, suppose that (X, d) is a metric space and $f : X \to X$ a self-map. Then a point $x \in X$ is called *fixed point* of f if $f(x) = x$. For example the function $\cos$ defines a self-map on the interval $[0, 1]$, and by starting with $x_1 = 0$ and inductively computing $x_{n+1} = \cos x_n$ one converges to the value roughly 0.739085 which is a fixed point of $\cos$, i.e. solves the equation $\cos(x) = x$. Under certain conditions one can show that such sequences always converge to a fixed point. This is the statement of the Banach fixed point theorem (contraction mapping principle).

Definition 16.1 (Contraction Mapping)**.** Let (X, d) be a metric space. Then a map $f : X \to X$ is called *contraction* if there exists a constant $C < 1$ such that

$$d(f(x), f(y)) \leqslant C d(x, y).$$

Note that any contraction is (uniformly) continuous.

Theorem 16.2 (Banach Fixed Point Theorem)**.** *Suppose that $f : X \to X$ is a contraction on a complete metric space (X, d). Then f has a unique fixed point y. Moreover, for any $x \in X$ the sequence (x_n) defined recursively by*

$$x_{n+1} = f(x_n), \quad x_1 = x,$$

converges to y.

Proof. Let us start with uniqueness. If x, y are both fixed points in X, then since f is a contraction:

$$d(x, y) \leqslant C d(x, y)$$

for some constant $C < 1$. Hence, $d(x, y) = 0$ and therefore $x = y$.
To prove the remaining claims we start with any x in X and we will show that the sequence x_n defined by $x_1 = x$ and $x_{n+1} = f(x_n)$ converges. Since f is continuous the limit of (x_n) must be a fixed point. Since (X, d) is complete we only need to show that (x_n) is Cauchy. To see this note that

$$d(x_{n+1}, x_n) \leqslant C d(x_n, x_{n-1})$$

and therefore inductively,

$$d(x_{n+1}, x_n) \leqslant C^{n-1} d(x_2, x_1).$$

By the triangle inequality we have for any $n, m > 0$

$$d(x_{N+m}, x_N) \leqslant (C^{N-1} + C^N + \ldots C^{N+m-2}) d(x_2, x_1) \leqslant C^{N-1} \frac{1}{1-C} d(x_2, x_1).$$

Since $C < 1$ this can be made arbitrarily small by choosing N large enough. □

Corollary 16.3. *Suppose that (X, d) is a complete metric space and $f : X \to X$ a map such that f^n is a contraction for some $n \in \mathbb{N}$. Then f has a unique fixed point.*

Proof. Since f^n is a contraction it has a unique fixed point $x \in X$, i.e.

$$\underbrace{f \circ f \ldots \circ f}_{n-\text{times}}(x) = x.$$

Now note that

$$f^n(f(x)) = f^n \circ f(x) = f^{n+1}(x) = f \circ f^n(x) = f(f^n(x)) = f(x)$$

and therefore $f(x)$ is also a fixed point of f^n. By uniqueness we must have $f(x) = x$. □

The question arises how to show that a given map f is a contraction. In subsets of $\mathbb{R}^m$ there is a simple criterion. Recall that an open set $\mathcal{U} \subset \mathbb{R}$ is called *convex* if for any two points $x, y \in \mathcal{U}$ the line $\{tx + (1-t)y \mid t \in [0,1]\}$ is contained in $\mathcal{U}$.

Theorem 16.4 (Mean Value Inequality)**.** *Suppose that $\mathcal{U} \subset \mathbb{R}^m$ is an open set with convex closure $\overline{\mathcal{U}}$ and let $f : \overline{\mathcal{U}} \to \mathbb{R}^m$ be a C^1-function. Let df be the total derivative (or Jacobian) understood as a function on $\overline{\mathcal{U}}$ with values in $m \times m$-matrices. Suppose that $\|df(x)\| \leqslant M$ for all $x \in \overline{\mathcal{U}}$. Then $f : \overline{\mathcal{U}} \to \mathbb{R}^m$ satisfies*

$$\|f(x) - f(y)\| \leqslant M\|x - y\|$$

for all $x, y \in \overline{\mathcal{U}}$.

Proof. Given $x, y \in \mathcal{U}$ let $\gamma(t) = tx + (1-t)y$. Then $\frac{d}{dt}\gamma(t) = x - y$.

$$f(x) - f(y) = \int_0^1 \frac{d}{dt} f(\gamma(t))\, dt = \int_0^1 (df) \cdot \frac{d\gamma}{dt}(t)\, dt.$$

Using the triangle inequality (this can be used for Riemann integrals too because these are limits of finite sums), one gets

$$\|f(x) - f(y)\| \leqslant \int_0^1 \|(df) \cdot \frac{d\gamma}{dt}(t)\| dt \leqslant M \int_0^1 \|x - y\| dt = M\|x - y\|.$$

By continuity this inequality extends to $\overline{\mathcal{U}}$. □

Example 16.5. Consider the map $f : \mathbb{R}^2 \supset \overline{B_1(0)} \to \overline{B_1(0)}$, $(x, y) \mapsto (\frac{x^2}{4} + \frac{y}{3} + \frac{1}{3}, \frac{y^2}{4} - \frac{x}{2})$. Then

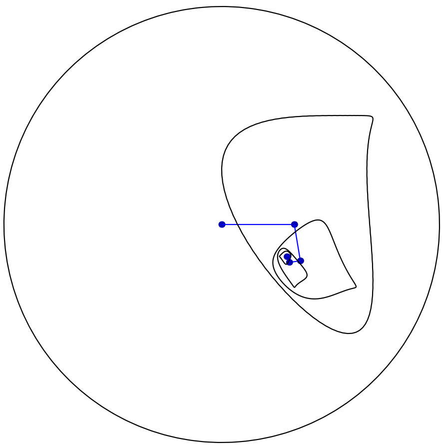
$$df = \begin{pmatrix} \frac{x}{2} & \frac{1}{3} \\ -\frac{1}{2} & \frac{y}{2} \end{pmatrix}.$$

The operator norm $\|df\|$ can be estimated by the Hilbert–Schmidt norm. Recall $\|A\|_{HS} = (\mathrm{tr}(A^*A))^{\frac{1}{2}}$, so we get

$$\|df\| \leqslant \|df\|_{HS} = (\frac{1}{4}(x^2 + y^2) + \frac{1}{4} + \frac{1}{9})^{1/2} < 1.$$

Therefore f is a contraction. We can find the fixed point by starting, for example, with the point $(0, 0)$ and iterating. We get iterations:

$$\begin{gathered}(0,0), (0.333333, 0.), (0.361111, -0.166667),\\ (0.310378, -0.173611), (0.299547, -0.147654),\\ (0.306547, -0.144323), (0.308719, -0.148066),\\ (0.307805, -0.148878), (0.307393, -0.148361),\\ (0.307502, -0.148194), (0.307575, -0.148261),\\ (0.307564, -0.148292), (0.307551, -0.148284),\\ (0.307552, -0.148279), (0.307554, -0.148279).\end{gathered}$$



Example 16.6. Put a map of the country of your current presence on the floor, there's a point on the map that is touching the actual point it refers to!

16.1.2. *Applications of fixed point theory: The Picard-Lindelöf Theorem.* Let $f : K \to \mathbb{R}$ be a function on a compact rectangle of the form $K = [T_1, T_2] \times [L_1, L_2]$ in $\mathbb{R}^2$. Consider the initial value problem (IVP)

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \tag{16.1}$$

where $y : [T_1, T_2] \to \mathbb{R}, t \mapsto y(t)$ is a function. The function f and the initial value $y_0 \in [L_1, L_2]$, and $t_0 \in [T_1, T_2]$ are given and we are looking for a function y satisfying the above equations.

Example 16.7. Let $f(t, x) = x$ and $y_0 = 1$, $t_0 = 0$. Then the initial value problem is

$$\frac{dy}{dt} = y, \quad y(0) = 1.$$

We know from other courses that there is a unique solution $y(t) = e^t$, see Fig. 18 top-left.

Example 16.8. Let $f(t, x) = x^2$ and $y_0 = 1$, $t_0 = 0$. Then the initial value problem is

$$\frac{dy}{dt} = y^2, \quad y(0) = 1.$$

We know from other courses that there is a unique solution $y(t) = \frac{1}{1-t}$ which exists only on the interval $(-\infty, 1)$, see Fig. 18 top-right.

Example 16.9. Let $f(t, x) = x^2 - t$ and $y_0 = 1$, $t_0 = 0$. Then the initial value problem is

$$\frac{dy}{dt} = y^2 - t, \quad y(0) = 1.$$

One can show that there exists a solution for small $|t|$, however this solution cannot be expressed in terms of elementary functions, see Fig. 18 bottom-left.

Example 16.10. Let $f(t, x) = x^{2/3}$ and $y_0 = 0$, $t_0 = 0$. Then the initial value problem is

$$\frac{dy}{dt} = y^{\frac{2}{3}}, \quad y(0) = 0.$$

It has at least two solutions, namely $y = 0$ and $y = \frac{t^3}{27}$, see Fig. 18 bottom-right.

Hence, there are two fundamental questions here: existence and uniqueness of solutions. The following theorem is one of the basic results in the theorem of ordinary differential equation and establishes existence and uniqueness under rather general assumptions.

Theorem 16.11 (Picard–Lindelöf theorem)**.** *Suppose that* $f : [T_1, T_2] \times [y_0 - C, y_0 + C] \to \mathbb{R}$ *is a continuous function such that for some* $M > 0$ *we have*

$$|f(t, y_1) - f(t, y_2)| \leqslant M|y_1 - y_2| \quad \text{(Lipschitz condition)}$$

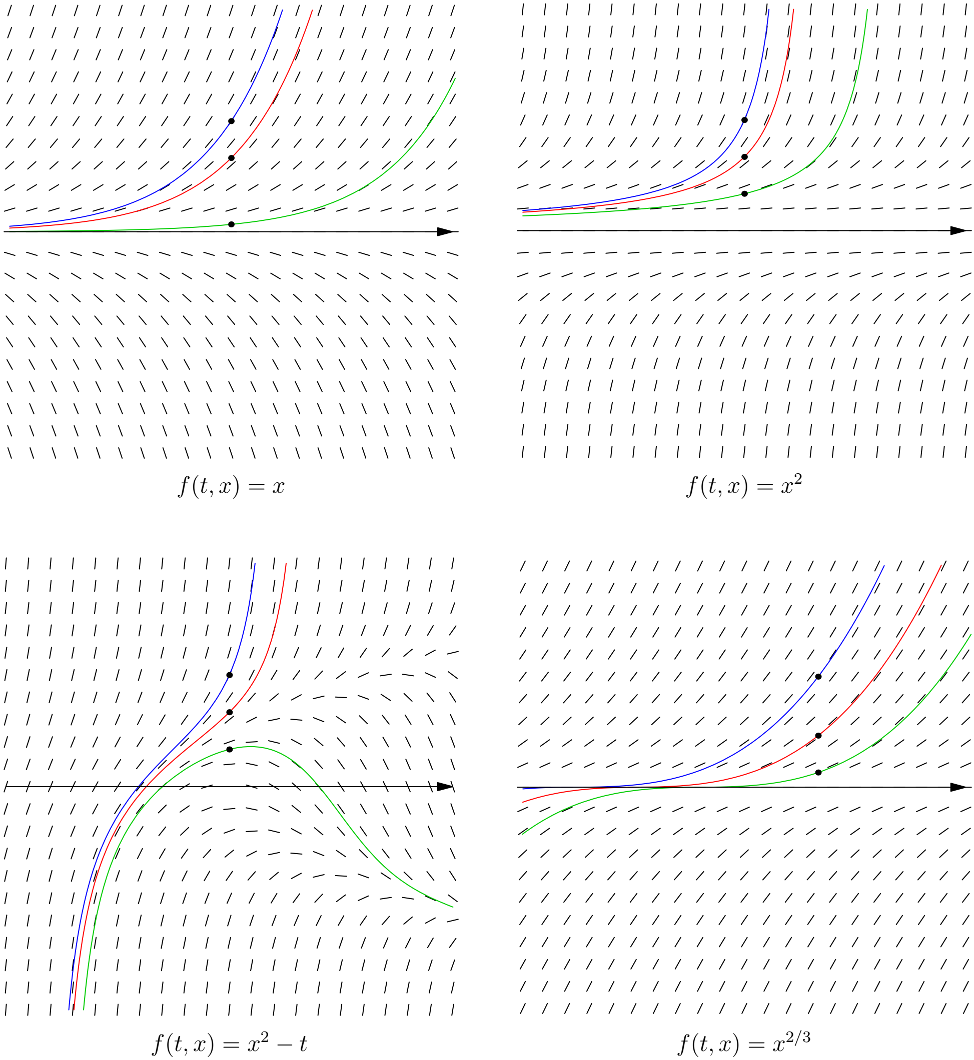


FIGURE 18. Vector fields and their integral curves from Ex. 16.7–16.10.

for all $t \in [T_1, T_2]$, $y_1, y_2 \in [y_0 - C, y_0 + C]$. *Then, for any* $t_0 \in [T_1, T_2]$ *the initial value problem*

$$\frac{dy}{dt}(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

has a unique solution y *in* $C^1[a,b]$*, where* $[a,b]$ *is the interval* $[t_0 - R, t_0 + R] \cap [T_1, T_2]$*, where*

$$R = \|f\|_\infty^{-1} C.$$

(The solution exists for all times t *such that* $|t - t_0| \leqslant R$*).*

Remark 16.12. Note, that the Lipschitz condition implies uniform continuity and is significantly stronger requirement.

Proof. Using the fundamental theorem of calculus we can write the IVP as a fixed point equation $F(y) = y$ for a map defined by

$$F(y)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds.$$

This is a map that will send a continuous function $y \in C[T_1, T_2]$ to a continuous function $F(y) \in C[T_1, T_2]$. As a metric space we take

$$X = C([a,b], [y_0 - C, y_0 + C])$$

that is, the set of continuous functions on $[a,b]$ taking values in the interval $[y_0 - C, y_0 + C]$. This is a closed (why?) subset of the Banach space $C[a,b]$ and is therefore a complete metric space.
First we show that $F : X \to X$, i.e. F maps X to itself. Indeed,

$$|F(y)(t) - y_0| = \left| \int_{t_0}^{t} f(s, y(s))\,ds \right| \leqslant R\|f\|_\infty \leqslant C.$$

Next we show that F^N is a contraction for N large enough and thus establish the existence of a unique fixed point. It is the place to use the Lipschitz condition. Observe that for two functions $y, \tilde{y} \in X$ we have

$$\begin{aligned} |F(y)(t) - F(\tilde{y})(t)| &= \left| \int_{t_0}^{t} f(s, y(s)) - f(s, \tilde{y}(s))\,ds \right| \\ &\leqslant \int_{t_0}^{t} |f(s, y(s)) - f(s, \tilde{y}(s))|\,ds \leqslant |t - t_0| M \|y - \tilde{y}\|_\infty. \end{aligned} \tag{16.2}$$

We did not assume that $(t - t_0)M \leqslant RM < 1$, so F will in general not be a contraction. There are several ways to resolve this situations. For example, we can argue in either of the following two manners:

(i) We use both the result and the method from (16.2) to compute distances for higher powers of F, starting from the squares:

$$\begin{aligned} |F^2(y)(t) - F^2(\tilde{y})(t)| &\leqslant \int_{t_0}^{t} |f(s, F(y)(s)) - f(s, F(\tilde{y})(s))| \, ds \\ &\leqslant \int_{t_0}^{t} |s - t_0| \cdot M \cdot \|F(y) - F(\tilde{y})\|_\infty \, ds \\ &\leqslant \int_{t_0}^{t} |s - t_0| \cdot M^2 \cdot \|y - \tilde{y}\|_\infty \, ds \\ &= \frac{|t - t_0|^2}{2} M^2 \|y - \tilde{y}\|_\infty, \end{aligned}$$

and iterating this gives for any natural N:

$$\|F^N(y) - F^N(\tilde{y})\|_\infty \leqslant \frac{|t - t_0|^N}{N!} M^N \|y - \tilde{y}\|_\infty.$$

Since the factorial will overgrow the respective power, for N large enough, F^N is a contraction and we deduce the existence of a unique solution from Cor. 16.3. This solution is in C^1 since it can be written as the integral of a continuous function.

(ii) The inequality (16.2) shows existence and uniqueness of solution only in the space of functions $C([t_0 - r, t_0 + r], [y_0 - C, y_0 + C])$ where $r < M^{-1}$ and therefore $|t - t_0|M < 1$ in (16.2). Now suppose we have two solutions y and $\tilde{y}$. They coincide at t_0. Application of (16.2) to other initial points where the solutions coincide shows that the set $E = \{x \in [a, b] \mid y(x) = \tilde{y}(x)\}$ is open. It is also the pre-image of the closed set $\{0\}$ under the continuous map $y - \tilde{y}$. So we have that E is a closed and open subset of $[a, b]$ that is non-empty. It must therefore be $[a, b]$. Hence, we get $y = \tilde{y}$, establishing uniqueness in the whole $C[a, b]$.

□

Note that this not only gives uniqueness and existence, but also gives a constructive method to compute the solution by iterating the map F starting for example with the constant function $y(t) = y_0$. The iteration

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s)) ds$$

is called *Picard iteration*. It will converge to the solution uniformly. See Fig. 19 for an illustration of few first iterations for the exponent functions.

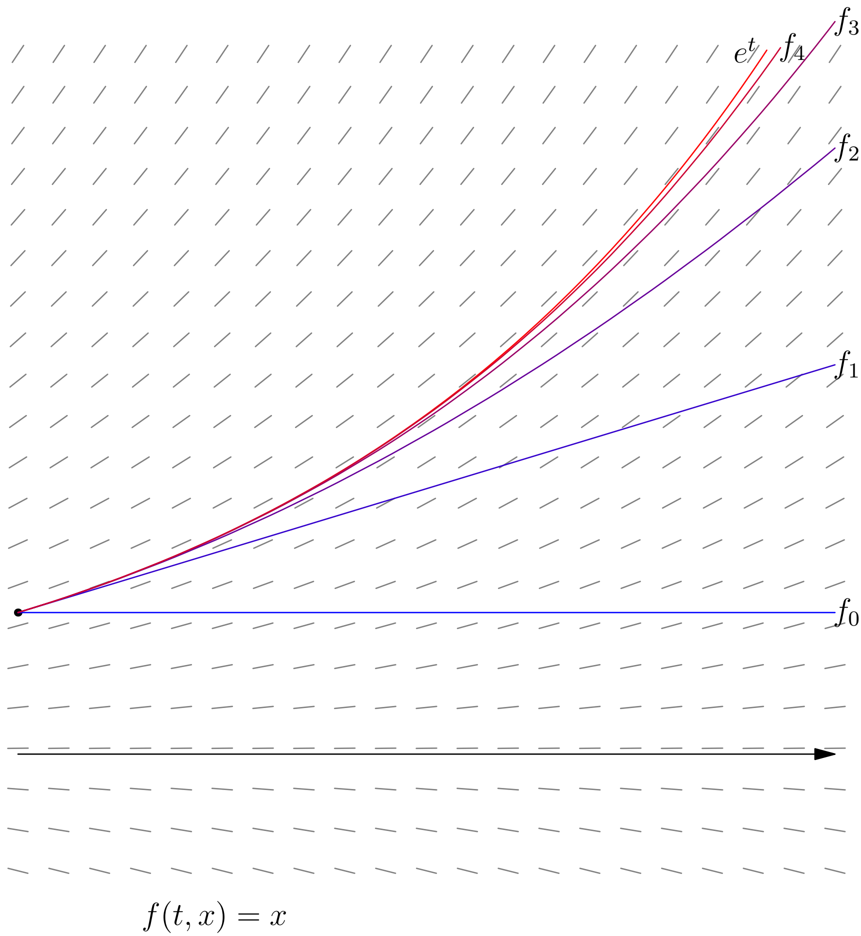


FIGURE 19. Few initial Picard iterations for the differential equation $y' = y$: constant f_0, linear f_1, quadratic f_2, etc.

Remark 16.13. The proof also gives a bound on the solution, namely if the assumptions are satisfied one gets $|y(t) - y_0| \leqslant C$ for $t \in [a, b]$.

Remark 16.14. The proof works in the same way if y takes values in $\mathbb{R}^m$ and therefore $f : \mathbb{R} \times \mathbb{R}^m \supset [T_1, T_2] \times \overline{B_C(0)} \to \mathbb{R}^m$. In fact, the target space may even be a Banach space (the derivative for Banach space-valued functions appropriately defined). Higher order differential equations may be written as systems of first order equations and hence the theorem applies to these as well. For example $y''(t) + y(t) = 0, y(0) = 1, y'(0) = 0$ can be written as

$$\frac{d}{dt}\begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} w \\ -y \end{pmatrix}, \quad \begin{pmatrix} y \\ w \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So here the function f is $f(t,(x_1,x_2)) = (x_2, -x_1)$.

Example 16.15. Consider the IVP

$$\frac{dy}{dt} = y^2 t + 1, \; y(0) = 1.$$

Hence, $f(t,x) = x^2 t + 1$. If we take f to be defined on the square $[-T,T] \times [1 - C, 1 + C]$ then we obtain $\|f\|_\infty = (1+C)^2 T + 1$ (the value at the top-right corner). In this case the solution will exist up to time

$$\min\left\{T, \frac{C}{(1+C)^2 T + 1}\right\}.$$

If we choose, for example $C = 2$ and $T = \frac{1}{2}$ we get that a unique solution exists up to time $|t| \leqslant \frac{4}{11}$. This solution will then satisfy $|y(t) - 1| \leqslant 2$ for $|t| \leqslant \frac{4}{11}$. In fact one can show that the solution can be expressed in a complicated way in terms of the Airy-Bi-function and it blows up at $t = 1$.

16.1.3. *Applications of fixed point theory: Inverse and Implicit Function Theorems.* It is an easy exercise in Analysis to show that if a function $f \in C^1[a,b]$ has nowhere vanishing derivative, then f is invertible on its image. To be more precise, $f^{-1} : \mathrm{Im}(f) \to [a,b]$ exists and has derivative $(f'(x))^{-1}$ at the point $y = f(x)$. In higher dimensions a statement like this can not be correct as the following counterexample shows. Let $0 < a < b$ and define

$$f : [a,b] \times \mathbb{R} \to \mathbb{R}^2,$$
$$(r,\theta) \mapsto (r\cos\theta, r\sin\theta).$$

This maps has invertible derivative

$$f'(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta. \end{pmatrix}, \quad \det f'(r,\theta) = r^2 > 0.$$

at any point, the map is however not injective, see Fig. 20 for a cartoon illustration of the difference between one- and two-dimensional cases. However, for any point we can restrict domain and co-domain, so that the restriction of the function is invertible. In such a case we say that f is locally invertible. This concept will be explained in more detail below.

Definition 16.16 (Local Invertibility)**.** Suppose $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{R}^m$ are open subsets of $\mathbb{R}^m$. Then a map $f : \mathcal{U}_1 \to \mathcal{U}_2$ is called *locally invertible at* $x \in \mathcal{U}_1$ if

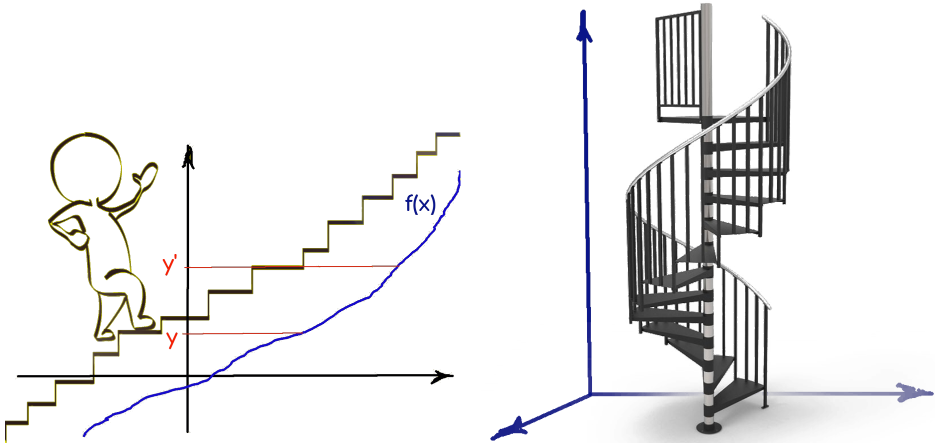


FIGURE 20. Flat and spiral staircases: can we return to the same value going just in one way?

there exists an open neighbourhood $\mathcal{U}$ of x such that $f|_{\mathcal{U}} : \mathcal{U} \to f(\mathcal{U})$ is invertible. The function f is said to be *locally invertible* it it is locally invertible at x for any $x \in \mathcal{U}_1$.

Often, say for differential equations, we need a map which preserves differentiability of functions in *both* directions.

Definition 16.17 (Diffeomorphism)**.** Suppose $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{R}^m$ are open subsets of $\mathbb{R}^m$. Then a map $f : \mathcal{U}_1 \to \mathcal{U}_2$ is called C^k*-diffeomorphism* if $f \in C^k(\mathcal{U}_1, \mathcal{U}_2)$ and if there exists a $g \in C^k(\mathcal{U}_2, \mathcal{U}_1)$ such that

$$f \circ g = \mathbb{1}_{\mathcal{U}_2}, \quad g \circ f = \mathbb{1}_{\mathcal{U}_1},$$

where $\mathbb{1}_{\mathcal{U}_1}$ and $\mathbb{1}_{\mathcal{U}_2}$ are the identity maps on $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively.

There is also a local version of the above definition.

Definition 16.18 (Local Diffeomorphism)**.** Suppose $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{R}^m$ are open subsets of $\mathbb{R}^m$. Then a map $f : \mathcal{U}_1 \to \mathcal{U}_2$ is called a *local*-C^k- *diffeomorphism at* $x \in \mathcal{U}_1$ if there exists an open neighbourhood $\mathcal{U}$ of x such that $f|_{\mathcal{U}} : \mathcal{U} \to f(\mathcal{U})$ is a C^k-diffeomorphism. It is called a *local*-C^k- *diffeomorphism* if it is a local diffeomorphism at any point $x \in \mathcal{U}_1$.

Not every invertible C^k-map is a diffeomorphism. An example is the function $f(x) = x^3$ whose inverse $g(x) = x^{\frac{1}{3}}$ fails to be differentiable.

Theorem 16.19 (Inverse Function Theorem). *Let $\mathcal{U} \subset \mathbb{R}^m$ be an open subset and suppose that $f \in C^k(\mathcal{U}, \mathbb{R}^m)$ such that $f'(x)$ is invertible at every point $x \in \mathcal{U}$. Then f is a local C^k-diffeomorphism.*

Before we can prove this theorem we need a Lemma, which basically says that under the assumptions of the inverse function theorem an inverse function must be in C^1. That is, differentiability is the *leading particular case* [10, § 4.4] for the general case of k-differentiable functions.

Lemma 16.20. *Suppose that $f \in C^1(\mathcal{U}_1, \mathcal{U}_2)$ is bijective with continuous inverse. Assume that the derivative of f is invertible at any point, then f is a C^1-diffeomorphism, and $g'(f(x)) = (f'(x))^{-1}$.*

Proof. Denote the inverse of f by $g : \mathcal{U}_2 \to \mathcal{U}_1$. The continuity of f and g imply that $x_n \to x_0$ if and only if $f(x_n) \to f(x_0)$. We will show that g is differentiable at the point $y_0 = f(x_0)$. If $y = f(x)$ is very close to y_0 (so that the line interval between x and x_0 is contained in $\mathcal{U}_1$) then, by the MVT there exists a ξ on this line such that $y - y_0 = f(x) - f(x_0) = f'(\xi) \cdot (x - x_0)$. Therefore, $g(y) - g(y_0) = (f'(\xi))^{-1} \cdot (y - y_0)$. If y tends to y_0, then ξ will tend to x_0, and therefore, by continuity of f' the value of $(f'(\xi))^{-1}$ will tend to $(f'(x_0))^{-1}$. Thus, the partial derivatives of g exist and are continuous, so $g \in C^1$. Note that we have used here that matrix inversion is continuous. □

Now we can proceed with the general situation.

Proof of the Inverse Function Theorem 16.19. Let $x_0 \in \mathcal{U}$ and let $y_0 = f(x_0)$. We need to show that there exists an open neighborhood $\mathcal{U}_1$ of $f(x_0)$ such that $f : f^{-1}(\mathcal{U}_1) \to \mathcal{U}_1$ is a C^k-diffeomorphism. As a first step we construct a continuous inverse. Since $f'(x_0) = A$ is an invertible $m \times m$-matrix we can change coordinates $x = A^{-1}y + x_0$, so that we can assume without loss of generality that $f'(x_0) = \mathbb{1}$ and $x_0 = 0$. Replacing f by $f - y_0$ we also assume w.l.o.g. that $y_0 = 0$. Since $f'(x)$ is continuous there exists an $\varepsilon > 0$ such that $\|f'(x) - \mathbb{1}\| \leqslant \frac{1}{2}$ for all $x \in \overline{B_\varepsilon(0)}$. This $\varepsilon > 0$ can also be chosen such that $B_\varepsilon(0) \subset \mathcal{U}$. Thus, $\|x - f(x)\| \leqslant \frac{1}{2}\|x\|$ for all $x \in \overline{B_\varepsilon(0)}$ by MVT, and for each $y \in \overline{B_{\varepsilon/2}(0)}$ the map

$$x \mapsto x + y - f(x)$$

is a contraction on $\overline{B_\varepsilon(0)}$. Indeed, by MVT again:

$$\begin{aligned} \|x+y-f(x)-(x'+y-f(x'))\| &= \|x-f(x)-(x'-f(x'))\| \\ &= \|(f'(\xi)-\mathbb{1})(x-x')\| \\ &\leqslant \frac{1}{2}\|x-x'\|, \end{aligned} \tag{16.3}$$

where $\|\cdot\|$ is the norm of vectors in $\mathbb{R}^m$. Consider the complete metric space $X = C(\overline{B_{\varepsilon/2}(0)}, \overline{B_\varepsilon(0)})$ and define the map

$$F : X \to X,\ u \mapsto F(u), \quad F(u)(y) = u(y) + y - f(u(y)).$$

By the above this map is well defined and it also is a contraction

$$\begin{aligned} \|F(u)(y)-F(v)(y)\| &= \|u(y)-f(u(y))-(v(y)-f(v(y)))\| \\ &\leqslant \frac{1}{2}\|u(y)-v(y)\| && \text{[by (16.3)]} \\ &\leqslant \frac{1}{2}\|u-v\|_\infty . \end{aligned}$$

Hence, there exists a unique fixed point g. This fixed point yields a continuous inverse g of $f|_{\mathcal{U}}$ defined on $\mathcal{U} = B_{\varepsilon/2}(0) \cap f^{-1}(B_{\varepsilon/2}(0))$. By the previous Lemma this implies that g is differentiable. Now simply note that $g' = (f')^{-1} \circ g$. Since matrix inversion is smooth and f' is in C^{k-1} this implies that for $m \leqslant k-1$ we get the conclusion $(g \in C^m) \implies (g \in C^{m+1})$. Hence, g is in C^k. □

The implicit function theorem is actually a rather simple consequence of the inverse function theorem. It gives a nice criterion for local solvability of equations in many variables.

Theorem 16.21 (Implicit Function Theorem). *Let $\mathcal{U}_1 \subset \mathbb{R}^n \times \mathbb{R}^m$ and $\mathcal{U}_2 \subset \mathbb{R}^m$ be open subsets and let*

$$F : \mathcal{U}_1 \to \mathcal{U}_2,\ (x_1,\ldots,x_n,y_1,\ldots,y_m) \mapsto F(x_1,\ldots,x_n,y_1,\ldots,y_m)$$

be a C^k-map. Suppose that $F(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{0}$ for some point $(\mathbf{x}_0,\mathbf{y}_0) \in \mathcal{U}_1$ and that the $m \times m$-matrix $\partial_{\mathbf{y}} F(\mathbf{x}_0,\mathbf{y}_0)$ is invertible. Then there exists an neighborhood $\mathcal{U}$ of $(\mathbf{x}_0,\mathbf{y}_0) \in \mathbb{R}^n \times \mathbb{R}^m$, an open neighborhood $\mathcal{V}$ of $\mathbf{x}_0$ in $\mathbb{R}^n$, and a C^k-function $f : \mathcal{V} \to \mathbb{R}^m$ such that

$$\{(\mathbf{x},\mathbf{y}) \in \mathcal{U} \mid F(\mathbf{x},\mathbf{y}) = \mathbf{0}\} = \{(\mathbf{x}, f(\mathbf{x})) \in \mathcal{U} \mid \mathbf{x} \in \mathcal{V}\}.$$

The function f has derivative

$$f'(\mathbf{x}_0) = -(\partial_{\mathbf{y}} F(\mathbf{x}_0,\mathbf{y}_0))^{-1} \partial_{\mathbf{x}} F(\mathbf{x}_0,\mathbf{y}_0)$$

at $\mathbf{x}_0$.

Proof. This is proved by reducing it to the inverse function theorem. Just design the map

$$G : \mathcal{U}_1 \to \mathbb{R}^n \times \mathbb{R}^m, \ (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, F(\mathbf{x}, \mathbf{y}))$$

and then note that

$$G'(\mathbf{x}_0, \mathbf{y}_0) = \begin{pmatrix} \mathbb{1} & 0 \\ \partial_{\mathbf{x}} F(\mathbf{x}_0, \mathbf{y}_0) & \partial_{\mathbf{y}} F(\mathbf{x}_0, \mathbf{y}_0) \end{pmatrix}$$

is invertible with inverse

$$(G'(\mathbf{x}_0, \mathbf{y}_0))^{-1} = \begin{pmatrix} \mathbb{1} & 0 \\ -(\partial_{\mathbf{y}} F(\mathbf{x}_0, \mathbf{y}_0))^{-1} \partial_{\mathbf{x}} F(\mathbf{x}_0, \mathbf{y}_0) & (\partial_{\mathbf{y}} F(\mathbf{x}_0, \mathbf{y}_0))^{-1} \end{pmatrix}.$$

By the inverse function theorem there exists a local inverse $G^{-1} : \mathcal{U}_3 \to \mathcal{U}_4$, where $\mathcal{U}_3$ is an open neighborhood of $\mathbf{0}$ and $\mathcal{U}_4$ an open neighborhood of $(\mathbf{x}_0, \mathbf{y}_0)$. Now define f by $(\mathbf{x}, f(\mathbf{x})) = G^{-1}(\mathbf{x}, \mathbf{0})$. □

Example 16.22. Consider the system of equations

$$x_1^2 + x_2^2 + y_1^2 + y_2^2 = 2,$$
$$x_1 + x_2^3 + y_1 + y_2^3 = 2.$$

We would like to know if this system implicitly determines functions $y_1(x_1, x_2)$ and $y_2(x_1, x_2)$ near the point $(0, 0, 1, 1)$, which solves the equation. For this one simply applies the implicit function theorem to

$$F(x_1, x_2, y_1, y_2) = (x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2, x_1 + x_2^3 + y_1 + y_2^3 - 2).$$

The derivatives are

$$\partial_{\mathbf{x}} F = \begin{pmatrix} 2x_1 & 2x_2 \\ 1 & 3x_2^2 \end{pmatrix}, \ \partial_{\mathbf{y}} F = \begin{pmatrix} 2y_1 & 2y_2 \\ 1 & 3y_2^2 \end{pmatrix}$$

The values of these derivatives at the point $(0, 0, 1, 1)$ are

$$\partial_{\mathbf{x}} F(0, 0, 1, 1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \partial_{\mathbf{y}} F(0, 0, 1, 1) = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

The latter matrix is invertible and one computes

$$-(\partial_{\mathbf{y}} F(\mathbf{x}_0, \mathbf{y}_0))^{-1} \partial_{\mathbf{x}} F(\mathbf{x}_0, \mathbf{y}_0)(0, 0, 1, 1) = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 0 \end{pmatrix}.$$

We conclude that there is an implicitly defined function $(y_1, y_2) = f(x_1, x_2)$ whose derivative at $(0, 0)$ is given by

$$\begin{pmatrix} 1/2 & 0 \\ -1/2 & 0 \end{pmatrix}.$$

The geometric meaning is that near the point $(0, 0, 1, 1)$ the system defines a two-dimensional manifold that is locally given by the graph of a function. Its tangent plane is spanned by the vectors $(1/2, 0, 1, 0)$ and $(-1/2, 0, 0, 1)$.

Example 16.23. Consider the system of equations

$$x^2 + y^2 + z^2 = 1,$$
$$x + yz + z^3 = 1.$$

This is the intersection of a sphere (drawn in light green on Figure 21) with some cubic surface defined by the second equation (drawn in light blue). The point $(0, 0, 1)$ solves the equation and is pictured as a little orange dot. By the implicit function theorem the intersection is a smooth curve (drawn in red) near this point which can be parametrised by x coordinate. Indeed, we can express y and z along the curve as functions of x because the resulting matrix

$$\partial_{(y,z)}F(0,1) = \begin{pmatrix} 2y & 2z \\ z & y + 3z^2 \end{pmatrix}\bigg|_{y=0,z=1} = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$$

is invertible.

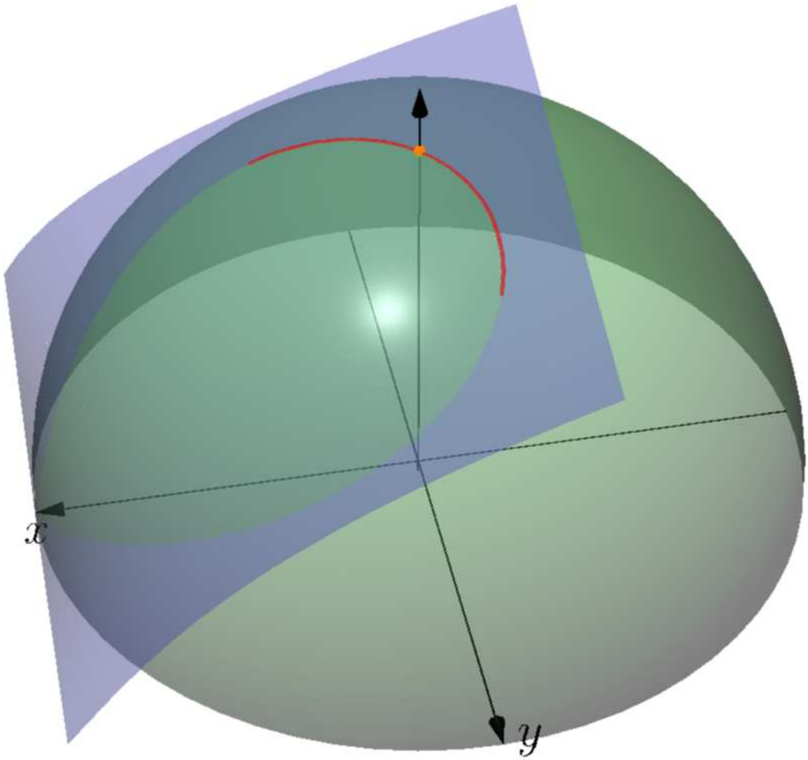


FIGURE 21. Example of the implicit theorem: the intersection (red) of the unit sphere (green) and a cubic surface (blue).

Exercise 16.24. Fig. 21 suggests that the intersection curve can be alternatively parametrised by the coordinates y and can*not* by z (why?). Check these claims by verifying conditions of Thm. 16.21.

16.2. **The Baire Category Theorem and Applications.** We are going to see another example of an abstract result which has several non-trivial consequences for real analysis.

16.2.1. *The Baire's Categories.* Let us first prove the following result and then discuss its meaning and name.

Theorem 16.25 (Baire's category theorem)**.** *Let* (X, d) *be a complete metric space and* U_n *a sequence of open dense sets. Then the intersection* $S = \bigcap_n U_n$ *is dense.*

Proof. The proof is rather straightforward. We need to show that any ball $B_\varepsilon(x_0)$ contains an element of S. Let us therefore fix x_0 and $\varepsilon > 0$. Since U_1 is dense the intersection of $B_\varepsilon(x_0)$ with U_1 is non-trivial. Thus there exists a point $x_1 \in B_\varepsilon(x_0) \cap U_1$. Now choose $\varepsilon_1 < \varepsilon/2$ so that $\overline{B_{\varepsilon_1}(x_1)} \subset B_\varepsilon(x) \cap U_1$ (note the closure of the ball). Since U_2 is dense, the intersection $B_{\varepsilon_1}(x_1) \cap U_2 \subset B_\varepsilon(x_0) \cap U_1 \cap U_2$ is non-empty. Choose a point x_2 and $\varepsilon_2 < \varepsilon_1/2$ such that $\overline{B_{\varepsilon_2}(x_2)} \subset B_{\varepsilon_1}(x_1) \cap U_2 \subset B_\varepsilon(x_0) \cap U_1 \cap U_2$. Continue inductively, to obtain a sequence x_n such that

$$\overline{B_{\varepsilon_n}(x_n)} \subset B_{\varepsilon_{n-1}}(x_{n-1}) \cap U_n \subset B_\varepsilon(x_0) \cap U_1 \cap U_2 \cap \ldots \cap U_n,$$

and $\varepsilon_n < 2^{-n}\varepsilon$. In particular, for any $n > N$ we have

$$x_n \in B_{2^{-N}\varepsilon}(x_N),$$

which implies that x_n is a Cauchy sequence. Hence x_n has a limit x, by completeness of (X, d). Consequently, x is contained in the closed ball $\overline{B_{\varepsilon_N}(x_N)}$ for any N, and therefore it is contained in $B_\varepsilon(x_0) \cap (\bigcap_n U_n)$, as claimed. □

Completeness is essential here. For example, the conclusion does not hold for the metric space $\mathbb{Q}$: take bijection $\psi : \mathbb{N} \to \mathbb{Q}$, and consider the open dense sets

$$U_n = \{\psi(1), \psi(1), \ldots, \psi(n)\}^c = \{\psi(n+1), \psi(n+2), \ldots\}.$$

The intersection $\cap_n U_n$ is empty.

The following historic terminology, due to Baire, is in use.

Definition 16.26 (Baire's categories). A subset Y of a metric space X is called

(i) *nowhere dense* if the interior of $\overline{Y}$ is empty;
(ii) *of first category* if there is a sequence (Y_k) of nowhere dense sets with $Y = \cup_k Y_k$;

(iii) *of second category* if it is not of first category.

Example of nowhere dense sets are $\mathbb{Z} \subset \mathbb{R}$, the circle in $\mathbb{R}^2$, or the set $\{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Note that the complement of a nowhere dense set is a dense open set.

Corollary 16.27. *In a complete metric space the complement of a set of the first category is dense.*

Proof. Follows from relations for complements

$$Y^c = (\cup_k Y_k)^c = \cap_k Y_k^c \supset \cap_k \overline{Y_k}^c$$

and the fact that $\overline{Y_k}^c$ is dense. □

The following corollary is also called Baire's category theorem in some sources:

Corollary 16.28. *A complete metric space is of second category in itself, or plainly speaking it is never the union of a countable number of nowhere dense sets.*

The theorem is often used to show abstract existence results. Here is an example.

Theorem 16.29. *There exists a function* $f \in C[0,1]$ *that is nowhere differentiable.*

Proof. For each $n \in \mathbb{N}$ define

$$U_n = \left\{ f \in C[0,1] \text{ s.t. } \sup \left\{ \left| \frac{f(x+h) - f(x)}{h} \right| \text{ over } 0 < |h| \leqslant \frac{1}{n} \right\} > n, \forall x \in [0,1] \right\}.$$

We will show that the U_n are open and dense. By the Category theorem their intersection is also dense.

U_n is open: Let $f \in U_n$. For each $x \in [0,1]$ choose $\delta_x > 0$ such that

$$\sup \left\{ \left| \frac{f(x+h) - f(x)}{h} \right| \text{ over } 0 < |h| \leqslant \frac{1}{n} \right\} > n + \delta_x,$$

hence there is a $h_x < \frac{1}{n}$ with

$$\left| \frac{f(x+h_x) - f(x)}{h_x} \right| > n + \delta_x.$$

By continuity of f there is an open neighborhood I_x of x such that

$$\left| \frac{f(y+h_x) - f(y)}{h_x} \right| > n + \delta_x.$$

for all $y \in I_x$. These I_x form an open cover. We choose a finite subcover $(I_{x_k})_{k=1,\dots,N}$. Let $\delta = \min\{\delta_{x_1}, \dots, \delta_{x_N}\} > 0$. Then, for $y \in I_{x_k}$:

$$\left| \frac{f(y + h_{x_k}) - f(y)}{h_{x_k}} \right| > n + \delta.$$

Now let $g \in B_\varepsilon(f)$, where $\varepsilon > 0$ is chosen so that $\varepsilon < \frac{1}{2}\delta h_{x_k}$ for all k. Then by an $\varepsilon/3$-style argument:

$$\left| \frac{g(y + h_{x_k}) - g(y)}{h_{x_k}} \right| \geqslant \left| \frac{f(y + h_{x_k}) - f(y)}{h_{x_k}} \right| - 2\frac{\|f - g\|_\infty}{h_{x_k}} > n + \delta - 2\varepsilon h_{x_k}^{-1} > n,$$

and therefore $g \in U_n$. We conclude that U_n is open.

U_n is dense: For each $\varepsilon > 0$ and $f \in C[0,1]$ choose a polynomial p such that $\|f - p\| < \frac{\varepsilon}{2}$ and a sequence of continuous function $g_m \in C[0,1]$ such that $\|g\|_\infty < \frac{\varepsilon}{2}$ and such that for all $x \in [0,1]$:

$$\sup\left\{ \frac{g_m(x + h) - g_m(x)}{h} \text{ over } 0 < |h| \leqslant \frac{1}{n} \right\} > m$$

by using a "zigzag" function. Then, for large enough m we have $p + g_m \in U_n$. □

The above proof actually shows much more, namely that the set of nowhere differentiable functions is dense in $C[0,1]$. It is also useful to compare it with the construction of the continuous nowhere differentiable Weierstrass function and identify some common elements.

16.2.2. *Banach–Steinhaus Uniform Boundedness Principle.* Another consequence of the Baire Category theorem is the Banach–Steinhaus uniform boundedness principle. Recall that, if X and Y are normed spaces, $T : X \to Y$ is called a bounded operator if it is a bounded linear map.

Theorem 16.30 (Banach–Steinhaus Uniform Boundedness Principle)**.** *Let* X *be a Banach space and* Y *a normed space, and let* $(T_\alpha)_{\alpha \in I}$ *be a family of bounded operators* $T_\alpha : X \to Y$*. Suppose that*

$$\forall x \in X : \sup_\alpha \|T_\alpha x\| < \infty.$$

Then we have $\sup_\alpha \|T_\alpha\| < \infty$*, i.e. the family* T_α *is bounded in the set* $\mathcal{B}(X, Y)$ *of bounded operators from* X *to* Y*.*

Proof. Define $X_n = \{x \in X \mid \sup_\alpha \|T_\alpha x\| \leqslant n\}$. By assumption $X = \cup_n X_n$. Note that all the X_n are closed. By the Baire category theorem at least one of these sets must have non-empty interior, since otherwise the Banach space X would be a countable union of nowhere dense sets. Hence, there exists $N \in \mathbb{N}$, $y \in X_N$, and $\varepsilon > 0$ such that $\overline{B_\varepsilon(y)} \in X_N$. Now X_N is symmetric under reflections $x \mapsto -x$

and convex. So we get the same statement for $-y$. Hence, $x \in \overline{B_\varepsilon(0)}$ implies

(16.4) $$x = \frac{1}{2}((x+y)+(x-y)) \in \frac{1}{2}(X_N + X_N) \subset X_N.$$

This means that $\|x\| \leqslant \varepsilon$ implies $\|T_\alpha x\| \leqslant N$, and therefore $\|T_\alpha\| \leqslant \varepsilon^{-1}N$ for all $\alpha \in I$. □

Recall that the Fourier series of a C^1-function on a circle (identified with 2π-periodic functions) converges uniformly to the function. We will now show that a statement like that can not hold for continuous functions.

Corollary 16.31. *There exist continuous periodic functions whose Fourier series do not converge point-wise.*

Proof. We will show that there exists a continuous function whose Fourier series does not converge at $x = 0$. Suppose by contradiction such functions would not exist, so we would have point-wise convergence of the Fourier series

$$\frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos(mx) + b_m \sin(mx)$$

for every $f \in C(S^1) = C_{per}(\mathbb{R})$. Here we identify continuous functions on the unit circle with continuous 2π-periodic functions $C_{per}(\mathbb{R})$. Hence we have a map

$$T_n : C(S^1) \to \mathbb{R}, f \mapsto \frac{1}{2}a_0 + \sum_{m=1}^{n} a_m$$

by mapping the function f to the n-th partial sum of its Fourier series at $x = 0$. This is a family of bounded operators $T_n : C(S^1) \to \mathbb{R}$ and by assumption we have for every f that

$$\sup_n |T_n(f)| < \infty.$$

By Banach–Steinhaus theorem we have $\sup_n \|T_n\| = \sup_{n,\|f\|_\infty=1} |T_n(f)| < \infty$. Now one computes the norm of the map

$$T_n : C(S^1) \to \mathbb{R}, \quad f \mapsto \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{1}{2} + \sum_{k=1}^{n} \cos(kx)\right) dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_n(x)dx$$

where

$$D_n(x) = \frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)}$$

is the *Dirichlet kernel* , cf. Lem. 5.6. This norm equals $\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(x)|dx = \frac{1}{2\pi}\int_{0}^{2\pi}|D_n(x)|dx$ (Exercise) which goes to ∞ as $n\to\infty$. Indeed, using $\sin(x/2)\leqslant x/2$ and substituting we get

$$\begin{aligned}
\int_0^{2\pi}|D_n(x)|dx &\geqslant \int_0^{2\pi}\frac{|\sin((n+\frac{1}{2})x)|}{x/2}dx && \text{[since } \sin s\leqslant s\text{]}\\
&= \int_0^{(2n+1)\pi}\frac{|\sin(t)|}{t}dt && \text{[change of variables } t=(n+\tfrac{1}{2})x\text{]}\\
&\geqslant \sum_{k=0}^{2n}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{t}dt && \text{[split integral into intervals]}\\
&\geqslant \left|\sum_{k=0}^{2n}\int_0^{\pi}\frac{\sin t}{(k+1)}dt\right| && \text{[since } t\leqslant k+1 \text{ for } t\in(k,k+1)\text{]}\\
&= 2\sum_{k=0}^{2n}\frac{1}{k+1} && \text{[evaluating the integral],}
\end{aligned}$$

which is the harmonic series divergent as $n\to\infty$. This gives a contradiction. □

Another corollary of the Banach–Steinhaus principle is an important continuity statement. Recall that of X and Y are normed spaces them so is the Cartesian product $X\times Y$ equipped with the norm $\|(x,y)\| = \left(\|x\|_X^2+\|y\|_Y^2\right)^{\frac{1}{2}}$. It is easy to see that a sequence (x_n,y_n) converges to (x,y) in this norm if and only if $x_n\to x$ and $y_n\to y$.

Theorem 16.32. *Suppose that* X, Y *are Banach spaces and suppose that* $B : X\times Y\to\mathbb{R}$ *is a bilinear form on* $X\times Y$ *that is separately continuous, i.e.* $B(\cdot,y)$ *is continuous on* X *for every* $y\in Y$ *and* $B(x,\cdot)$ *is continuous on* Y *for every* $x\in X$. *Then* B *is continuous.*

Proof. Suppose that (x_n,y_n) is a sequence that converges to (x,y). First note that

$$B(x_n-x,y_n-y) = B(x_n,y_n)-B(x_n,y)-B(x,y_n)+B(x,y),$$

where $B(x_n,y)\to B(x,y)$ as well as $B(x,y_n)\to B(x,y)$. So it is sufficient to show that $B(x_n-x,y_n-y)\to 0$ or, equivalently, $B(\tilde{x}_n,\tilde{y}_n)\to 0$ for any $\tilde{x}_n\to 0$

and $\tilde{y}_n \to 0$. Now. the linear mappings $T_n(x) = B(x, \tilde{y}_n) : X \to \mathbb{R}$ are bounded, by assumption. Since $\|\tilde{y}_n\| \to 0$ the sequence $T_n(x) \to 0$ and is bounded for every $x \in X$. Then, by the Banach–Steinhaus theorem there exists a constant C such that $\|T_n\| \leqslant C$ for all n. That is $|T_n(x)| = B(x, \tilde{y}_n) \leqslant C\|x\|$ for all n and $x \in X$. Therefore, $|B(\tilde{x}_n, \tilde{y}_n)| \leqslant C\|\tilde{x}_n\| \to 0$. □

Remark 16.33. Recall that already on $\mathbb{R}^2$ separate continuity does not imply joint continuity for any function. The standard example from Analysis is the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = 0, \end{cases}$$

which is continuous in x or y separately but is not jointly continuous.

16.2.3. *The open mapping theorem.* Recall that for a continuous map the pre-image of any open set is open. This does of course not mean that the image of any open set is open (for example, $\sin : \mathbb{R} \to \mathbb{R}$ has image $[-1, 1]$, which is not open). A map $f : X \to Y$ between metric space is called *open* if the image of every open set is open. If a map is invertible then it is open if and only if its inverse is continuous. We start with a simple observation for linear maps. We will denote open balls in normed spaces X and Y by $B_r^X(x)$ and $B_s^Y(y)$ respectively, or simply B_r^X and B_s^Y if they are centred at the origin.

Lemma 16.34. *Let X and Y be normed spaces. Then a linear map $T : X \to Y$ is open if and only if there exists $\varepsilon > 0$ such that $B_\varepsilon^Y(0) \subset T(B_1^X(0))$, i.e. the image of the unit ball contains a zero's neighbourhood.*

Proof. If the map T is open it clearly has this property. Suppose conversely, that $B_\varepsilon^Y(0) \subset T(B_1^X(0))$ for some $\varepsilon > 0$. Then, by scaling, $B_{\varepsilon\delta}^Y(0) \subset T(B_\delta^X(0))$ for any $\delta > 0$. Suppose that U is open. Suppose that $y \in f(U)$, that is there exists $x \in U$ such that $y = f(x)$. Then there exists $\delta > 0$ with $x + B_\delta^X(0) \subset U$ and therefore

$$TU \supset TB_\delta^X(x) = \{Tx\} + TB_\delta^X(0) \supset \{y\} + B_{\delta\varepsilon}^Y(0) = B_{\delta\varepsilon}^Y(y).$$

□

Theorem 16.35 (Open Mapping Theorem). *Let $T : X \to Y$ be a continuous surjective linear operator between Banach spaces. Then T is open.*

Proof. Since T is surjective we have $Y = \cup_n TB_n^X$. Therefore trivially, $Y = \cup_n \overline{TB_n^X}$. By the Baire category theorem one of the $\overline{TB_n^X}$ must have an interior point. Rescaling implies that $\overline{TB_1^X}$ has an interior point y_0. Since $\overline{TB_1^X}$ is symmetric under reflection $y \to -y$, the point $-y_0$ must also be an interior point. Therefore, by

convexity of $\overline{TB_1^X}$ there exists a $\delta > 0$ with $B_\delta^Y \subset \overline{TB_1^X}$, cf. (16.4). By linearity this means $B_{\delta 2^{-n}}^Y \subset \overline{TB_{2^{-n}}^X}$ for any natural n.
We will show that $\overline{TB_1^X} \subset TB_2^X$, with the implication from above that $B_\delta^Y \subset TB_2^X$, which will complete the proof by the previous Lemma. So, let $y \in \overline{TB_1^X}$ be arbitrary. Then, there exists $x_1 \in B_1^X$ such that $y - Tx_1 \in B_{\delta/2}^Y \subset \overline{TB_{1/2}^X}$. Repeating this, there exists $x_2 \in B_{1/2}^X$ such that $y - Tx_1 - Tx_2 \in B_{\delta/4}^Y$.
Continuing inductively, we obtain a sequence (x_n) with the property that $\|x_n\| < 2^{-n+1}$ and

$$y - \sum_{k=1}^{n} Tx_n \in B_{\delta 2^{1-n}}^Y. \tag{16.5}$$

By completeness of X, the absolute convergent series $\sum_1^n x_n$ converges to an element $x \in X$ of norm $\|x\| < 2$. By linearity an continuity of T we get from (16.5) that $y = Tx$. Thus $y \in TB_2$. □

If the map T is also injective (and, therefore, bijective with the inverse T^{-1}) we can quickly conclude continuity of T^{-1}.

Corollary 16.36. *Suppose that $T : X \to Y$ is a bijective bounded linear map between Banach spaces. Then T has a bounded inverse T^{-1}.*

It is not rare that we may have two different norms $\|\cdot\|$ and $\|\cdot\|_*$ on the same Banach space X. We say that $\|\cdot\|$ and $\|\cdot\|_*$ are *equivalent* if there are constants $c > 0$ and $C > 0$ such that:

$$c\|x\| \leqslant \|x\|_* \leqslant C\|x\| \qquad \text{for all } x \in X. \tag{16.6}$$

Exercise 16.37. (i) Check that (16.6) defines an equivalence relations on the set of all norms on X.
(ii) If a sequence is Cauchy/convergent/bounded in a norm then it is also Cauchy/convergent/bounded in any equivalent norm.

The Cor. 16.36 implies that if the identity map $(X, \|\cdot\|) \to (X, \|\cdot\|_*)$ is bounded then both norms are equivalent.

Corollary 16.38. *Let $(X, \|\cdot\|)$ be a Banach space and $\|\cdot\|_*$ be a norm on X in which X is complete. If $\|\cdot\| \leqslant C\|\cdot\|_*$ for some $C > 0$ the norms are equivalent.*

16.2.4. *The closed graph theorem.* Suppose that X, Y are Banach spaces and suppose that $D \subset X$ is a linear subspace (not necessarily closed). Now suppose that $T : D \to Y$ is a linear operator. Then the *graph* $\mathrm{gr}(T)$ is defined as the subset $\{(x, Tx) \mid x \in$

$D\} \subset X \times Y$. This is a linear subspace in the Banach space $X \times Y$, which can be equipped with the norm $\|(x,y)\|^2 = \|x\|_X^2 + \|y\|_Y^2$. One often uses the equivalent norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$ but the first choice makes sure that the product $X \times Y$ is also a Hilbert space if X and Y are Hilbert spaces. We will refer to T as an operator from X to Y with *domain* D.

Definition 16.39. The operator T is called *closed* if and only if its graph is a closed subset of $X \times Y$.

It is easy to see that T is closed if an only if $x_n \to x$ and $Tx_n \to y$ imply that $Tx_n \to Tx$. Note the difference with continuity of T!!!

If T is an operator $T : D \to Y$ then its graph is a subset of $X \times Y$. If we close this subset the resulting set may fail to be the graph of an operator. If the closure is the graph as well, we say that T is *closable* and its *closure* is the operator whose graph is obtained by closing the graph of T.

Differential operators are often closed but not bounded. Let $L^2[a,b]$ be the Hilbert space obtained by abstract completion of $(C[a,b], \|\cdot\|_2)$, cf. Prop. 1.59. Then $D = C^1[a,b]$ is a dense subspace in $L^2[a,b]$ and the operator $\frac{d}{dx} : C^1[a,b] \to L^2[a,b]$ is of the above type. This operator is not closed, however it is closable and its closure therefore defines a closed operator with dense domain. We have already seen that this operator is unbounded and therefore it cannot be continuous.

Of course, the map $D \to (x, Tx)$ is a bijection from D to $\mathrm{gr}(T)$. We can use the norm on $\mathrm{gr}(T)$ to define a norm on D, which is then

$$\|x\|_D = \left(\|x\|_X^2 + \|Tx\|_Y^2\right)^{\frac{1}{2}}.$$

Obviously, T is closed if and only of D with norm $\|\cdot\|_D$ is a Banach space. We are now ready to state the closed graph theorem. It is easy to check that T continuously maps $(D, \|\cdot\|_D)$ to Y.

Theorem 16.40 (Closed Graph Theorem)**.** *Suppose that X and Y are Banach spaces and suppose that $T : X \to Y$ is closed. Then T is bounded.*

Proof. Since in this case we have $D = X$ with have two norms $\|\cdot\|_X$ and $\|\cdot\|_D$ on X that are both complete. Clearly,

$$\|\cdot\|_X \leqslant \|\cdot\|_D,$$

and by Cor. 16.38 the norms are therefore equivalent. Hence,

$$\|Tx\|_Y \leqslant \|x\|_D \leqslant C\|x\|_X$$

for some constant $C > 0$. □

16.3. **Semi-norms and locally convex topological vector spaces.**

Definition 16.41 (Semi-Norm)**.** Let X be a vector space, then a map $p : X \to \mathbb{R}$ is called *semi-norm* if
(i) $p(x) \geqslant 0$ for all $x \in X$,
(ii) $p(\lambda x) = |\lambda| p(x)$, for all $\lambda \in \mathbb{R}, x \in X$,
(iii) $p(x + y) \leqslant p(x) + p(y)$, for all $x, y \in X$.

An example of a semi-norm on $C^1[0, 1]$ is $p(f) := \|f'\|_\infty$. If $(p_\alpha)_\alpha$ is a family of semi-norms with the property that

$$(\forall \alpha \in I, p_\alpha(x) = 0) \implies x = 0$$

then we say X with that family is a *locally convex topological vector space*. There is a topology (that is, a description of all open sets) on such a vector space, by declaring a subset $U \subset X$ to be open if and only if for every point $x \in U$ and any index $\alpha \in I$ there exists $\varepsilon > 0$ such that $\{y \mid p_\alpha(y - x) < \varepsilon\} \subset U$. The notion of convergence one gets is $x_n \to x$ if and only of $p_\alpha(x_n - x) \to 0$ for all α. The topology of point-wise convergence on the space of functions $S \to \mathbb{R}$ is for example of this type, with the family of semi-norms given by $(p_x)_{sx \in S}, p_x(f) = |f(x)|$.

Another example is the vector space $C^\infty(\mathbb{R}^m)$ with the topology of uniform convergence of all derivatives on compact sets. Here the family of semi-norms $p_{\alpha,K}$ is indexed by all multi-indices $\alpha \in {\mathbb{N}_0}^m$ and all compact subsets $K \subset \mathbb{R}$ and is given by

$$p_{\alpha,K}(f) = \sup_{x \in K} |\partial^\alpha f(x)|.$$

If the family of semi-norms is countable then this topology is actually coming from a metric (so the space is a metric space)

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x - y)}{1 + p_k(x - y)}.$$

Such a metric space is called Frechet space. Note that $C^\infty(\mathbb{R}^m)$ is a Frechet space because the family of semi-norms above can be replaced by a countable one by taking a countable exhaustion of $\mathbb{R}^m$ by compact subsets.

Appendix A. Tutorial Problems

These are tutorial problems intended for self-assessment of the course understanding.

A.1. **Tutorial problems I.** *All spaces are complex, unless otherwise specified.*

A.1. Show that $\|f\| = |f(0)| + \sup |f'(t)|$ defines a norm on $C^1[0, 1]$, which is the space of (real) functions on $[0, 1]$ with continuous derivative.

A.2. Show that the formula $\langle (x_n), (y_n) \rangle = \sum_{n=1}^\infty x_n \bar{y}_n / n^2$ defines an inner product on ℓ_∞, the space of bounded (complex) sequences. What norm does it produce?

A.3. Use the Cauchy–Schwarz inequality for a suitable inner product to prove that for all $f \in C[0,1]$ the inequality

$$\left| \int_0^1 f(x)x\,dx \right| \leqslant C \left(\int_0^1 |f(x)|^2\,dx \right)^{1/2}$$

holds for some constant $C > 0$ (independent of f) and find the smallest possible C that holds for all functions f (hint: consider the cases of equality).

A.4. We define the following norm on ℓ_∞, the space of bounded complex sequences:

$$\|(x_n)\|_\infty = \sup_{n \geqslant 1} |x_n|.$$

Show that this norm makes ℓ_∞ into a Banach space (i.e., a complete normed space).

A.5. Fix a vector $(w_1, \ldots, w_n)$ whose components are strictly positive real numbers, and define an inner product on $\mathbb{C}^n$ by

$$\langle x, y \rangle = \sum_{k=1}^{n} w_k x_k \bar{y}_k.$$

Show that this makes $\mathbb{C}^n$ into a Hilbert space (i.e., a complete inner-product space).

A.2. **Tutorial problems II.**

A.6. Show that the supremum norm on $C[0,1]$ isn't given by an inner product, by finding a counterexample to the parallelogram law.

A.7. In ℓ_2 let $e_1 = (1,0,0,\ldots)$, $e_2 = (0,1,0,0,\ldots)$, $e_3 = (0,0,1,0,0,\ldots)$, and so on. Show that $\mathrm{Lin}\,(e_1, e_2, \ldots) = c_{00}$, and that $\mathrm{CLin}\,(e_1, e_2, \ldots) = \ell_2$. What is $\mathrm{CLin}\,(e_2, e_3, \ldots)$?

A.8. Let $C[-1,1]$ have the standard L_2 inner product, defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt.$$

Show that the functions 1, t and $t^2 - 1/3$ form an orthogonal (not orthonormal!) basis for the subspace P_2 of polynomials of degree at most 2 and hence calculate the best L_2-approximation of the function t^4 by polynomials in P_2.

A.9. Define an inner product on $C[0,1]$ by

$$\langle f, g \rangle = \int_0^1 \sqrt{t}\, f(t)\,\overline{g(t)}\,dt.$$

Use the Gram–Schmidt process to find the first 2 terms of an orthonormal sequence formed by orthonormalising the sequence $1, t, t^2, \ldots$.

A.10. Consider the plane P in $\mathbb{C}^4$ (usual inner product) spanned by the vectors $(1,1,0,0)$ and $(1,0,0,-1)$. Find orthonormal bases for P and $P^\perp$, and verify directly that $(P^\perp)^\perp = P$.

A.3. **Tutorial Problems III.**

A.11. Let a and b be arbitrary real numbers with $a < b$. By using the fact that the functions $\frac{1}{\sqrt{2\pi}}e^{inx}$, $n \in \mathbb{Z}$, are orthonormal in $L_2[0,2\pi]$, together with the change of variable $x = 2\pi(t-a)/(b-a)$, find an orthonormal basis in $L_2[a,b]$ of the form $e_n(t) = \alpha e^{in\lambda t}$, $n \in \mathbb{Z}$, for suitable real constants α and λ.

A.12. For which real values of α is

$$\sum_{n=1}^{\infty} n^\alpha e^{int}$$

the Fourier series of a function in $L_2[-\pi,\pi]$?

A.13. Calculate the Fourier series of $f(t) = e^t$ on $[-\pi,\pi]$ and use Parseval's identity to deduce that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{\tanh\pi}.$$

A.14. Using the fact that (e_n) is a complete orthonormal system in $L_2[-\pi,\pi]$, where $e_n(t) = \exp(int)/\sqrt{2\pi}$, show that $e_0, s_1, c_1, s_2, c_2, \ldots$ is a complete orthonormal system, where $s_n(t) = \sin nt/\sqrt{\pi}$ and $c_n(t) = \cos nt/\sqrt{\pi}$. Show that every $L_2[-\pi,\pi]$ function f has a Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt,$$

converging in the L_2 sense, and give a formula for the coefficients.

A.15. Let $C(\mathbb{T})$ be the space of continuous (complex) functions on the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the supremum norm. Show that, for any polynomial $f(z)$ in $C(\mathbb{T})$

$$\int_{|z|=1} f(z)\,dz = 0.$$

Deduce that the function $f(z) = \bar{z}$ is *not* the uniform limit of polynomials on the circle (i.e., Weierstrass's approximation theorem doesn't hold in this form).

A.4. **Tutorial Problems IV.**

A.16. Define a linear functional on $C[0,1]$ (continuous functions on $[0,1]$) by $\alpha(f) = f(1/2)$. Show that α is bounded if we give $C[0,1]$ the supremum norm. Show that α is not bounded if we use the L_2 norm, because we can find a sequence (f_n) of continuous functions on $[0,1]$ such that $\|f_n\|_2 \leqslant 1$, but $f_n(1/2) \to \infty$.

A.17. The *Hardy space* H_2 is the Hilbert space of all power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, where the inner product is given by

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

Show that the sequence $1, z, z^2, z^3, \ldots$ is an orthonormal basis for H_2.

Fix w with $|w| < 1$ and define a linear functional on H_2 by $\alpha(f) = f(w)$. Write down a formula for the function $g(z) \in H_2$ such that $\alpha(f) = \langle f, g \rangle$. What is $\|\alpha\|$?

A.18. The *Volterra operator* $V : L_2[0,1] \to L_2[0,1]$ is defined by

$$(Vf)(x) = \int_0^x f(t)\, dt.$$

Use the Cauchy–Schwarz inequality to show that $|(Vf)(x)| \leqslant \sqrt{x}\|f\|_2$ (hint: write $(Vf)(x) = \langle f, J_x \rangle$ where J_x is a function that you can write down explicitly).

Deduce that $\|Vf\|_2^2 \leqslant \frac{1}{2}\|f\|_2^2$, and hence $\|V\| \leqslant 1/\sqrt{2}$.

A.19. Find the adjoints of the following operators:

(i) $A : \ell_2 \to \ell_2$, defined by $A(x_1, x_2, \ldots) = (0, \frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)$;
and, on a general Hilbert space H:

(ii) The rank-one operator R, defined by $Rx = \langle x, y \rangle z$, where y and z are fixed elements of H;

(iii) The projection operator P_M, defined by $P_M(m + n) = m$, where $m \in M$ and $n \in M^{\perp}$, and $H = M \oplus M^{\perp}$ as usual.

A.20. Let $U \in B(H)$ be a unitary operator. Show that (Ue_n) is an orthonormal basis of H whenever (e_n) is.

Let $\ell_2(\mathbb{Z})$ denote the Hilbert space of two-sided sequences $(a_n)_{n=-\infty}^{\infty}$ with

$$\|(a_n)\|^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty.$$

Show that the *bilateral right shift*, $V : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ defined by $V((a_n)) = (b_n)$, where $b_n = a_{n-1}$ for all $n \in \mathbb{Z}$, is unitary, whereas the usual right shift S on $\ell_2 = \ell_2(\mathbb{N})$ is not unitary.

A.5. **Tutorial Problems V.**

A.21. Let $f \in C[-\pi, \pi]$ and let M_f be the multiplication operator on $L_2(-\pi, \pi)$, given by $(M_f g)(t) = f(t)g(t)$, for $g \in L_2(-\pi, \pi)$. Find a function $\tilde{f} \in C[-\pi, \pi]$ such that $M_f^* = M_{\tilde{f}}$.

Show that M_f is always a normal operator. When is it Hermitian? When is it unitary?

A.22. Let T be any operator such that $T^n = 0$ for some integer n (such operators are called *nilpotent*). Show that $I - T$ is invertible (hint: consider $I + T + T^2 + \ldots + T^{n-1}$). Deduce that $I - T/\lambda$ is invertible for any $\lambda \neq 0$.

What is $\sigma(T)$? What is $r(T)$?

A.23. Let (λ_n) be a fixed bounded sequence of complex numbers, and define an operator on ℓ_2 by $T((x_n)) = ((y_n))$, where $y_n = \lambda_n x_n$ for each n. Recall that T is a bounded operator and $\|T\| = \|(\lambda_n)\|_\infty$. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$. Prove the following:

(i) Each λ_k is an eigenvalue of T, and hence is in $\sigma(T)$.
(ii) If $\lambda \notin \overline{\Lambda}$, then the inverse of $T - \lambda I$ exists (and is bounded).

Deduce that $\sigma(T) = \overline{\Lambda}$. Note, that then *any non-empty compact set could be a spectrum of some bounden operator*.

A.24. Let S be an *isomorphism* between Hilbert spaces H and K, that is, $S : H \to K$ is a linear bijection such that S and S^{-1} are bounded operators. Suppose that $T \in B(H)$. Show that T and STS^{-1} have the same spectrum and the same eigenvalues (if any).

A.25. Define an operator $U : \ell_2(\mathbb{Z}) \to L_2(-\pi, \pi)$ by $U((a_n)) = \sum_{n=-\infty}^{\infty} a_n e^{int}/\sqrt{2\pi}$. Show that U is a bijection and an isometry, i.e., that $\|Ux\| = \|x\|$ for all $x \in \ell_2(\mathbb{Z})$.

Let V be the bilateral right shift on $\ell_2(\mathbb{Z})$, the unitary operator defined on Question A.20. Let $f \in L_2(-\pi, \pi)$. Show that $(UVU^{-1}f)(t) = e^{it}f(t)$, and hence, using Question A.24, show that $\sigma(V) = \mathbb{T}$, the unit circle, but that V has no eigenvalues.

A.6. **Tutorial Problems VI.**

A.26. Show that $K(X)$ is a closed linear subspace of $B(X)$, and that AT and TA are compact whenever $T \in K(X)$ and $A \in B(X)$. (This means that $K(X)$ is a closed ideal of $B(X)$.)

A.27. Let A be a Hilbert–Schmidt operator, and let $(e_n)_{n\geqslant 1}$ and $(f_m)_{m\geqslant 1}$ be orthonormal bases of A. By writing each Ae_n as $Ae_n = \sum_{m=1}^{\infty} \langle Ae_n, f_m\rangle f_m$, show that

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{m=1}^{\infty} \|A^* f_m\|^2.$$

Deduce that the quantity $\|A\|_{HS}^2 = \sum_{n=1}^{\infty} \|Ae_n\|^2$ is independent of the choice of orthonormal basis, and that $\|A\|_{HS} = \|A^*\|_{HS}$. ($\|A\|_{HS}$ is called the *Hilbert–Schmidt norm* of A.)

A.28. (i) Let $T \in K(H)$ be a compact operator. Using Question A.26, show that T^*T and TT^* are compact Hermitian operators.

(ii) Let $(e_n)_{n\geqslant 1}$ and $(f_n)_{n\geqslant 1}$ be orthonormal bases of a Hilbert space H, let $(\alpha_n)_{n\geqslant 1}$ be any bounded complex sequence, and let $T \in B(H)$ be an operator defined by

$$Tx = \sum_{n=1}^{\infty} \alpha_n \langle x, e_n\rangle f_n.$$

Prove that T is Hilbert–Schmidt precisely when $(\alpha_n) \in \ell_2$. Show that T is a compact operator if and only if $\alpha_n \to 0$, and in this case write down spectral decompositions for the compact Hermitian operators T^*T and TT^*.

A.29. Solve the Fredholm integral equation $\phi - \lambda T\phi = f$, where $f(x) = x$ and

$$(T\phi)(x) = \int_0^1 xy^2\phi(y)\,dy \qquad (\phi \in L_2(0,1)),$$

for small values of λ by means of the Neumann series.

For what values of λ does the series converge? Write down a solution which is valid for all λ apart from one exception. What is the exception?

A.30. Suppose that h is a 2π-periodic $L_2(-\pi,\pi)$ function with Fourier series $\sum\limits_{n=-\infty}^{\infty} a_n e^{int}$. Show that each of the functions $\phi_k(y) = e^{iky}$, $k \in \mathbb{Z}$, is an eigenvector of the integral operator T on $L_2(-\pi,\pi)$ defined by

$$(T\phi)(x) = \int_{-\pi}^{\pi} h(x-y)\phi(y)\,dy,$$

and calculate the corresponding eigenvalues.

Now let $h(t) = -\log(2(1-\cos t))$. Assuming, without proof, that $h(t)$ has the Fourier series $\sum_{n\in\mathbb{Z}, n\neq 0} e^{int}/|n|$, use the Hilbert–Schmidt method to solve the Fredholm equation $\phi - \lambda T\phi = f$, where $f(t)$ has Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{int}$ and $1/\lambda \notin \sigma(T)$.

A.7. **Tutorial Problems VII.**

A.31. Use the Gram–Schmidt algorithm to find an orthonormal basis for the subspace X of $L_2(-1,1)$ spanned by the functions t, t^2 and t^4.

Hence find the best $L_2(-1,1)$ approximation of the constant function $f(t) = 1$ by functions from X.

A.32. For $n = 1, 2, \ldots$ let ϕ_n denote the linear functional on ℓ_2 defined by

$$\phi_n(x) = x_1 + x_2 + \ldots + x_n,$$

where $x = (x_1, x_2, \ldots) \in \ell_2$. Use the Riesz–Fréchet theorem to calculate $\|\phi_n\|$.

A.33. Let T be a bounded linear operator on a Hilbert space, and suppose that $T = A + iB$, where A and B are self-adjoint operators. Express T^* in terms of A and B, and hence solve for A and B in terms of T and T^*.

Deduce that every operator T can be written $T = A + iB$, where A and B are self-adjoint, in a unique way.

Show that T is normal if and only if $AB = BA$.

A.34. Let P_n be the subspace of $L_2(-\pi, \pi)$ consisting of all polynomials of degree at most n, and let T_n be the subspace consisting of all trigonometric polynomials of the form $f(t) = \sum_{k=-n}^{n} a_k e^{ikt}$. Calculate the spectrum of the differentiation operator D, defined by $(Df)(t) = f'(t)$, when

(i) D is regarded as an operator on P_n, and
(ii) D is regarded as an operator on T_n.

Note that both P_n and T_n are finite-dimensional Hilbert spaces.

Show that T_n has an orthonormal basis of eigenvectors of D, whereas P_n does not.

A.35. Use the Neumann series to solve the Volterra integral equation $\phi - \lambda T\phi = f$ in $L_2[0, 1]$, where $\lambda \in \mathbb{C}$, $f(t) = 1$ for all t, and $(T\phi)(x) = \int_0^x t^2\phi(t)\,dt$. (You should be able to sum the infinite series.)

Appendix B. Solutions of Tutorial Problems

Solutions of the tutorial problems will be distributed due in time on the paper.

APPENDIX C. COURSE IN THE NUTSHELL

C.1. **Some useful results and formulae (1).**

C.1. A *norm* on a vector space, $\|x\|$, satisfies $\|x\| \geqslant 0$, $\|x\| = 0$ if and only if $x = 0$, $\|\lambda x\| = |\lambda| \, \|x\|$, and $\|x+y\| \leqslant \|x\| + \|y\|$ (*triangle inequality*). A norm defines a metric and a complete normed space is called a *Banach space*.

C.2. An *inner-product space* is a vector space (usually complex) with a scalar product on it, $\langle x, y\rangle \in \mathbb{C}$ such that $\langle x, y\rangle = \overline{\langle y, x\rangle}$, $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$, $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, $\langle x, x\rangle \geqslant 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$. This defines a norm by $\|x\|^2 = \langle x, x\rangle$. A complete inner-product space is called a *Hilbert space*. A Hilbert space is automatically a Banach space.

C.3. The *Cauchy–Schwarz inequality*. $|\langle x, y\rangle| \leqslant \|x\| \, \|y\|$ with equality if and only if x and y are linearly dependent.

C.4. Some *examples of Hilbert spaces*. (i) Euclidean $\mathbb{C}^n$. (ii) ℓ_2, sequences (a_k) with $\|(a_k)\|_2^2 = \sum |a_k|^2 < \infty$. In both cases $\langle (a_k), (b_k)\rangle = \sum a_k \overline{b_k}$. (iii) $L_2[a, b]$, functions on $[a, b]$ with $\|f\|_2^2 = \int_a^b |f(t)|^2 \, dt < \infty$. Here $\langle f, g\rangle = \int_a^b f(t)\overline{g(t)} \, dt$. (iv) Any closed subspace of a Hilbert space.

C.5. Other *examples of Banach spaces*. (i) $C_b(X)$, continuous bounded functions on a topological space X. (ii) $\ell_\infty(X)$, all bounded functions on a set X. The supremum norms on $C_b(X)$ and $\ell_\infty(X)$ make them into Banach spaces. (iii) Any closed subspace of a Banach space.

C.6. On *incomplete spaces*. The inner-product (L_2) norm on $C[0, 1]$ is incomplete. c_{00} (sequences eventually zero), with the ℓ_2 norm, is another incomplete i.p.s.

C.7. The *parallelogram identity*. $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ in an inner-product space. Not in general normed spaces.

C.8. On *subspaces*. Complete $\Longrightarrow$ closed. The closure of a linear subspace is still a linear subspace. $\mathrm{Lin}\,(A)$ is the smallest subspace containing A and $\mathrm{CLin}\,(A)$ is its closure, the smallest closed subspace containing A.

C.9. *From now on we work in inner-product spaces.*

C.10. The *orthogonality*. $x \perp y$ if $\langle x, y\rangle = 0$. An *orthogonal sequence* has $\langle e_n, e_m\rangle = 0$ for $n \neq m$. If all the vectors have norm 1 it is an *orthonormal sequence (o.n.s.)*, e.g. $e_n = (0, \ldots, 0, 1, 0, 0, \ldots) \in \ell_2$ and $e_n(t) = (1/\sqrt{2\pi})e^{int}$ in $L_2(-\pi, \pi)$.

C.11. *Pythagoras's theorem*: if $x \perp y$ then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

C.12. The *best approximation* to x by a linear combination $\sum_{k=1}^n \lambda_k e_k$ is $\sum_{k=1}^n \langle x, e_k\rangle e_k$ if the e_k are orthonormal. Note that $\langle x, e_k\rangle$ is the Fourier coefficient of x w.r.t. e_k.

C.13. *Bessel's inequality*. $\|x\|^2 \geqslant \sum_{k=1}^n |\langle x, e_k\rangle|^2$ if $e_1, \ldots, e_n$ is an o.n.s.

C.14. *Riesz–Fischer theorem*. For an o.n.s. (e_n) in a Hilbert space, $\sum \lambda_n e_n$ converges if and only if $\sum |\lambda_n|^2 < \infty$; then $\| \sum \lambda_n e_n\|^2 = \sum |\lambda_n|^2$.

C.15. A *complete o.n.s.* or *orthonormal basis (o.n.b.)* is an o.n.s. (e_n) such that if $\langle y, e_n\rangle = 0$ for all n then $y = 0$. In that case every vector is of the form $\sum \lambda_n e_n$ as in the R-F theorem. Equivalently: the closed linear span of the (e_n) is the whole space.

C.16. *Gram–Schmidt orthonormalization process*. Start with $x_1, x_2, \ldots$ linearly independent. Construct $e_1, e_2, \ldots$ an o.n.s. by inductively setting $y_{n+1} = x_{n+1} - \sum_{k=1}^{n} \langle x_{n+1}, e_k\rangle e_k$ and then normalizing $e_{n+1} = y_{n+1}/\|y_{n+1}\|$.

C.17. On *orthogonal complements*. $M^\perp$ is the set of all vectors orthogonal to everything in M. If M is a closed linear subspace of a Hilbert space H then $H = M \oplus M^\perp$. There is also a linear map, P_M the projection from H onto M with kernel $M^\perp$.

C.18. *Fourier series*. Work in $L_2(-\pi, \pi)$ with o.n.s. $e_n(t) = (1/\sqrt{2\pi})e^{int}$. Let $CP(-\pi, \pi)$ be the continuous periodic functions, which are dense in L_2. For $f \in CP(-\pi, \pi)$ write $f_m = \sum_{n=-m}^{m} \langle f, e_n\rangle e_n$, $m \geqslant 0$. We wish to show that $\|f_m - f\|_2 \to 0$, i.e., that (e_n) is an o.n.b.

C.19. The *Fejér kernel*. For $f \in CP(-\pi, \pi)$ write $F_m = (f_0 + \ldots + f_m)/(m + 1)$. Then $F_m(x) = (1/2\pi) \int_{-\pi}^{\pi} f(t)K_m(x - t)\, dt$ where $K_m(t) = (1/(m + 1)) \sum_{k=0}^{m} \sum_{n=-k}^{k} e^{int}$ is the Fejér kernel. Also $K_m(t) = (1/(m + 1))[\sin^2 (m + 1)t/2]/[\sin^2 t/2]$.

C.20. *Fejér's theorem*. If $f \in CP(-\pi, \pi)$ then its Fejér sums tend uniformly to f on $[-\pi, \pi]$ and hence in L_2 norm also. Hence $\mathrm{CLin}\,((e_n)) \supseteq CP(-\pi, \pi)$ so must be all of $L_2(-\pi, \pi)$. Thus (e_n) is an o.n.b.

C.21. *Corollary*. If $f \in L_2(-\pi, \pi)$ then $f(t) = \sum c_n e^{int}$ with convergence in L_2, where $c_n = (1/2\pi) \int_{-\pi}^{\pi} f(t)e^{-int}\, dt$.

C.22. *Parseval's formula*. If f, $g \in L_2(-\pi, \pi)$ have Fourier series $\sum c_n e^{int}$ and $\sum d_n e^{int}$ then $(1/2\pi)\langle f, g\rangle = \sum c_n \bar{d}_n$.

C.23. *Weierstrass approximation theorem*. The polynomials are dense in $C[a, b]$ for any $a < b$ (in the supremum norm).

C.2. **Some useful results and formulae (2).**

C.24. On *dual spaces*. A *linear functional* on a vector space X is a linear mapping $\alpha : X \to \mathbb{C}$ (or to $\mathbb{R}$ in the real case), i.e., $\alpha(ax + by) = a\alpha(x) + b\alpha(y)$. When X is a normed space, α is continuous if and only if it is *bounded*, i.e., $\sup\{|\alpha(x)| : \|x\| \leqslant 1\} < \infty$. Then we define $\|\alpha\|$ to be this sup, and it is a norm on the space X^* of bounded linear functionals, making X^* into a Banach space.

C.25. *Riesz–Fréchet theorem.* If $\alpha : H \to \mathbb{C}$ is a bounded linear functional on a Hilbert space H, then there is a unique $y \in H$ such that $\alpha(x) = \langle x, y \rangle$ for all $x \in H$; also $\|\alpha\| = \|y\|$.

C.26. On *linear operator*. These are linear mappings $T : X \to Y$, between normed spaces. Defining $\|T\| = \sup\{\|T(x)\| : \|x\| \leqslant 1\}$, finite, makes the bounded (i.e., continuous) operators into a normed space, $B(X, Y)$. When Y is complete, so is $B(X, Y)$. We get $\|Tx\| \leqslant \|T\| \, \|x\|$, and, when we can compose operators, $\|ST\| \leqslant \|S\| \, \|T\|$. Write $B(X)$ for $B(X, X)$, and for $T \in B(X)$, $\|T^n\| \leqslant \|T\|^n$. *Inverse* $S = T^{-1}$ when $ST = TS = I$.

C.27. On *adjoints*. $T \in B(H, K)$ determines $T^* \in B(K, H)$ such that $\langle Th, k \rangle_K = \langle h, T^*k \rangle_H$ for all $h \in H$, $k \in K$. Also $\|T^*\| = \|T\|$ and $T^{**} = T$.

C.28. On *unitary operator*. Those $U \in B(H)$ for which $UU^* = U^*U = I$. Equivalently, U is surjective and an isometry (and hence preserves the inner product).

Hermitian operator or *self-adjoint operator*. Those $T \in B(H)$ such that $T = T^*$.

On *normal operator*. Those $T \in B(H)$ such that $TT^* = T^*T$ (so including Hermitian and unitary operators).

C.29. On *spectrum*. $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not invertible in } B(X)\}$. Includes all *eigenvalues* λ where $Tx = \lambda x$ for some $x \neq 0$, and often other things as well. On *spectral radius:* $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. Properties: $\sigma(T)$ is closed, bounded and nonempty. Proof: based on the fact that $(I - A)$ is invertible for $\|A\| < 1$. This implies that $r(T) \leqslant \|T\|$.

C.30. The *spectral radius formula*. $r(T) = \inf_{n \geqslant 1} \|T^n\|^{1/n} = \lim_{n \to \infty} \|T^n\|^{1/n}$.

Note that $\sigma(T^n) = \{\lambda^n : \lambda \in \sigma(T)\}$ and $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$. The spectrum of a unitary operator is contained in $\{|z| = 1\}$, and the spectrum of a self-adjoint operator is real (proof by *Cayley transform*: $U = (T - iI)(T + iI)^{-1}$ is unitary).

C.31. On *finite rank operator*. $T \in F(X, Y)$ if $\operatorname{Im} T$ is finite-dimensional.

On *compact operator*. $T \in K(X, Y)$ if: whenever (x_n) is bounded, then (Tx_n) has a convergent subsequence. Now $F(X, Y) \subseteq K(X, Y)$ since bounded sequences in a finite-dimensional space have convergent subsequences (because when Z is f.d., Z is isomorphic to ℓ_2^n, i.e., $\exists S : \ell_2^n \to Z$ with S, S^{-1} bounded). Also limits of compact operators are compact, which shows that a diagonal operator $Tx = \sum \lambda_n \langle x, e_n \rangle e_n$ is compact iff $\lambda_n \to 0$.

C.32. *Hilbert–Schmidt operators*. T is H–S when $\sum \|Te_n\|^2 < \infty$ for some o.n.b. (e_n). All such operators are compact—write them as a limit of finite rank operators T_k with $T_k \sum_{n=1}^{\infty} a_n e_n = \sum_{n=1}^{k} a_n (Te_n)$. This class includes integral operators $T : L_2(a, b) \to L_2(a, b)$ of the form

$$(Tf)(x) = \int_a^b K(x, y) f(y) \, dy,$$

where K is continuous on $[a, b] \times [a, b]$.

C.33. On *spectral properties of normal operators*. If T is normal, then (i) $\ker T = \ker T^*$, so $Tx = \lambda x \Longrightarrow T^* x = \bar{\lambda} x$; (ii) eigenvectors corresponding to distinct eigenvalues are orthogonal; (iii) $\|T\| = r(T)$.

If $T \in B(H)$ is compact normal, then its set of eigenvalues is either finite or a sequence tending to zero. The eigenspaces are finite-dimensional, except possibly for $\lambda = 0$. All nonzero points of the spectrum are eigenvalues.

C.34. On *spectral theorem for compact normal operators*. There is an orthonormal sequence (e_k) of eigenvectors of T, and eigenvalues (λ_k), such that $Tx = \sum_k \lambda_k \langle x, e_k \rangle e_k$. If (λ_k) is an infinite sequence, then it tends to 0. All operators of the above form are compact and normal.

Corollary. In the spectral theorem we can have the same formula with an orthonormal *basis*, adding in vectors from $\ker T$.

C.35. On *general compact operators*. We can write $Tx = \sum \mu_k \langle x, e_k \rangle f_k$, where (e_k) and (f_k) are orthonormal sequences and (μ_k) is either a finite sequence or an infinite sequence tending to 0. *Hence* $T \in B(H)$ *is compact if and only if it is the norm limit of a sequence of finite-rank operators.*

C.36. On *integral equations*. Fredholm equations on $L_2(a, b)$ are $T\phi = f$ or $\phi - \lambda T\phi = f$, where $(T\phi)(x) = \int_a^b K(x, y)\phi(y)\, dy$. Volterra equations similar, except that T is now defined by $(T\phi)(x) = \int_a^x K(x, y)\phi(y)\, dy$.

C.37. *Neumann series*. $(I - \lambda T)^{-1} = 1 + \lambda T + \lambda^2 T^2 + \ldots$, for $\|\lambda T\| < 1$.

On *separable kernel*. $K(x, y) = \sum_{j=1}^{n} g_j(x) h_j(y)$. The image of T (and hence its eigenvectors for $\lambda \neq 0$) lies in the space spanned by $g_1, \ldots, g_n$.

C.38. *Hilbert–Schmidt theory*. Suppose that $K \in C([a, b] \times [a, b])$ and $K(y, x) = \overline{K(x, y)}$. Then (in the Fredholm case) T is a self-adjoint Hilbert–Schmidt operator and eigenvectors corresponding to nonzero eigenvalues are continuous functions. If $\lambda \neq 0$ and $1/\lambda \notin \sigma(T)$, the the solution of $\phi - \lambda T\phi = f$ is

$$\phi = \sum_{k=1}^{\infty} \frac{\langle f, \nu_k \rangle}{1 - \lambda \lambda_k} \nu_k.$$

C.39. *Fredholm alternative*. Let T be compact and normal and $\lambda \neq 0$. Consider the equations (i) $\phi - \lambda T\phi = 0$ and (ii) $\phi - \lambda T\phi = f$. Then EITHER (A) The only solution of (i) is $\phi = 0$ and (ii) has a unique solution for all f OR (B) (i) has nonzero solutions ϕ and (ii) can be solved if and only if f is orthogonal to every solution of (i).

Appendix D. Supplementary Sections

D.1. **Reminder from Complex Analysis.** The analytic function theory is the most powerful tool in the operator theory. Here we briefly recall few facts of complex analysis used in this course. Use any decent textbook on complex variables for a concise exposition. The only difference with our version that we consider function $f(z)$ of a complex variable z *taking value in an arbitrary normed space* V over the field $\mathbb{C}$. By the direct inspection we could check that all standard proofs of the listed results work as well in this more general case.

Definition D.1. A function $f(z)$ of a complex variable z taking value in a normed vector space V is called *differentiable* at a point z_0 if the following limit (called *derivative* of $f(z)$ at z_0) exists:

(D.1)
$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Definition D.2. A function $f(z)$ is called *holomorphic* (or *analytic*) in an open set $\Omega \subset \mathbb{C}$ it is differentiable at any point of Ω.

Theorem D.3 (Laurent Series)**.** *Let a function* $f(z)$ *be analytical in the annulus* $r < z < R$ *for some real* $r < R$*, then it could be uniquely represented by the Laurent series:*

(D.2)
$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k, \qquad \text{for some } c_k \in V.$$

Theorem D.4 (Cauchy–Hadamard)**.** *The radii* r' *and* R'*,* $(r' < R')$ *of convergence of the Laurent series* (D.2) *are given by*

(D.3)
$$r' = \liminf_{n\to\infty} \|c_n\|^{1/n} \qquad \text{and} \qquad \frac{1}{R'} = \limsup_{n\to\infty} \|c_n\|^{1/n}.$$

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