Wavelets in Applied and Pure Mathematics

Vladimir V. Kisil

ABSTRACT. The course gives an overview of *wavelets* (or *coherent states*) construction and its realisations in applied and pure mathematics. After a short introduction to wavelets based on the representation theory of groups we will consider:

- Spaces of analytic functions with reproducing kernels: the Hardy and the Bergman spaces, etc.;
- The Fock-Segal-Bargmann space and Berezin-Toeplitz quantisation;
- Functional calculus of self-adjoint operators;
- Elements of signal processing.

The variety of applications is essentially grouped just around three groups: the Heisenberg group, $\mathrm{SL}_2(\mathbb{R})$, and $ax + b$ group.

Contents

List of Figures 5

Preface 7

Preface 7

Chapter 1. What Are Wavelets and What Are They Good for? 9
1. Fourier Transform and Bases in Hilbert Spaces 9
2. Complex Analysis and Reproducing Kernels 10
3. Qantum Mechanics and Quantisation 12
4. Signal Prosessing 13
5. Functional Calculus 14
6. Discussion 14

Chapter 2. Groups and Homogeneous Spaces 17
1. Groups and Transformations 17
2. Subgroups and Homogeneous Spaces 19
3. Differentiation on Lie Groups and Lie Algebras 21
4. Integration on Groups 24

Chapter 3. Elements of the Representation Theory 27
1. Representations of Groups 27
2. Decomposition of Representations 29
3. Invariant Operators and Schur's Lemma 31
4. Induced Representations 31

Chapter 4. Wavelets on Groups and Square Integrable Representations 33
1. Wavelet Transform on Groups 33
2. Square Integrable Representations 36

Chapter 5. Wavelets on Homogeneous Spaces: the Segal-Bargmann Space 39
1. Quantum Mechanical Setting 39
2. Fundamentals of Wavelets on Homogeneous Spaces 40
3. Advanced Properties 45

Chapter 6. Wavelets in Banach Spaces and Functional Calculus 47
1. Coherent States for Banach Spaces 47
2. Wavelets in Operator Algebras 53
3. Examples 59
Acknowledgements 65

Bibliography 67

Index 71

List of Figures

1 An example of Windowed Fourier Transform 13

1 The Gaussian function 34

Preface

The purpose of this course is to sketch in ten lectures a huge area related to *wavelets*. There are no precise boundaries of this area: it overlaps with many other subjects in pure mathematics and many applications in physics and engendering. Moreover there are many different approaches to wavelets based on rather different techniques. However this course is not intended to be complete and encyclopedic. Our main goal is to generate an interest in wavelets and to show that much of the theory and applications are related to *groups* and *symmetries*.

The word "wavelets" came to fashion about 15 years ago and is very popular now. On the other hand the notion of wavelets resemble *coherent states* used in quantum mechanics for 75 years already. Future analysis shows that many classic objects (e.g. from complex analysis) known at least from XIX century are essentially wavelets-coherent states too. This indicates that a significance of wavelets is above just a current fashion.

We apply the name "wavelets" to the whole range of related objects to stress their common origin and nature. Meanwhile the common usage of this term is much narrower. Objects called "wavelets" by us usually appear as *coherent states* (CS) in the literature. While commonly "wavelets" are coherent states related to $ax + b$ group.

CHAPTER 1

What Are Wavelets and What Are They Good for?

In this introductory lecture one would only sketch an answer to the above question: even the whole course could not be enough for that. Now we just list several instances of *wavelets* appeared in different areas. All mentioned topics will be considered in greater details in following lectures.

Most of the listed facts should be well known to reader, we are just presenting them in a way highlighting the common structure. Similarities and differences of these instances of wavelets will be discussed in the final section 6.

1. Fourier Transform and Bases in Hilbert Spaces

We start from two basic examples which were at the beginning of harmonic and functional analysis.

1.1. Fourier Series and Basis in Hilbert Space. Consider the space $L_2[-\pi,\pi]$ of square integrable functions on $[-\pi,\pi]$ with the Lebesgue measure. It is a *Hilbert space* with a scalar product

$$\langle f_1, f_2\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\bar{f}_2(x)\,dx. \tag{1.1}$$

Let us introduce the set of functions

$$e_0(x)=\frac{1}{2},\ e_{2n}(x)=\cos nx,\ e_{2n-11}=\sin nx, \qquad \text{where} \qquad n\in\mathbb{N}. \tag{1.2}$$

It is a straightforward calculation that

$$\langle e_i, e_j\rangle = \delta_{i-j}, \tag{1.3}$$

where δ_{i-j} is the Kronecker delta. Moreover the set (1.2) is a maximal family of functions in $L_2[-\pi,\pi]$ with the above property. In fact (1.3) could be taken as a definition of *orthonormal base* in a Hilbert space H.

It worths to state main properties of the family (1.2) in the generality of an arbitrary orthonormal base. For such a base e_j the following is true [31, § III.5.1]:

- A base e_i define a *linear continuous mapping* $\mathcal{W}: H \to \ell_2(Z)$ of a vector $f \in H$ to a sequence of coefficients in $\ell_2(Z)$ by the formula:

$$\hat{f}_n = \langle f, e_n\rangle\,. \tag{1.4}$$

- The above mapping is an *isometry* of the Hilbert spaces (the *Parseval(-Pythagoras) identity*):

$$\langle f, f'\rangle = \sum_{j=-\infty}^{\infty} \hat{f}_j\overline{\hat{f}'_j}. \tag{1.5}$$

- The mapping $\mathcal{W}$ could be inverted by an operator $\mathcal{M} : \ell_2(Z) \to H$, namely we could reconstruct a vector f from its sequence of coefficients as a linear combination of e_j:

$$f = \sum_{j=-\infty}^{\infty} f_j e_j. \tag{1.6}$$

- From the above we could define a reproducing operator $P = \mathcal{M}\mathcal{W}$ on H, symbolically written by the Dirac *bra and ket* notations:

$$P = \sum_{j=-\infty}^{\infty} |e_j\rangle \langle e_j| .$$

1.2. Fourier Transform. It is useful to compare the above properties of the Fourier series with the Fourier integral transform. The later is defined in $L_2(\mathbb{R})$ with the scalar product

$$\langle f_1, f_2 \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(t)\bar{f}_2(t)\, dt. \tag{1.7}$$

by means of functions:

$$e_a(t) = e^{iat}, \qquad a \in \mathbb{R}. \tag{1.8}$$

A replacement of the orthonormal property (1.3) is the following identity (cf. (1.10)):

$$\langle e_a, e_b \rangle = \delta(a - b), \tag{1.9}$$

where $\delta(a-b)$ is the Dirac delta function and the identity is true in the sense of distributions [31, § III.4.4].

The following is true [31, § IV.2.3]:

- Functions e_a define a *linear continuous mapping* $\mathcal{W} : L_2(Z) \to L_2(Z)$ by the formula:

$$\hat{f}(a) = \langle f, e_a \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-iat}\, dt.$$

- The above mapping is an *isometry* of the Hilbert spaces (the *Plancherel identity*):

$$\langle f, f' \rangle = \left\langle \hat{f}, \hat{f}' \right\rangle .$$

- The mapping $\mathcal{W}$ could be inverted by an operator $\mathcal{M} : L_2(Z) \to L_2(Z)$, i.e. we could reconstruct a function f from its Fourier transform as a continuous linear combination of $e_a(t)$:

$$f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(a) e^{iat}\, da.$$

- The composition of the above two operators $P = \mathcal{M}\mathcal{W}$ gives an integral resolution (in the distributional sense) of the Dirac delta function $\delta(u-t)$:

$$f(u) = \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} e^{i(u-t)a}\, da\, dt. \tag{1.10}$$

2. Complex Analysis and Reproducing Kernels

We move to the classic Hilbert spaces in complex analysis which are examples of wavelets in pure mathematics. Particularly the first example named after G.H. Hardy, probably the purest mathematician of all times and nations.

2.1. The Hardy Space. Let $H_2(\mathbb{T})$ be the *Hardy space* of L_2 functions on the *unit circle* $\mathbb{T}$ with an analytic continuation inside the *unit disk* $\mathbb{D}$. The scalar product is defined as follows:

$$\langle f, f' \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \bar{f}'(t)\, dt. \tag{2.1}$$

We could consider a set of functions in $H_2(\mathbb{T})$ parametrised by a point a of $\mathbb{D}$:

$$e_a(t) = \frac{1}{\bar{a}e^{it} - 1}. \tag{2.2}$$

Then we could find similarly to cases of the Fourier series and integral that:

- Functions $e_a(t)$ define a *linear continuous mapping* $\mathcal{W} : H_2(\mathbb{T}) \to H_2(\mathbb{D})$ of a function on $\mathbb{T}$ to an analytic function in $\mathbb{D}$:

$$\begin{aligned} \hat{f}(a) = [\mathcal{W}f](a) &= \langle f, e_a \rangle \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \overline{\left(\frac{1}{\bar{a}e^{it} - 1} \right)}\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(t)}{a - e^{it}} ie^{it}\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(t)}{a - z}\, dz, \end{aligned} \tag{2.3}$$

  This is the *Cauchy integral formula*, of course.
- The above mapping is an *isometry* of the Hilbert spaces $H_2(\mathbb{T})$ and $H_2(\mathbb{D})$, where the scalar product on $H_2(\mathbb{D})$ defined as usual:

$$\langle f, f' \rangle = \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) \bar{f}'(re^{it})\, dt. \tag{2.4}$$

- The mapping $\mathcal{W}$ could be inverted by an operator $\mathcal{M} : H_2(\mathbb{D}) \to H_2(\mathbb{T})$, with a very simple definition:

$$f(t) = \lim_{r \to 1} \hat{f}(re^{it}). \tag{2.5}$$

- From the above we could define a reproducing operator $P = \mathcal{M}\mathcal{W}$ on $H_2(\mathbb{T})$, which is essentially the *Szegö singular integral operator*. Considered on $L_2(\mathbb{T})$ the operator P is an orthogonal projection on its closed subspace $H_2(\mathbb{T})$.

2.2. The Bergman Space. We consider the *Bergman space* in a way very similar to the Hardy space above. Let $L_2(\mathbb{D})$ be the space of square integrable function on $\mathbb{D}$. There is a closed linear subspace—the Bergman space $B_2(\mathbb{D})$—of analytic functions in $L_2(\mathbb{D})$. We define a family of functions

$$e_a(z) = \frac{1}{(\bar{a}z - 1)^2}, \qquad a \in \mathbb{D}.$$

- Functions $e_a(z)$ define a *linear continuous mapping* $\mathcal{W} : L_2(\mathbb{D}) \to B_2(\mathbb{D})$ of a square integrable function on $\mathbb{D}$ to an analytic function in $\mathbb{D}$:

$$\begin{aligned} \hat{f}(a) = [\mathcal{W}f](a) &= \langle f, e_a \rangle \\ &= \int_{\mathbb{T}} \frac{f(t)}{(a\bar{z} - 1)^2}\, dz, \end{aligned} \tag{2.6}$$

- The above mapping is an *isometry* of the Hilbert spaces if restricted to $B_2(\mathbb{D}) \subset L_2(\mathbb{D})$, in fact it is the identity operator on $B_2(\mathbb{D})$.

- Consequently $\mathcal{W}$ could be trivially "inverted" by the identity operator $\mathcal{M} : B_2(\mathbb{D}) \to B_2(\mathbb{D})$.
- It is follows from the above that the operator $P = \mathcal{M}\mathcal{W} = \mathcal{M} = \mathcal{W}$ is reproducing on $B_2(\mathbb{D})$ and is orthogonal projection $L_2(\mathbb{D})$ onto $B_2(\mathbb{D})$. This is the *Bergman projection*.

REMARK 2.1. In both cases of the Hardy and the Bergman spaces we meet orthogonal projection P from the spaces of square integrable functions onto their subspace of analytic functions. Let M_b be a (bounded) operator on L_2 of multiplication by a bounded function b. It is easy to see that for any such b the *Töplitz operator* $T_b = PM_b$ is a bounded operator on the subspace of analytical functions. We will link later such operators with the wavelet theory.

3. Qantum Mechanics and Quantisation

Now we turn to the object which combines the beauty of the mentioned above classic spaces of complex analysis and importance in applied area of quantum mechanics. As in case of the Fourier integral we start from $L_2(\mathbb{R})$ with the scalar product (1.7). Let us consider the family of functions:

$$e_z(t) = e^{-(\bar{z}^2+t^2)/2+\sqrt{2}\bar{z}t}, \qquad t \in \mathbb{R}, \quad z \in \mathbb{C}.$$

Note that $e_0(t) = e^{-t^2/2}$ is the celebrated Gaussian shown on Figure 1. All other functions obtained from it by horizontal shifts and multiplication by a function e^{ipt} which takes value on the unit circle in $\mathbb{C}$. In quantum mechanical language the function $e_z(t)$ with $z = q + ip$ describes a state of a particle with an expectation of its coordinate equal to q, an expectation of its momentum—p, and the minimal value of product of coordinate and momentum dispersions [24, § 1.3]. We will discuss a physical meaning in details letter on.

We again find a similar structure:

- Functions $e_z(t)$ define a *linear continuous mapping* $\mathcal{W} : L_2(\mathbb{R}) \to SB_2(\mathbb{C})$ of square integrable function $f(t)$ on $\mathbb{R}$ to an analytic function in $\mathbb{C}$:

$$\begin{aligned}\hat{f}(z) = [\mathcal{W}f](z) &= \langle f, e_z \rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-(z^2+t^2)/2+\sqrt{2}zt}\, dt.\end{aligned} \tag{3.1}$$

  Such analytic functions are square integrable on $\mathbb{C}$ with respect to the Gaussian measure $d\beta(z) = e^{-|z|^2} dz$ and form *Segal-Bargmann space* $SB_2(\mathbb{C})$.
- The above mapping is an *isometry* of the Hilbert spaces $L_2(\mathbb{R})$ and $SB_2(\mathbb{C})$, where the scalar product on $SB_2(\mathbb{C})$ defined as follows:

$$\langle f, f' \rangle = \int_{\mathbb{C}} f(z) \bar{f}'(z)\, d\beta(z). \tag{3.2}$$

- The mapping $\mathcal{W}$ could be inverted by an operator $\mathcal{M} : SB_2(\mathbb{C}) \to L_2(\mathbb{R})$ such that the original functions is a linear combination of $e_z(t)$:

$$f(t) = \int_{\mathbb{C}} \hat{f}(z) e^{-(\bar{z}^2+t^2)/2+\sqrt{2}\bar{z}t}\, d\beta(z). \tag{3.3}$$

- From the above we could define a reproducing operator $P = \mathcal{M}\mathcal{W}$ on $L_2(\mathbb{R})$ and $P' = \mathcal{W}\mathcal{M}$ on $SB_2(\mathbb{C})$. The former gives yet another integral resolution of the delta function, cf. the Fourier integral case. The later is *Segal-Bargmann projection*. Considered on $L_2(\mathbb{C}, d\beta(z))$ the operator P' is an orthogonal projection on its closed subspace $SB_2(\mathbb{C})$. We again could consider Töplitz operator of the form $T_b = P'M_b$ for a bounded function b, cf. Remark 2.1.

4. Signal Prosessing

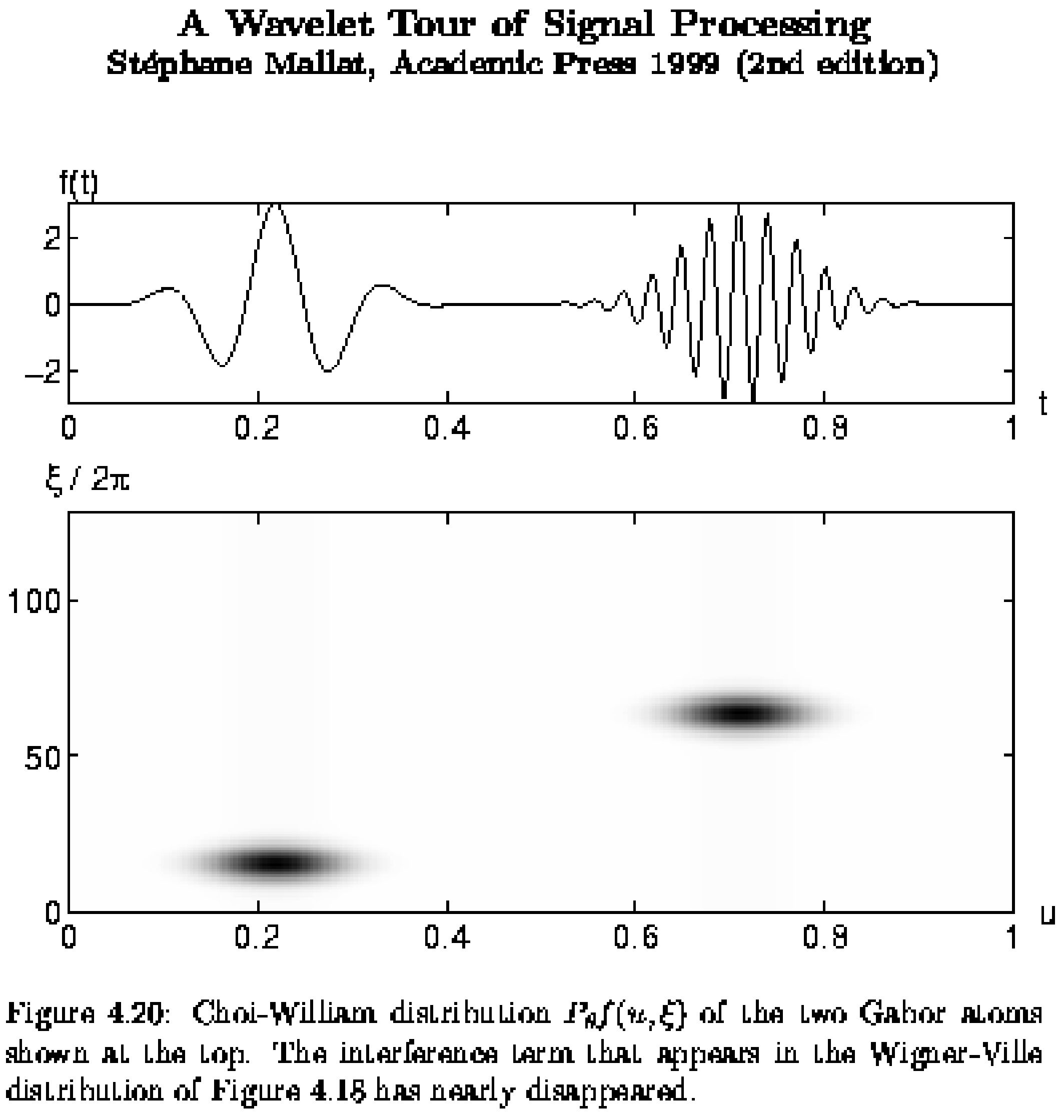

FIGURE 1. An example of Windowed Fourier Transform

The Fourier series and integral appeared as a tool for decomposition of an arbitrary oscillation (or signals) into a superposition of harmonic oscillations with a fixed frequencies. This technique is quite successful in the cases then spectrum of frequencies is independent from time or changes very slowly. But in many common situation like music, speech, etc. this is not true and the Fourier transformation is out of help.

To improve performance it is useful to introduce *Windowed Fourier Transform* (WFT). It analyses the spectrum of frequence of not entire signal but only a part "seen" through a small windows. The position and size of the windows are among parameters of WFT. An example of such a transformation is shown on Figure 1 which is taken from the book [40], it is also instructional to view other pictures from this book on-line.

The word "wavelets" is commonly attributed to the area of signal processing. Decompositions of that type are of huge importance in *signal processing* and are under active investigation. We will discuss this topic in details due to course.

5. Functional Calculus

In the above consideration we oftenly meet a decomposition of an arbitrary function in to linear superposition of elementary ones, cf. (1.6), (2.6), (3.3). Because functions are used as models for operators such formulas could be employed for constructions of functional calculi. Particularly the Cauchy integral formula (2.2) inspires the *Riesz-Dunford functional calculus* defined by the integral formula:

$$f(A) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(t)}{A - z}\, dz, \tag{5.1}$$

for an operator A.

6. Discussion

The above consideration could rise many questions. We list now our answers to some of them:

- *Why is there a common pattern in the above different examples?*
  Opinions vary. Our feeling that the common structure related to the symmetries. In each of the above case there is a group (or even several groups) which is represented by transformations in the function spaces. The groups are:

the Fourier series	the group of integers $\mathbb{Z}$;
the Fourier integral	the group of reals $\mathbb{R}$;
the Hardy space	the $SL_2(\mathbb{R})$ group;
the Bergman space	the $SL_2(\mathbb{R})$ group;
the Segal-Bargmann space	the Heisenberg group $\mathbb{H}^1$;
the signal processing	the $ax + b$ group.

  For example all functions a_x, which are essentially *wavelets* or *coherent states*, could be obtained from the function e_0 (*mother wavelet* or *vacuum vector*) by means of the corresponding group.
- *Why are there significant differences?* (e.g. in the Hardy space the inverse operator $\mathcal{M}$ (2.5) is not defined as an integral)
  The above group are different with different properties, therefore pictures generated by them even within a common scheme could significant differences. Even the same group could have representations with very different properties. For example we will see later that the same group $SL_2(\mathbb{R})$ group could generate analytic function theories of "elliptic" and "hyperbolic" types. By the way the mentioned operator $\mathcal{M}$ (2.5) *could be* expressed as integral similar to the scalar product (2.4).
- *Do groups provide the ultimate explanations in the above examples?*
  Probably not. One could expect the "ultimate explanation" only in a very simple situation and we hope that the above examples are more complicated and consequently interesting. But group do explain many fundamental properties of the mentioned objects and allow to put many different cases within a common framework (cf. with the Erlangen program of F. Klein ;-).

- *In section 2 we meet the Cauchy integral formula. Are wavelets related to other objects of complex analysis (Cauchy-Riemann equation, Laplacian, Taylor and Lorant expansion, etc.)?*
  Yes. We will see it later. For the moment we will mention that the Taylor expansion is a close relative of the Fourier series from the range of our examples.
- *Are groups useful in classification of known types of wavelets or they could help to discover new one?*
  We already mentioned above few new objects derived from the group approach: hyperbolic complex analysis and new types of functional calculi of operators.

The following lecture should give answer to more questions. But before we could proceed we will need a short overview of the representation theory.

CHAPTER 2

Groups and Homogeneous Spaces

Group theory and representation theory are themselves two enormous and interesting subjects. However, they are auxiliary in our presentation and we are forced to restrict our consideration to a brief overview.

Besides introduction to that areas presented in [41, 55] we recommend additionally the books [30, 54]. The representation theory intensively uses tools of functional analysis and on the other hand inspires its future development. We use the book [31] for references on functional analysis here and recommend it as a nice reading too.

1. Groups and Transformations

We start from the definition of the central object, which formalises the universal notion of symmetries [30, § 2.1].

DEFINITION 1.1. A *transformation group* G is a non-void set of mappings of a certain set X into itself with the following properties:

(i) The identical map is included in G.
(ii) If $g_1 \in G$ and $g_2 \in G$ then $g_1 g_2 \in G$.
(iii) If $g \in G$ then g^{-1} exists and belongs to G.

EXERCISE 1.2. List all transformation groups on a set of three elements.

EXERCISE 1.3. Verify that the following sets are transformation groups:

(i) The group of permutations of n elements.
(ii) The group of rotations of the unit circle $\mathbb{T}$.
(iii) The groups of shifts of the real line $\mathbb{R}$ and the plane $\mathbb{R}^2$.
(iv) The group of one-to-one linear maps of an n-dimensional vector space over a field $\mathbb{F}$ onto itself.
(v) The group of linear-fractional (Möbius) transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}, \tag{1.1}$$

of the extended complex plane such that $ad - bc \neq 0$.

It is worth (and often done) to push abstraction one level higher and to keep the group alone without the underlying space:

DEFINITION 1.4. An *abstract group* (or simply *group*) is a non-void set G on which there is a law of *group multiplication* (i.e. mapping $G \times G \to G$) with the properties:

(i) *Associativity*: $g_1(g_2 g_3) = (g_1 g_2) g_3$.
(ii) The existence of the *identity*: $e \in G$ such that $eg = ge = g$ for all $g \in G$.
(iii) The existence of the *inverse*: for every $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

EXERCISE 1.5. Check that

(i) any transformation group is an abstract group; and

(ii) any abstract group is isomorphic to a transformation group. HINT: Use the action of the abstract group on itself by *left* (or *right*the *right shift*) shifts from Exercise .⋄

If we forget the nature of the elements of a transformation group G as transformations of a set X then we need to supply a separate "multiplication table" for elements of G. By the previous Exercise both concepts are mathematically equivalent. However, an advantage of a transition to abstract groups is that the same abstract group can act by transformations of apparently different sets.

EXERCISE 1.6. Check that the following transformation groups (cf. Example 1.3) have the same law of multiplication, i.e. are equivalent as abstract groups:

(i) The group of isometric mapping of an equilateral triangle onto itself.
(ii) The group of all permutations of a set of three elements.
(iii) The group of invertible matrices of order 2 with coefficients in the field of integers modulo 2.
(iv) The group of linear fractional transformations of the extended complex plane generated by the mappings $z \mapsto z^{-1}$ and $z \mapsto 1 - z$.

HINT: Recall that linear fractional transformations are represented by matrices (1.1). Furthermore, a linear fractional transformation is completely defined by the images of any three different points (say, 0, 1 and ∞), see Exercise **??**. What are images of 0, 1 and ∞ under the maps specified in 1.6(iv)?⋄

EXERCISE* 1.7. Expand the list in the above exercise.

It is much simpler to study groups with the following additional property.

DEFINITION 1.8. A group G is *commutative* (or *abelian*) if, for all $g_1, g_2 \in G$, we have $g_1 g_2 = g_2 g_1$.

However, most of the interesting and important groups are non-commutative.

EXERCISE 1.9. Which groups among those listed in Exercises 1.2 and 1.3 are commutative?

Groups may have some additional analytical structures, e.g. they can be a topological space with a corresponding notion of limit and respective continuity. We also assume that our topological groups are always *locally compact* [30, § 2.4], that is there exists a compact neighbourhood of every point. It is common to assume that the topological and group structures are in agreement:

DEFINITION 1.10. If, for a group G, group multiplication and inversion are continuous mappings, then G is *continuous group*.

EXERCISE 1.11. (i) Describe topologies which make groups from Exercises 1.2 and 1.3 continuous.
(ii) Show that a continuous group is locally compact if there exists a compact neighbourhood of its identity.

An even better structure can be found among *Lie groups* [30, § 6], e.g. groups with a differentiable law of multiplication. In the investigation of such groups, we could employ the whole arsenal of analytical tools. Hereafter, most of the groups studied will be Lie groups.

EXERCISE 1.12. Check that the following are non-commutative Lie (and, thus, continuous) groups:

(i) The $ax + b$ *group* (or the *affine group*) [54, Ch. 7] of the real line: the set of elements (a, b), $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ with the group law:

$$(a, b) * (a', b') = (aa', ab' + b).$$

The identity is $(1,0)$ and $(a,b)^{-1} = (a^{-1}, -b/a)$.

(ii) The *Heisenberg group* $\mathbb{H}^1$ [Ch. 1]MTaylor86; 25; ?[]: a set of triples of real numbers (s,x,y) with the group multiplication:

$$(s,x,y) * (s',x',y') = (s+s'+\frac{1}{2}(x'y-xy'), x+x', y+y'). \tag{1.2}$$

The identity is $(0,0,0)$ and $(s,x,y)^{-1} = (-s,-x,-y)$.

(iii) The $\mathrm{SL}_2(\mathbb{R})$ *group* [27,38]: a set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries $a,b,c,d \in \mathbb{R}$ and the determinant $\det = ad-bc$ equal to 1 and the group law coinciding with matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}.$$

The identity is the unit matrix and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The above three groups are behind many important results of real and complex analysis [25,27,38,?Kisil06a] and we meet them many times later.

2. Subgroups and Homogeneous Spaces

A study of any mathematical object is facilitated by a decomposition into smaller or simpler blocks. In the case of groups, we need the following:

DEFINITION 2.1. A *subgroup* of a group G is subset $H \subset G$ such that the restriction of multiplication from G to H makes H a group itself.

EXERCISE 2.2. Show that the $ax+b$ group is a subgroup of $\mathrm{SL}_2(\mathbb{R})$.

HINT: Consider matrices $\frac{1}{\sqrt{a}}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.$\diamond$

While abstract groups are a suitable language for investigation of their general properties, we meet groups in applications as transformation groups acting on a set X. We will describe the connections between those two viewpoints. It can be approached either by having a homogeneous space build the class of isotropy subgroups or by having a subgroup define respective homogeneous spaces. The next two subsections explore both directions in detail.

2.1. From a Homogeneous Space to the Isotropy Subgroup. Let X be a set and let us define, for a group G, an operation $G : X \to X$ of G on X. We say that a subset $S \subset X$ is G-invariant if $g\cdot s \in S$ for all $g \in G$ and $s \in S$.

EXERCISE 2.3. Show that if $S \subset X$ is G-invariant then its complement $X \setminus S$ is G-invariant as well.

Thus, if X has a non-trivial invariant subset, we can split X into disjoint parts. The finest such decomposition is obtained from the following equivalence relation on X, say, $x_1 \sim x_2$, if and only if there exists $g \in G$ such that $gx_1 = x_2$, with respect to which X is a disjoint union of distinct *orbits* [37, § I.5], that is subsets of all gx_0 with a fixed $x_0 \in X$ and arbitrary $g \in G$.

EXERCISE 2.4. Let the group $\mathrm{SL}_2(\mathbb{R})$ act on $\mathbb{C}$ by means of linear-fractional transformations (1.1). Show that there exist three orbits: the real axis $\mathbb{R}$, the *upper* $\mathbb{R}^2_+$ and *lower* $\mathbb{R}^2_-$ half-planes:

$$\mathbb{R}^2_+ = \{x \pm iy \mid x,y \in \mathbb{R},\ y>0\} \quad\text{and}\quad \mathbb{R}^2_- = \{x \pm iy \mid x,y\in\mathbb{R},\ y<0\}.$$

Thus, from now on, without loss of generality, we assume that the action of G on X is *transitive*, i.e. for every $x \in X$ we have

$$Gx := \bigcup_{g \in G} gx = X.$$

In this case, X is G-*homogeneous space*.

EXERCISE 2.5. Show that either of the following conditions define a transitive action of G on X:

(i) For two arbitrary points $x_1, x_2 \in X$, there exists $g \in G$ such that $gx_1 = x_2$.
(ii) There is a point $x_0 \in X$ with the property that for an arbitrary point $x \in X$, there exists $g \in G$ such that $gx_0 = x$.

EXERCISE 2.6. Show that, for any group G, we can define its action on $X = G$ as follows:

(i) The *conjugation* $g : x \mapsto gxg^{-1}$.
(ii) The *left shift* $\Lambda(g) : x \mapsto gx$ and the *right shift* $R(g) : x \mapsto xg^{-1}$.

The above actions define group homomorphisms from G to the transformation group of G. However, the conjugation is trivial for all commutative groups.

EXERCISE 2.7. Show that:

(i) The set of elements $G_x = \{g \in G \mid gx = x\}$ for a fixed point $x \in X$ forms a subgroup of G, which is called the *isotropy (sub)group* of x in G [37, § I.5].
(ii) For any $x_1, x_2 \in X$, isotropy subgroups G_{x_1} and G_{x_2} are *conjugated*, that is, there exists $g \in G$ such that $G_{x_1} = g^{-1} G_{x_2} g$.

This provides a transition from a G-action on a homogeneous space X to a subgroup of G, or even to an equivalence class of such subgroups under conjugation.

EXERCISE 2.8. Find a subgroup which corresponds to the given action of G on X:

(i) Action of $ax + b$ group on $\mathbb{R}$ by the formula: $(a, b) : x \mapsto ax + b$ for the point $x = 0$.
(ii) Action of $SL_2(\mathbb{R})$ group on one of three orbit from Exercise 2.4 with respective points $x = 0$, i and $-i$.

2.2. From a Subgroup to the Homogeneous Space. We can also go in the opposite direction—given a subgroup of G, find the corresponding homogeneous space. Let G be a group and H be its subgroup. Let us define the space of *cosets* $X = G/H$ by the equivalence relation: $g_1 \sim g_2$ if there exists $h \in H$ such that $g_1 = g_2 h$.

There is an important type of subgroups:

DEFINITION 2.8.1. A subgroup H of a group G is said to be *normal* if H is invariant under conjugation, that is $g^{-1}hg \in H$ for all $g \in G$, $h \in H$.

The special role of normal subgroups is explained by the following property:

EXERCISE 2.8.2. Check that, the binary operation $g_1H \cdot g_2H = (g_1g_2)H$, where $g_1, g_2 \in G$, is well-defined on $X = G/H$. Furthermore, this operation turns X into a group, called the *quotient group*.

In our studies normal subgroup will not appear and the set $X = G/H$ will not be a group. However, for any subgroup $H \subset G$ the set $X = G/H$ is a homogeneous space under the left G-action $g : g_1H \mapsto (gg_1)H$. For practical purposes it is more convenient to have a parametrisation of X and express the above G-action through those parameters, as shown below.

We define a function (*section*) [30, § 13.2] $s : X \to G$ such that it is a right inverse to the natural projection $\mathbf{p} : G \to G/H$, i.e. $p(s(x)) = x$ for all $x \in X$. Depending on situation some additional properties of s may be required, e.g. continuity. In our work we will usually need only that the section s is a measurable function.

EXERCISE 2.9. Check that, for any $g \in G$, we have $s(p(g)) = gh$, for some $h \in H$ depending on g.

Then, any $g \in G$ has a unique decomposition of the form $g = s(x)h$, where $x = p(g) \in X$ and $h \in H$. We define a map r associated to s through the identities:

$$x = p(g), \qquad h = r(g) := s(x)^{-1}g.$$

EXERCISE 2.10. Show that:

(i) X is a left G-space with the G-action defined in terms of maps s and p as follows:

$$g : x \mapsto g \cdot x = p(g * s(x)), \tag{2.1}$$

where $*$ is the multiplication on G. This is illustrated by the diagram:

$$\begin{tikzcd} G \arrow[r, "g*"] \arrow[d, shift left, "p"] & G \arrow[d, shift left, "p"] \\ X \arrow[r, "g\cdot"] \arrow[u, shift left, "s"] & X \arrow[u, shift left, "s"] \end{tikzcd} \tag{2.2}$$

(ii) The above action of $G : X \to X$ is transitive on X, thus X is a G-homogeneous space.

(iii) The choice of a section s is not essential in the following sense. Let s_1 and s_2 be two maps, such that $p(s_i(x)) = x$ for all $x \in X$, $i = 1, 2$. Then, $p(g * s_1(x)) = p(g * s_2(x))$ for all $g \in G$.

Thus, starting from a subgroup H of a group G, we can define a G-homogeneous space $X = G/H$.

3. Differentiation on Lie Groups and Lie Algebras

To do some analysis on groups, we need suitably-defined basic operations: differentiation and integration.

Differentiation is naturally defined for Lie groups. If G is a Lie group and G_x is its closed subgroup, then the homogeneous space G/G_x considered above is a smooth manifold (and a *loop* as an algebraic object) for every $x \in X$ [30, Thm. 2 in § 6.1]. Therefore, the one-to-one mapping $G/G_x \to X$ from § 2.2 induces a structure of C^∞-manifold on X. Thus, the class $C_0^\infty(X)$ of smooth functions with compact supports on X has the natural definition.

For every Lie group G there is an associated *Lie algebra* $\mathfrak{g}$. This algebra can be realised in many different ways. We will use the following two out of four listed in [30, § 6.3].

3.1. One-parameter Subgroups and Lie Algebras. For the first realisation, we consider a *one-dimensional continuous subgroup* $x(t)$ of G as a group homomorphism of $x : (\mathbb{R}, +) \to G$. For such a homomorphism x, we have $x(s+t) = x(s)x(t)$ and $x(0) = e$.

EXERCISE 3.1. Check that the following subsets of elements parametrised by $t \in \mathbb{R}$ are one-parameter subgroups:

(i) For the affine group: $a(t) = (e^t, 0)$ and $n(t) = (1, t)$.

(ii) For the Heisenberg group $\mathbb{H}^1$:

$$s(t) = (t, 0, 0), \qquad x(t) = (0, t, 0) \quad \text{and} \quad y(t) = (0, 0, t).$$

(iii) For the group $\mathrm{SL}_2(\mathbb{R})$:

$$a(t)=\begin{pmatrix}e^{-t/2} & 0\\ 0 & e^{t/2}\end{pmatrix},\qquad n(t)=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix},\tag{3.1}$$

$$b(t)=\begin{pmatrix}\cosh\frac{t}{2} & \sinh\frac{t}{2}\\ \sinh\frac{t}{2} & \cosh\frac{t}{2}\end{pmatrix},\qquad z(t)=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}.\tag{3.2}$$

The one-parameter subgroup $x(t)$ defines a tangent vector $X=x'(0)$ belonging to the tangent space T_e of G at $e=x(0)$. The Lie algebra $\mathfrak{g}$ can be identified with this *tangent space*. The important *exponential map* $\exp:\mathfrak{g}\rightarrow G$ works in the opposite direction and is defined by $\exp X=x(1)$ in the previous notations. For the case of a matrix group, the exponent map can be explicitly realised through the exponentiation of the matrix representing a tangent vector:

$$\exp(A)=I+A+\frac{A^2}{2}+\frac{A^3}{3!}+\frac{A^4}{4!}+\dots.$$

EXERCISE 3.2. (i) Check that subgroups $a(t)$, $n(t)$, $b(t)$ and $z(t)$ from Exercise 3.1(iii) are generated by the exponent map of the following zero-trace matrices:

$$a(t)=\exp\begin{pmatrix}-\frac{t}{2} & 0\\ 0 & \frac{t}{2}\end{pmatrix},\qquad n(t)=\exp\begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix},\tag{3.3}$$

$$b(t)=\exp\begin{pmatrix}0 & \frac{t}{2}\\ \frac{t}{2} & 0\end{pmatrix},\qquad z(t)=\exp\begin{pmatrix}0 & t\\ -t & 0\end{pmatrix}.\tag{3.4}$$

(ii) Check that for any $g\in\mathrm{SL}_2(\mathbb{R})$ there is a unique (up to a parametrisation) one-parameter subgroup passing g. Alternatively, the identity $e^{tX}=e^{sY}$ for some $X,Y\in\mathfrak{sl}_2$ and $t,s\in\mathbb{R}$ implies $X=uY$ for some $u\in\mathbb{R}$.

3.2. Invariant Vector Fields and Lie Algebras. In the second realisation of the Lie algebra, $\mathfrak{g}$ is identified with the left (right) *invariant vector fields* on the group G, that is, first-order differential operators X defined at every point of G and invariant under the left (right) shifts: $X\Lambda=\Lambda X$ ($XR=RX$). This realisation is particularly usable for a Lie group with an appropriate parametrisation. The following examples describe different techniques for finding such invariant fields.

EXAMPLE 3.3. Let us build left (right) invariant vector fields on G—the $ax+b$ group using the plain definition. Take the basis $\{\partial_a,\partial_b\}$ ($\{-\partial_a,-\partial_b\}$) of the tangent space T_e to G at its identity. We will propagate these vectors to an arbitrary point through the invariance under shifts. That is, to find the value of the invariant field at the point $g=(a,b)$, we

(i) make the left (right) shift by g,
(ii) apply a differential operator from the basis of T_e,
(iii) make the inverse left (right) shift by $g^{-1}=(\frac{1}{a},-\frac{b}{a})$.

Thus, we will obtain the following invariant vector fields:

$$A^l=a\partial_a,\quad N^l=a\partial_b;\qquad\text{and}\qquad A^r=-a\partial_a-b\partial_b,\quad N^r=-\partial_b.\tag{3.5}$$

EXAMPLE 3.4. An alternative calculation for the same Lie algebra can be done as follows. The Jacobians at $g=(a,b)$ of the left and the right shifts

$$\Lambda(u,v):f(a,b)\mapsto f\left(\frac{a}{u},\frac{b-v}{u}\right),\qquad\text{and}\qquad R(u,v):f(a,b)\mapsto f(ua,va+b)$$

by $h=(u,v)$ are:

$$J_\Lambda(h)=\begin{pmatrix}\frac{1}{u} & 0\\ 0 & \frac{1}{u}\end{pmatrix},\qquad\text{and}\qquad J_R(h)=\begin{pmatrix}u & 0\\ v & 1\end{pmatrix}.$$

Then the invariant vector fields are obtained by the transpose of Jacobians:

$$\begin{aligned}\begin{pmatrix}A^l\\ N^l\end{pmatrix} &= J^t_\Lambda(g^{-1})\begin{pmatrix}\partial_a\\ \partial_b\end{pmatrix}=\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}\partial_a\\ \partial_b\end{pmatrix}=\begin{pmatrix}a\partial_a\\ a\partial_b\end{pmatrix}\\ \begin{pmatrix}A^r\\ N^r\end{pmatrix} &= J^t_R(g)\begin{pmatrix}-\partial_a\\ -\partial_b\end{pmatrix}=\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}-\partial_a\\ -\partial_b\end{pmatrix}=\begin{pmatrix}-a\partial_a-b\partial_b\\ -\partial_b\end{pmatrix}\end{aligned}$$

This rule is a very special case of the general theorem on the change of variables in the calculus of *pseudo-differential operators* (PDO), cf. [§ 4.2]Shubin87 Thm. 18.1.17]Hormander85; ? [; ? [].

EXAMPLE 3.5. Finally, we calculate the invariant vector fields on the $ax + b$ group through a connection to the above one-parameter subgroups. The left-invariant vector field corresponding to the subgroup $a(t)$ from Exercise 3.1(i) is obtained through the differentiation of the right action of this subgroup:

$$\begin{aligned}[A^l f](a,b) &= \left.\frac{d}{dt}f((a,b)*(e^t,0))\right|_{t=0} = \left.\frac{d}{dt}f(ae^t,b)\right|_{t=0} = af'_a(a,b),\\ [N^l f](a,b) &= \left.\frac{d}{dt}f((a,b)*(1,t))\right|_{t=0} = \left.\frac{d}{dt}f(a,at+b)\right|_{t=0} = af'_b(a,b).\end{aligned}$$

Similarly, the right-invariant vector fields are obtained by the derivation of the left action:

$$\begin{aligned}[A^r f](a,b) &= \left.\frac{d}{dt}f((e^{-t},0)*(a,b))\right|_{t=0}\\ &= \left.\frac{d}{dt}f(e^{-t}a,e^{-t}b)\right|_{t=0} = -af'_a(a,b)-bf'_b(a,b),\\ [N^r f](a,b) &= \left.\frac{d}{dt}f((1,-t)*(a,b))\right|_{t=0} = \left.\frac{d}{dt}f(a,b-t)\right|_{t=0} = -f'_b(a,b).\end{aligned}$$

EXERCISE 3.6. Use the above techniques to calculate the following left (right) invariant vector fields on the Heisenberg group:

$$S^{l(r)} = \pm\partial_s,\quad X^{l(r)} = \pm\partial_x - \tfrac{1}{2}y\partial_s,\quad Y^{l(r)} = \pm\partial_y + \tfrac{1}{2}x\partial_s. \tag{3.6}$$

3.3. Commutator in Lie Algebras. The important operation in a Lie algebra is a *commutator*. If the Lie algebra of a matrix group is realised by matrices, e.g. Exercise 3.2, then the commutator is defined by the expression $[A,B] = AB-BA$ in terms of the respective matrix operations. If the Lie algebra is realised through left (right) invariant first-order differential operators, then the commutator $[A,B] = AB - BA$ again defines a left (right) invariant first-order operator—an element of the same Lie algebra.

Among the important properties of the commutator are its anti-commutativity ($[A,B] = -[B,A]$) and the *Jacobi identity*

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0. \tag{3.7}$$

EXERCISE 3.7. Check the following commutation relations:

(i) For the Lie algebra (3.5) of the $ax + b$ group

$$[A^{l(r)}, N^{l(r)}] = N^{l(r)}.$$

(ii) For the Lie algebra (3.6) of Heisenberg group

$$[X^{l(r)}, Y^{l(r)}] = S^{l(r)}, \qquad [X^{l(r)}, S^{l(r)}] = [Y^{l(r)}, S^{l(r)}] = 0. \tag{3.8}$$

These are the celebrated *Heisenberg commutation relations*, which are very important in quantum mechanics.

(iii) Denote by A, B and Z the generators of the one-parameter subgroups $a(t)$, $b(t)$ and $z(t)$ in (3.3) and (3.4). The commutation relations in the Lie algebra $\mathfrak{sl}_2$ are

$$[Z, A] = 2B, \qquad [Z, B] = -2A, \qquad [A, B] = -\tfrac{1}{2}Z. \tag{3.9}$$

The procedure from Example 3.5 can also be used to calculate the *derived action* of a G-action on a homogeneous space.

EXAMPLE 3.8. Consider the action of the $ax + b$ group on the real line associated with group's name:

$$(a, b) : x \mapsto ax + b, \qquad x \in \mathbb{R}.$$

Then, the derived action on the real line is:

$$\begin{aligned} [A^d f](x) &= \left.\frac{d}{dt} f(e^{-t}x)\right|_{t=0} = -xf'(x), \\ [N^d f](a, b) &= \left.\frac{d}{dt} f(x - t)\right|_{t=0} = -f'(x). \end{aligned}$$

4. Integration on Groups

In order to perform an integration we need a suitable *measure*. A measure $d\mu$ on X is called (left) *invariant measure* with respect to an operation of G on X if

$$\int_X f(x)\, d\mu(x) = \int_X f(g \cdot x)\, d\mu(x), \quad \text{for all } g \in G,\ f(x) \in C_0^\infty(X). \tag{4.1}$$

EXERCISE 4.1. Show that measure $y^{-2}dy\, dx$ on the upper half-plane $\mathbb{R}^2_+$ is invariant under action from Exercise 2.4.

Left invariant measures on $X = G$ is called the (left) *Haar measure*. It always exists and is uniquely defined up to a scalar multiplier [54, § 0.2]. An equivalent formulation of (4.1) is: G *operates on* $L_2(X, d\mu)$ *by unitary operators*. We will transfer the Haar measure $d\mu$ from G to $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \to G$ and will call it as the *invariant measure on a Lie algebra* $\mathfrak{g}$.

EXERCISE 4.2. Check that the following are Haar measures for corresponding groups:

(i) The *Lebesgue measure* dx on the real line $\mathbb{R}$.
(ii) The Lebesgue measure $d\phi$ on the unit circle $\mathbb{T}$.
(iii) dx/x is a Haar measure on the multiplicative group $\mathbb{R}_+$;
(iv) $dx\, dy/(x^2 + y^2)$ is a Haar measure on the multiplicative group $\mathbb{C} \setminus \{0\}$, with coordinates $z = x + iy$.
(v) $a^{-2}\, da\, db$ and $a^{-1}\, da\, db$ are the left and right invariant measure on $ax+b$ group.
(vi) The Lebesgue measure $ds\, dx\, dy$ of $\mathbb{R}^3$ for the Heisenberg group $\mathbb{H}^1$.

In this notes we assume *all integrations on groups performed over the Haar measures*.

EXERCISE 4.3. Show that invariant measure on a compact group G is finite and thus can be normalised to total measure 1.

The above simple result has surprisingly important consequences for representation theory of compact groups.

DEFINITION 4.4. The left *convolution* $f_1 * f_2$ of two functions $f_1(g)$ and $f_2(g)$ defined on a group G is

$$f_1 * f_2(g) = \int_G f_1(h)\, f_2(h^{-1}g)\, dh$$

EXERCISE 4.5. Let $k(g) \in L_1(G, d\mu)$ and operator K on $L_1(G, d\mu)$ is the left *convolution operator* with k, .i.e. $K : f \mapsto k * f$. Show that K commutes with all right shifts on G.

The following Lemma characterizes *linear subspaces* of $L_1(G, d\mu)$ invariant under shifts in the term of *ideals of convolution algebra* $L_1(G, d\mu)$ and is of the separate interest.

LEMMA 4.6. *A closed linear subspace* H *of* $L_1(G, d\mu)$ *is invariant under left (right) shifts if and only if* H *is a left (right) ideal of the right group convolution algebra* $L_1(G, d\mu)$.

PROOF. Of course we consider only the "right-invariance and right-convolution" case. Then the other three cases are analogous. Let H be a closed linear subspace of $L_1(G, d\mu)$ invariant under right shifts and $k(g) \in H$. We will show the inclusion

$$[f * k]_r(h) = \int_G f(g)k(hg)\, d\mu(g) \in H, \tag{4.2}$$

for any $f \in L_1(G, d\mu)$. Indeed, we can treat integral (4.2) as a limit of sums

$$\sum_{j=1}^{N} f(g_j)k(hg_j)\Delta_j. \tag{4.3}$$

But the last sum is simply a linear combination of vectors $k(hg_j) \in H$ (by the invariance of H) with coefficients $f(g_j)$. Therefore sum (4.3) belongs to H and this is true for integral (4.2) by the closeness of H.

Otherwise, let H be a right ideal in the group convolution algebra $L_1(G, d\mu)$ and let $\phi_j(g) \in L_1(G, d\mu)$ be an approximate unit of the algebra [20, § 13.2], i.e. for any $f \in L_1(G, d\mu)$ we have

$$[\phi_j * f]_r(h) = \int_G \phi_j(g)f(hg)\, d\mu(g) \to f(h), \text{ when } j \to \infty.$$

Then for $k(g) \in H$ and for any $h' \in G$ the right convolution

$$[\phi_j * k]_r(hh') = \int_G \phi_j(g)k(hh'g)\, d\mu(g) = \int_G \phi_j(h'^{-1}g')k(hg')\, d\mu(g'),\ g' = h'g,$$

from the first expression is tensing to $k(hh')$ and from the second one belongs to H (as a right ideal). Again the closeness of H implies $k(hh') \in H$ that proves the assertion. $\square$

CHAPTER 3

Elements of the Representation Theory

1. Representations of Groups

Objects unveil their nature in actions. Groups act on other sets by means of *representations*. A representation of a group G is a group homomorphism of G in a transformation group of a set. It is a fundamental observation that *linear* objects are easer to study. Therefore we begin from linear representations of groups.

DEFINITION 1.1. A linear continuous *representation of a group* G is a continuous function $T(g)$ on G with values in the group of non-degenerate linear continuous transformation in a linear space H (either finite or infinite dimensional) such that $T(g)$ satisfies to the functional identity:

$$T(g_1 g_2) = T(g_1)\, T(g_2). \tag{1.1}$$

REMARK 1.2. If we have a representation of a group G by its action on a set X we can use the following *linearization procedure*. Let us consider a linear space $L(X)$ of functions $X \to \mathbb{C}$ which may be restricted by some additional requirements (e.g. integrability, boundedness, continuity, etc.). There is a natural representation of G on $L(X)$ which produced by its action on X:

$$g : f(x) \mapsto \rho_g f(x) = f(g^{-1} \cdot x), \qquad \text{where } g \in G,\ x \in X. \tag{1.2}$$

Clearly this representation is already linear. However in many practical cases the formula for linearization (1.2) has some additional terms which are required to make it, for example, unitary.

EXERCISE 1.3. Show that $T(g^{-1}) = T^{-1}(g)$ and $T(e) = I$, where I is the identity operator on H.

EXERCISE 1.4. Show that these are linear continuous representations of corresponding groups:

(i) Operators $T(x)$ such that $[T(x)\, f](t) = f(t + x)$ form a representation of $\mathbb{R}$ in $L_2(\mathbb{R})$.

(ii) Operators $T(n)$ such that $T(n)a_k = a_{k+n}$ form a representation of $\mathbb{Z}$ in ℓ_2.

(iii) Operators $T(a, b)$ defined by

$$[T(a, b)\, f](x) = \sqrt{a} f(ax + b), \qquad a \in \mathbb{R}_+,\ b \in \mathbb{R} \tag{1.3}$$

form a representation of $ax + b$ group in $L_2(\mathbb{R})$.

(iv) Operators $T(s, x, y)$ defined by

$$[T(s, x, y)\, f](t) = e^{i(2s - \sqrt{2} y t + xy)} f(t - \sqrt{2} x) \tag{1.4}$$

form *Schrödinger representation* of the Heisenberg group $\mathbb{H}^1$ in $L_2(\mathbb{R})$.

(v) Operators $T(g)$ defined by

$$[T(g)f](t) = \frac{1}{ct + d} f\left(\frac{at + b}{ct + d}\right), \qquad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1.5}$$

form a representation of $SL_2(\mathbb{R})$ in $L_2(\mathbb{R})$.

In the sequel a representation *always means* linear continuous representation. $T(g)$ is an *exact representation* (or *faithful representation* if $T(g) = I$ only for $g = e$. The opposite case when $T(g) = I$ for all $g \in G$ is a *trivial representation*. The space H is *representation space* and in most cases will be a *Hilber space* [31, § III.5]. If dimensionality of H is finite then T is a *finite dimensional representation*, in the opposite case it is *infinite dimensional representation*.

We denote the *scalar product* on H by $\langle\cdot,\cdot\rangle$. Let $\{\mathbf{e}_j\}$ be an (finite or infinite) *orthonormal basis* in H, i.e.

$$\langle \mathbf{e}_j, \mathbf{e}_j\rangle = \delta_{jk},$$

where δ_{jk} is the *Kroneker delta*, and linear span of $\{\mathbf{e}_j\}$ is dense in H.

DEFINITION 1.5. The *matrix elements* $t_{jk}(g)$ of a representation T of a group G (with respect to a basis $\{\mathbf{e}_j\}$ in H) are complex valued functions on G defined by

$$t_{jk}(g) = \langle T(g)\mathbf{e}_j, \mathbf{e}_k\rangle\,. \tag{1.6}$$

EXERCISE 1.6. Show that [55, § 1.1.3]

(i) $T(g)\,\mathbf{e}_k = \sum_j t_{jk}(g)\,\mathbf{e}_j$.
(ii) $t_{jk}(g_1 g_2) = \sum_n t_{jn}(g_1)\,t_{nk}(g_2)$.

It is typical mathematical questions to determine identical objects which may have a different appearance. For representations it is solved in the following definition.

DEFINITION 1.7. Two representations T_1 and T_2 of the same group G in spaces H_1 and H_2 correspondingly are *equivalent representations* if there exist a linear operator $A : H_1 \rightarrow H_2$ with the continuous inverse operator A^{-1} such that:

$$T_2(g) = A\,T_1(g)\,A^{-1}, \qquad \forall g \in G.$$

EXERCISE 1.8. Show that representation $T(a,b)$ of $ax+b$ group in $L_2(\mathbb{R})$ from Exercise 1.4(iii) is equivalent to the representation

$$[T_1(a,b)\,f](x) = \frac{e^{i\frac{b}{a}}}{\sqrt{a}}\,f\left(\frac{x}{a}\right). \tag{1.7}$$

HINT. Use the Fourier transform. □

The *relation of equivalence* is reflexive, symmetric, and transitive. Thus it splits the set of all representations of a group G into *classes of equivalent representations*. In the sequel we study *group representations up to their equivalence classes only*.

EXERCISE 1.9. Show that equivalent representations have the same matrix elements in appropriate basis.

DEFINITION 1.10. Let T be a representation of a group G in a Hilbert space H The *adjoint representation* $T'(g)$ of G in H is defined by

$$T'(g) = \left(T(g^{-1})\right)^*,$$

where * denotes the adjoint operator in H.

EXERCISE 1.11. Show that

(i) T' is indeed a representation.
(ii) $t'_{jk}(g) = \bar{t}_{kj}(g^{-1})$.

Recall [31, § III.5.2] that a bijection $U : H \rightarrow H$ is a *unitary operator* if

$$\langle Ux, Uy\rangle = \langle x, y\rangle\,, \qquad \forall x, y \in H.$$

EXERCISE 1.12. Show that $UU^* = I$.

DEFINITION 1.13. T is a *unitary representation* of a group G in a space H if $T(g)$ is a unitary operator for all $g \in G$. T_1 and T_2 are *unitary equivalent representations* if $T_1 = UT_2U^{-1}$ for a unitary operator U.

EXERCISE 1.14. (i) Show that all representations from Exercises 1.4 are unitary.
(ii) Show that representations from Exercises 1.4(iii) and 1.8 are unitary equivalent.

HINT. Take that the Fourier transform is unitary for granted. □

EXERCISE 1.15. Show that if a Lie group G is represented by unitary operators in H then its Lie algebra $\mathfrak{g}$ is represented by selfadjoint (possibly unbounded) operators in H.

The following definition have a sense for *finite* dimensional representations.

DEFINITION 1.16. A *character of representation* T is equal $\chi(g) = \operatorname{tr}(T(g))$, where tr is the *trace* [31, § III.5.2 (Probl.)] of operator.

EXERCISE 1.17. Show that
(i) Characters of a representation T are constant on the adjoint elements $g^{-1}hg$, for all $g \in G$.
(ii) Character is an algebra homomorphism from an algebra of representations with Kronecker's (tensor) multiplication [55, § 1.9] to complex numbers.

HINT. Use that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$, and $\operatorname{tr}(A \otimes B) = \operatorname{tr} A \, \operatorname{tr} B$. □

For *infinite* dimensional representation characters can be defined either as distributions [30, § 11.2] or in infinitesimal terms of Lie algebras [30, § 11.3].

The characters of a representation should not be confused with the following notion.

DEFINITION 1.18. A *character of a group* G is a one-dimensional representation of G.

EXERCISE 1.19. (i) Let χ be a character of a group G. Show that a character of representation χ coincides with it and thus is a character of G.
(ii) A matrix element of a group character χ coincides with χ.
(iii) Let χ_1 and χ_1 be characters of a group G. Show that $\chi_1 \otimes \chi_2 = \chi_1\chi_2$ and $\chi'(g) = \chi_1(g^{-1})$ are again characters of G. In other words *characters of a group form a group themselves*.

2. Decomposition of Representations

The important part of any mathematical theory is classification theorems on structural properties of objects. Very well known examples are:
(i) The main theorem of arithmetic on unique representation an integer as a product of powers of prime numbers.
(ii) *Jordan's normal form of a matrix*.

The similar structural results in the representation theory are very difficult. The easiest (but still rather difficult) questions are on classification of unitary representations up to unitary equivalence.

DEFINITION 2.1. Let T be a representation of G in H. A linear subspace $L \subset H$ is *invariant subspace* for T if for any $\mathbf{x} \in L$ and any $g \in G$ the vector $T(g)\mathbf{x}$ again belong to L.

There are always two trivial invariant subspaces: the null space and entire H. All other are *non-trivial invariant subspaces*.

DEFINITION 2.2. If there are only two trivial invariant subspaces then T is *irreducible representation*. Otherwise we have *reducible representation*.

For any non-trivial invariant subspace we can define the *restriction of representation* of T on it. In this way we obtain a *subrepresentation* of T.

EXAMPLE 2.3. Let $T(a)$, $a \in \mathbb{R}_+$ be defined as follows: $[T(a)]f(x) = f(ax)$. Then spaces of even and odd functions are invariant.

DEFINITION 2.4. If the closure of liner span of all vectors $T(g)v$ is dense in H then v is called *cyclic vector* for T.

EXERCISE 2.5. Show that for an irreducible representation any non-zero vector is cyclic.

The following important result of representation theory of compact groups is a consequence of the Exercise 4.3 and we state here it without a proof.

THEOREM 2.6. [30, § 9.2]

(i) *Every topologically irreducible representation of a compact group* G *is finite-dimensional and unitarizable.*
(ii) *If* T_1 *and* T_2 *are two inequivalent irreducible representations, then every matrix element of* T_1 *is orthogonal in* $L_2(G)$ *to every matrix element of* T_2.
(iii) *For a compact group* G *its dual space* $\hat{G}$ *is discrete.*

The important property of unitary representation is complete reducibility.

EXERCISE 2.7. Let a unitary representation T has an invariant subspace $L \subset H$, then its orthogonal completion $L^{\perp}$ is also invariant.

DEFINITION 2.8. A representation on H is called decomposable if there are two non-trivial invariant subspaces H_1 and H_2 of H such that $H = H_1 \oplus H_2$.

If a representation is not decomposable then its primary.

THEOREM 2.9. [30, § 8.4] *Any unitary representation* T *of a locally compact group* G *can be decomposed in a (continuous) direct sum irreducible representations:* $T = \int_X T_x \, d\mu(x)$.

The necessity of continuous sums appeared in very simple examples:

EXERCISE 2.10. Let T be a representation of $\mathbb{R}$ in $L_2(\mathbb{R})$ as follows: $[T(a)f](x) = e^{iax}f(x)$. Show that

(i) Any measurable set $E \subset \mathbb{R}$ define an invariant subspace of functions vanishing outside E.
(ii) T does not have invariant irreducible subrepresentations.

DEFINITION 2.11. The set of equivalence classes of unitary irreducible representations of a group G is denoted by $\hat{G}$ and called *dual object* (or *dual space*) of the group G.

DEFINITION 2.12. A *left regular representation* $\Lambda(g)$ of a group G is the representation by left shifts in the space $L_2(G)$ of square-integrable function on G with the left Haar measure

$$\Lambda g : f(h) \mapsto f(g^{-1}h). \tag{2.1}$$

The *main problem of representation theory* is to decompose a left regular representation $\Lambda(g)$ into irreducible components.

3. Invariant Operators and Schur's Lemma

It is a pleasant feature of an abstract theory that we obtain important general statements from simple observations. Finiteness of invariant measure on a compact group is one such example. Another example is Schur's Lemma presented here.

To find different classes of representations we need to compare them each other. This is done by *intertwining operators*.

DEFINITION 3.1. Let T_1 and T_2 are representations of a group G in a spaces H_1 and H_2 correspondingly. An operator $A : H_1 \to H_2$ is called an *intertwining operator* if

$$A\,T_1(g) = T_2(g)\,A, \qquad \forall g \in G.$$

If $T_1 = T_2 = T$ then A is *interntwinig operator* or *commuting operator* for T.

EXERCISE 3.2. Let G, H, $T(g)$, and A be as above. Show the following: [55, § 1.3.1]

(i) Let $\mathbf{x} \in H$ be an eigenvector for A with eigenvalue λ. Then $T(g)\mathbf{x}$ for all $g \in G$ are eigenvectors of A with the same eigenvalue λ.
(ii) All eigenvectors of A with a fixed eigenvalue λ for a linear subspace invariant under all $T(g)$, $g \in G$.
(iii) If an operator A is commuting with irreducible representation T then $A = \lambda I$.

HINT. Use the spectral decomposition of selfadjoint operators [31, § V.2.2]. □

The next result have very important applications.

LEMMA 3.3 (Schur). [30, § 8.2] *If two representations* T_1 *and* T_2 *of a group* G *are irreducible, then every intertwining operator between them is either zero or invertible.*

HINT. Consider subspaces $\ker A \subset H_1$ and $\mathrm{im} A \subset H_2$. □

EXERCISE 3.4. Show that

(i) Two irreducible representations are either equivalent or disjunctive.
(ii) All operators commuting with an irreducible representation form a field.
(iii) Irreducible representation of commutative group are one-dimensional.
(iv) If T is unitary irreducible representation in H and $B(\cdot,\cdot)$ is a bounded semi linear form in H invariant under T: $B(T(g)\mathbf{x}, T(g)\mathbf{y}) = B(\mathbf{x},\mathbf{y})$ then $B(\cdot,\cdot) = \lambda\,\langle\cdot,\cdot\rangle$.

HINT. Use that $B(\cdot,\cdot) = \langle A\cdot,\cdot\rangle$ for some A [31, § III.5.1]. □

4. Induced Representations

The general scheme of induced representations is as follows, see [30, § 13.2; 33, § 3.1; 54, Ch. 5; ?Folland95, Ch. 6] and subsection 2.2. Let G be a group and let H be its subgroup. Let $X = G/H$ be the corresponding left homogeneous space and $s : X \to G$ be a continuous function (section) [30, § 13.2] which is a right inverse to the natural projection $\mathbf{p} : G \to G/H$.

Then any $g \in G$ has a unique decomposition of the form $g = s(x)h^{-1}$ where $x = \mathbf{p}(g) \in X$ and $h \in H$. We define the map $\mathbf{r} : G \to H$:

$$\mathbf{r}(g) = s(x)^{-1}g, \qquad \text{where } x = \mathbf{p}(g). \tag{4.1}$$

Note that X is a left homogeneous space with the G-action defined in terms of $\mathbf{p}$ and s as follows, see Ex. 2.10:

$$g : x \mapsto g \cdot x = \mathbf{p}(g * s(x)), \tag{4.2}$$

where $*$ is the multiplication on G. A useful consequences of the above formulae is:

$$\mathbf{s}(x) = g * \mathbf{s}(y) * (\mathbf{r}(g * \mathbf{s}(y)))^{-1}, \tag{4.3}$$

$$\mathbf{r}(g^{-1} * \mathbf{s}(x)) = \mathbf{r}(g * \mathbf{s}(y)), \qquad \text{where } y = g^{-1} \cdot x \text{ for } x, y \in X \text{ and } g \in G. \tag{4.4}$$

Let $\chi : H \to B(V)$ be a linear representation of H in a vector space V, e.g. by unitary rotations in the algebra of either complex, dual or double numbers. Then χ induces a linear representation of G, which is known as *induced representation* in the sense of Mackey [30, § 13.2]. This representation has the canonical realisation ρ in a space of V-valued functions on X. It is given by the formula (cf. [30, § 13.2.(7)–(9)]):

$$[\rho_\chi(g)f](x) = \chi(\mathbf{r}(g^{-1} * \mathbf{s}(x)))\, f(g^{-1} \cdot x), \tag{4.5}$$

where $g \in G$, $x \in X$, $h \in H$ and $\mathbf{r} : G \to H$, $\mathbf{s} : X \to G$ are maps defined above; $*$ denotes multiplication on G and $\cdot$ denotes the action (4.2) of G on X from the left.

In the case of complex numbers this representation automatically becomes unitary in the space $L_2(X)$ of the functions square integrable with respect to a measure $d\mu$ if instead of the representation χ one uses the following substitute:

$$\chi_0(h) = \chi(h) \left(\frac{d\mu(h \cdot x)}{d\mu(x)} \right)^{\frac{1}{2}}. \tag{4.6}$$

However in our study the unitarity of representations or its proper replacements is a more subtle issue and we will consider it separately.

An alternative construction of induced representations is realised on the space of functions on G which have the following property:

$$F(gh) = \chi(h)F(g), \qquad \text{for all } h \in H. \tag{4.7}$$

This space is invariant under the left shifts. The restriction of the left regular representation to this subspace is equivalent to the induced representation described above.

EXERCISE 4.1. (i) Write the intertwining operator for this equivalence.
(ii) Define the corresponding inner product on the space of functions 4.7 in such a way that the above intertwining operator becomes unitary.

HINT. Use the map $s : X \to G$. □

CHAPTER 4

Wavelets on Groups and Square Integrable Representations

A matured mathematical theory looks like a tree. There is a solid trunk which supports all branches and leaves but could not be alive without them. In the case of group approach to wavelets the trunk of the theory is a construction of wavelets from a *square integrable representation* [**10**], [?AliAntGaz00, Chap. 8]. We begin from this trunk which is a model for many different generalisations and will continue with some smaller "generalising" branches later.

1. Wavelet Transform on Groups

Let G be a group with a left Haar measure $d\mu$ and let ρ be a unitary irreducible representation of a group G by operators ρ_g, $g \in G$ in a Hilbert space H.

DEFINITION 1.1. Let us fix a vector $w_0 \in H$. We call $w_0 \in H$ a *vacuum vector* or a *mother wavelet* (other less-used names are *ground state*, *fiducial vector*, etc.). We will say that set of vectors $w_g = \rho(g)w_0$, $g \in G$ form a family of *coherent states* (*wavelets*).

EXERCISE 1.2. If ρ is irreducible then w_g, $g \in G$ is a *total set* in H, i.e. the linear span of these vectors is dense in H.

The *wavelet transform* can be defined as a mapping from H to a space of functions over G via its *representational coefficients* (also known as *matrix coefficients*):

$$\mathcal{W} : v \mapsto \hat{v}(g) = \left\langle \rho(g^{-1})v, w_0 \right\rangle = \langle v, \rho(g)w_0 \rangle = \langle v, w_g \rangle . \tag{1.1}$$

EXERCISE 1.3. Show that the wavelet transform $\mathcal{W}$ is a continuous linear mapping and the image of a vector is a bounded continuous function on G. The liner space of all such images is denoted by $W(G)$.

EXERCISE 1.4. Let a Hilbert space H has a basis e_j, $j \in \mathbb{Z}$ and a unitary representation ρ of $G = \mathbb{Z}$ defined by $\rho(k)e_j = e_{j+k}$. Write a formula for wavelet transform with $w_0 = e_0$ and characterise $W(\mathbb{Z})$.

ANSWER. $\hat{v}(n) = \langle v, e_n \rangle$. □

EXERCISE 1.5. Let G be $ax + b$ group and ρ is given by (cf. (1.3)):

$$[T(a, b)\, f](x) = \frac{1}{\sqrt{a}} f\left(\frac{x - b}{a}\right), \tag{1.2}$$

in $L_2(\mathbb{R})$. Show that

(i) The representation is reducible and describe its irreducible components.
(ii) for $w_0(x) = \frac{1}{2\pi i (x+i)}$ coherent states are $v_{(a,b)}(x) = \frac{\sqrt{a}}{2\pi i (x-(b-ia))}$.
(iii) Wavelet transform is given by

$$\hat{v}(a, b) = \frac{\sqrt{a}}{2\pi i} \int_{\mathbb{R}} \frac{v(x)}{x - (b + ia)} dx,$$

which resembles the *Cauchy integral formula*.

(iv) Give a characteristic of $W(G)$.
(v) Write the wavelet transform for the same representation of the group $ax+b$ and the *Gaussian* (or *Gauss function*)$e^{-x^2/2}$ (see Fig. 1) as a mother wavelets.

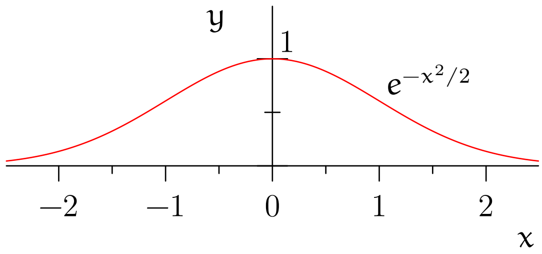

FIGURE 1. The Gaussian function $e^{-x^2/2}$.

PROPOSITION 1.6. *The wavelet transform* $\mathcal{W}$ *intertwines* ρ *and the left regular representation* Λ (2.1) *of* G*:*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}.$$

PROOF. We have:

$$\begin{aligned}[\mathcal{W}(\rho(g)v)](h) &= \left\langle \rho(h^{-1})\rho(g)v, w_0 \right\rangle \\ &= \left\langle \rho((g^{-1}h)^{-1})v, w_0 \right\rangle \\ &= [\mathcal{W}v](g^{-1}h) \\ &= [\Lambda(g)\mathcal{W}v](h).\end{aligned}$$

□

COROLLARY 1.7. *The function space* $W(G)$ *is invariant under the representation* Λ *of* G.

Wavelet transform maps vectors of H to functions on G. We can consider a map in the opposite direction sends a function on G to a vector in H.

DEFINITION 1.8. The *inverse wavelet transform* $\mathcal{M}_{w_0'}$ associated with a vector $w_0' \in H$ maps $L_1(G)$ to H and is given by the formula:

$$\begin{aligned}\mathcal{M}_{w_0'} : L_1(G) \to H : \hat{v}(g) \mapsto \mathcal{M}[\hat{v}(g)] &= \int_G \hat{v}(g) w_g' \, d\mu(g) \\ &= \int_G \hat{v}(g)\rho(g)\, d\mu(g)\, w_0', \qquad (1.3)\end{aligned}$$

where in the last formula the integral express an operator acting on vector w_0'.

EXERCISE 1.9. Write inverse wavelet transforms for Exercises 1.4 and 1.5.

ANSWER. (i) For Exercises 1.4: $v = \sum_{-\infty}^{\infty} \hat{v}(n)e_n$.
(ii) For Exercises 1.5:

$$v(x) = \frac{1}{2\pi i} \int_{\mathbb{R}_+^2} \frac{\hat{v}(a,b)}{x-(b-ia)} \frac{da\, db}{a^{\frac{3}{2}}}.$$

□

LEMMA 1.10. *If the wavelet transform* $\mathcal{W}$ *and inverse wavelet transform* $\mathcal{M}$ *are defined by the same vector* w_0 *then they are adjoint operators:* $\mathcal{W}^* = \mathcal{M}$.

PROOF. We have:

$$\begin{aligned}\langle \mathcal{M}\hat{v}, w_g\rangle &= \left\langle \int_G \hat{v}(g')w_{g'}\, d\mu(g'), w_g \right\rangle \\ &= \int_G \hat{v}(g')\, \langle w_{g'}, w_g\rangle \, d\mu(g') \\ &= \int_G \hat{v}(g')\overline{\langle w_g, w_{g'}\rangle}\, d\mu(g') \\ &= \langle \hat{v}, \mathcal{W}w_g\rangle\,,\end{aligned}$$

where the scalar product in the first line is on H and in the last line is on $L_2(G)$. Now the result follows from the totality of coherent states w_g in H. □

PROPOSITION 1.11. *The inverse wavelet transform $\mathcal{M}$ intertwines the representation Λ (2.1) on $L_2(G)$ and ρ on H:*

$$\mathcal{M}\Lambda(g) = \rho(g)\mathcal{M}.$$

PROOF. We have:

$$\begin{aligned}\mathcal{M}[\Lambda(g)\hat{v}(h)] &= \mathcal{M}[\hat{v}(g^{-1}h)] \\ &= \int_G \hat{v}(g^{-1}h)w_h\, d\mu(h) \\ &= \int_G \hat{v}(h')w'_{gh'}\, d\mu(h') \\ &= \rho(g)\int_G \hat{v}(h')w'_{h'}\, d\mu(h') \\ &= \rho(g)\mathcal{M}[\hat{v}(h')],\end{aligned}$$

where $h' = g^{-1}h$. □

COROLLARY 1.12. *The image $\mathcal{M}(L_1(G)) \subset H$ of subspace under the inverse wavelet transform $\mathcal{M}$ is invariant under the representation ρ.*

An important particular case of such an invariant subspace is Gårding space.

DEFINITION 1.13. Let $C_\infty^0(G)$ be the space of infinitely differentiable functions with compact supports. Then for the given representation ρ in H the Gårding space $\mathcal{G}(\rho) \subset H$ is the image of $C_\infty^0(G)$ under the inverse wavelet transform with all possible reconstruction vectors:

$$\mathcal{G}(\rho) = \{\mathcal{M}_w\phi \mid w \in H, \phi \in C_\infty^0(G)\}.$$

COROLLARY 1.14. *The Gårding space is invariant under the derived representation $d\rho$.*

The following proposition explain the usage of the name "inverse" (not "adjoint" as it could be expected from Lemma 1.10) for $\mathcal{M}$.

THEOREM 1.15. *The operator*

$$\mathcal{P} = \mathcal{M}\mathcal{W} : H \to H \tag{1.4}$$

maps H into its linear subspace for which w'_0 is cyclic. Particularly if ρ is an irreducible representation then P is cI for some constant c depending from w_0 and w'_0.

PROOF. It follows from Propositions 1.6 and 1.11 that operator $\mathcal{M}\mathcal{W} : H \to H$ intertwines ρ with itself. Then Corollaries 1.7 and 1.12 imply that the image $\mathcal{M}\mathcal{W}$ is a ρ-invariant subspace of H containing w_0. From irreducibility of ρ by Schur's Lemma [30, § 8.2] one concludes that $\mathcal{M}\mathcal{W} = cI$ on C for a constant $c \in \mathbb{C}$. □

REMARK 1.16. From Exercises 1.4 and 1.9 it follows that irreducibility of ρ is not necessary for $\mathcal{M}\mathcal{W} = cI$, it is sufficient that w_0 and w_0' are cyclic only.

We have similarly

THEOREM 1.17. *Operator $\mathcal{W}\mathcal{M}$ is up to a complex multiplier a projection of $L_1(G)$ to $W(G)$.*

2. Square Integrable Representations

So far our consideration of wavelets was mainly algebraic. Usually in analysis we wish that the wavelet transform can preserve an analytic structure, e.g. values of scalar product in Hilbert spaces. This accomplished if a representation ρ possesses the following property.

DEFINITION 2.1. [30, § 9.3] Let a group G with a left Haar measure $d\mu$ have a unitary representation $\rho : G \rightarrow \mathcal{L}(H)$. A vector $w \in H$ is called *admissible vector* if the function $\hat{w}(g) = \langle \rho(g)w, w \rangle$ is non-void and square integrable on G with respect to $d\mu$:

$$0 < c^2 = \int_G \langle \rho(g)w, w \rangle \langle w, \rho(g)w \rangle \, d\mu(g) < \infty. \tag{2.1}$$

If an admissible vector exists then ρ is a *square integrable representation*.

Square integrable representations of groups have many interesting properties (see [21, § 14] for unimodular groups and [22], [?AliAntGaz00, Chap. 8] for not unimodular generalisation) which are crucial in the construction of wavelets. For example, for a square integrable representation all functions $\langle \rho(g)v_1, v_2 \rangle$ with an admissible vector v_1 and any $v_2 \in H$ are square integrable on G; such representation belong to *dicrete series*; etc.

EXERCISE 2.2. Show that

(i) Admissible vectors form a linear space.
(ii) For an irreducible ρ the set of admissible vectors is dense in H or empty.

HINT. The set of all admissible vectors is an ρ-invariant subspace of H. □

EXERCISE 2.3. (i) Find a condition for a vector to be admissible for the representation (1.2) (and therefore the representation is square integrable).
(ii) Show that $w_0(x) = \frac{1}{2\pi i(x+i)}$ is admissible for $ax + b$ group.
(iii) Show that the Gaussian e^{-x^2} is *not* admissible for $ax + b$ group.

For an admissible vector w we take its normalisation $w_0 = \frac{\|w\|}{c} w$ to obtain:

$$\int_G |\langle \rho(g)w_0, w_0 \rangle|^2 \, d\mu(g) = \|w_0\|^2 . \tag{2.2}$$

Such a w_0 as a vacuum state produces many useful properties.

PROPOSITION 2.4. *If both wavelet transform $\mathcal{W}$ and inverse wavelet transform $\mathcal{M}$ for an irreducible square integrable representation ρ are defined by the same admissible vector w_0 then the following three statements are equivalent:*

(i) *w_0 satisfy (2.2);*
(ii) *$\mathcal{M}\mathcal{W} = I$;*
(iii) *for any vectors $v_1, v_2 \in H$:*

$$\langle v_1, v_2 \rangle = \int_G \hat{v}_1(g)\overline{\hat{v}_2(g)} \, d\mu(g). \tag{2.3}$$

PROOF. We already knew that $\mathcal{M}\mathcal{W} = cI$ for a constant $c \in \mathbb{C}$. Then (2.2) exactly says that $c = 1$. Because $\mathcal{W}$ and $\mathcal{M}$ are adjoint operators it follows from $\mathcal{M}\mathcal{W} = I$ on H that:

$$\langle v_1, v_2\rangle = \langle \mathcal{M}\mathcal{W}v_1, v_2\rangle = \langle \mathcal{W}v_1, \mathcal{M}^* v_2\rangle = \langle \mathcal{W}v_1, \mathcal{W}v_2\rangle\,,$$

which is exactly the isometry of $\mathcal{W}$ (2.3). Finally condition (2.2) is a partticular case of general isometry of $\mathcal{W}$ for vector w_0. □

EXERCISE 2.5. Write the isometry conditions (2.3) for wavelet transforms for $\mathbb{Z}$ and $ax + b$ groups (Exercises 1.4 and 1.5).

Wavelets from square integrable representation closely related to the following notion:

DEFINITION 2.6. A *reproducing kernel* on a set X with a measure is a function $K(x, y)$ such that:

$$K(x,x) > 0, \quad \forall x \in X, \tag{2.4}$$

$$K(x,y) = \overline{K(y,x)}, \tag{2.5}$$

$$K(x,z) = \int_X K(x,y)K(y,z)\,dy. \tag{2.6}$$

PROPOSITION 2.7. *The image $W(G)$ of the wavelet transform $\mathcal{W}$ has a reproducing kernel $K(g, g') = \langle w_g, w_{g'}\rangle$. The reproducing formula is in fact a convolution:*

$$\begin{aligned}\hat{v}(g') &= \int_G K(g', g)\hat{v}(g)\,d\mu(g)\\ &= \int_G \hat{w}_0(g^{-1}g')\hat{v}(g)\,d\mu(g)\end{aligned} \tag{2.7}$$

with a wavelet transform of the vacuum vector $\hat{w}_0(g) = \langle w_0, \rho(g)w_0\rangle$.

PROOF. Again we have a simple application of the previous formulas:

$$\begin{aligned}\hat{v}(g') &= \left\langle \rho(g'^{-1})v, w_0\right\rangle\\ &= \int_G \left\langle \rho(h^{-1})\rho(g'^{-1})v, w_0\right\rangle \overline{\left\langle \rho(h^{-1})w_0, w_0\right\rangle}\,d\mu(h) \qquad (2.8)\\ &= \int_G \left\langle \rho((g'h)^{-1})v, w_0\right\rangle \left\langle \rho(h)w_0, w_0\right\rangle\,d\mu(h)\\ &= \int_G \hat{v}(g'h)\,\hat{w}_0(h^{-1})\,d\mu(h)\\ &= \int_G \hat{v}(g)\,\hat{w}_0(g^{-1}g')\,d\mu(g),\end{aligned}$$

where transformation (2.8) is due to (2.3). □

EXERCISE 2.8. Write reproducing kernels for wavelet transforms for $\mathbb{Z}$ and $ax + b$ groups (Exercises 1.4 and 1.5.

EXERCISE* 2.9. Operator (2.7) of convolution with $\hat{w}_0$ is an orthogonal projection of $L_2(G)$ onto $W(G)$.

HINT. Use that an left invariant subspace of $L_2(G)$ is in fact an right ideal in convolution algebra, see Lemma 4.6. □

REMARK 2.10. To possess a reproducing kernel—is a well-known property of spaces of analytic functions. The space $W(G)$ shares also another important property of analytic functions: it belongs to a kernel of a certain first order differential operator with Clifford coefficients (the Dirac operator) and a second order operator with scalar coefficients (the Laplace operator) [4, 33–35], which we will consider that later too.

We consider only fundamentals of the wavelet construction here. There are much results which can be stated in an abstract level. To avoid repetition we will formulate it later on together with an interesting examples of applications.

The construction of wavelets from square integrable representations is general and straightforward. However we can not use it everywhere we may wish:

(i) Some important representations are not square integrable.
(ii) Some groups, e.g. $\mathbb{H}^n$, do not have square representations at all.
(iii) Even if representation is square integrable, some important vacuum vectors are not admissible, e.g. the Gaussian e^{-x^2} in 2.3(iii).
(iv) Sometimes we are interested in Banach spaces, while unitary square integrable representations are acting only on Hilbert spaces.

To be vivid the trunk of the wavelets theory should split into several branches adopted to particular cases and we describe some of them in the next lectures.

CHAPTER 5

Wavelets on Homogeneous Spaces: the Segal-Bargmann Space

We investigate a situation when a representation ρ of G is not square integrable in the sense of the previous Lecture but is *square integrable modulo subgroup*. An example which we use for illustration is the classic construction from quantum mechanics and is origin of coherent states.

1. Quantum Mechanical Setting

We begin from a statement of quantum problem [5, 45] which could be naturally solved in the terms of wavelet transform. Mathematical formulation of quantum mechanics could be founded for example in [39], [31, § V.3].

The *states* of a quantum mechanical system of n degrees of freedom (e.g. particle) are usually described by a function from the space $H = L_2(\mathbb{R}^n)$. Depending on a physical interpretation it could be considered either as *configuration space* with real variables $(q_1, q_2, \ldots, q_n)$ describing coordinates of the particle or *momentum space* with real variables $(p_1, p_2, \ldots, p_n)$ describing its momenta. One could move from one description to another by the Fourier transform. Elements of H are also called *wave functions*.

The *observables* of the system are self-adjoint (possibly unbounded) operators on H. The result of *measurement* of an observable A on a state (wave function) ϕ with $\|\phi\| = 1$ is a random distribution with an expectation $\langle A\rangle_\phi$:

$$\langle A\rangle_\phi = \langle A\phi, \phi\rangle$$

EXERCISE 1.1. Let we could find all expectations $\langle A\rangle_\phi$ for a fixed ϕ and any self-adjoint A. Show that we may calculate the random distributions of measurement.

HINT. Use expectation $\left\langle \chi_{[a,b]}(A)\phi, \phi\right\rangle$, where $\chi_{[a,b]}(A)$ is a spectral projection of A on the interval $[a, b]$ [31, § V.1.3] to find a probability that the result of measurement of A on ϕ will be within interval $[a, b]$. □

Among observables there is a special set of $2n$ primary ones: these are n observables of coordinates $q_1, \ldots, q_n$ and n observables of momentum $p_1, \ldots, p_n$. All other observables usually could be expressed by means of primary ones: either as functions [39] or as wavelets [34]. The relation between primary observables are given by the *Heisenberg commutation relations*, i.e. the only non-trivial commutators among them are:

$$[q_j, p_k] = i\hbar\delta_{jk}I. \tag{1.1}$$

EXERCISE 1.2. Check that the Heisenberg commutation relations (1.1) define a representation of the Lie Algebra of the Heisenberg group $\mathbb{H}^n$.

Therefore a realisation of primary observables as self-adjoint operators H is connected with a unitary representation of the Heisenberg group in $L_2(\mathbb{R}^n)$. We already met it, this is the *Schrödinger representation* (1.4):

$$[\rho_\hbar(s,x,y)\,f](q) = e^{i(2\hbar s - \sqrt{2\hbar}xq + \hbar xy)} f(q - \sqrt{2\hbar}y). \tag{1.2}$$

(the representation (1.4) correspond to the case $\hbar = 1$).

The important result is the following theorem which asserts that we know all possible realisation of the Heisenberg commutation relations (1.1).

THEOREM 1.3 (Stone-von Neumann). [30, § 18.4], [54, § 1.2] *All unitary irreducible representations of the Heisenberg group $\mathbb{H}^n$ up to unitary equivalence are as follows*

(i) *For any $\hbar \in (0,\infty)$ the Schrödinger irreducible noncommutative unitary representations in $L_2(\mathbb{R}^n)$*

$$\rho_{\pm\hbar}(s,x,y) = e^{i(\pm s\cdot\hbar I \pm x\cdot\hbar^{1/2}M + y\cdot\hbar^{1/2}D)}, \tag{1.3}$$

where xM and yD are such unbounded self-adjoint operators on $L_2(\mathbb{R}^n)$:

$$(x\cdot\hbar^{1/2}M)u(q) = \hbar^{1/2}\sum x_j q_j u(q), \tag{1.4}$$

$$(y\cdot\hbar^{1/2}D)u(q) = \frac{\hbar^{1/2}}{i}\sum y_j \frac{\partial u(q)}{\partial q_j}. \tag{1.5}$$

(ii) *For $(q,p) \in \mathbb{R}^{2n}$ commutative one-dimensional representations on $\mathbb{C}$:*

$$\rho_{(q,p)}(s,x,y)u = e^{i(qx+py)}u,\ u \in \mathbb{C}. \tag{1.6}$$

Therefore there is essentially unique model for a quantum mechanical particle. Nevertheless it is worthwhile to look for some models which can act as alternatives for the Schrödinger representation. In particular, the Segal-Bargmann representation [5, 45] serves to

- give a realisation of states by "true" functions, not an equivalent classes from $L_2(\mathbb{R}^n$.
- give a geometric representation of the dynamics of the harmonic oscillators;
- present a nice model for the *creation* and *annihilation operator*:

$$a_j^+ = \frac{1}{\sqrt{2}}(q_j + ip_j), \qquad a_j^- = \frac{1}{\sqrt{2}}(q_j - ip_j), \tag{1.7}$$

  which are important for quantum field theory;
- allow applying tools of analytic function theory.

The huge abilities of the Segal-Bargmann (or Fock [23]) model are not yet completely employed, see for example new ideas in a recent paper [42].

Since the Segal-Bargmann model should give a representation $\mathbb{H}^n$ which is unitary equivalent to the Schrödinger one then it is naturally to construct an intertwining operator between them as a wavelet transform.

2. Fundamentals of Wavelets on Homogeneous Spaces

Let G be a group and G_0 be its closed subgroup. Let $X = G/G_0$ be the corresponding homogeneous space with a left invariant measure $d\mu$. Let $s: X \to G$ be a Borel section in the principal bundle $G \to G/g_0$. Let ρ be a continuous representation of a group G by invertible unitary operators $\rho(g)$, $g \in G$ in a Hilbert space H.

For any $g \in G$ there is a unique decomposition of the form $g = s(x)h$, $h \in G_0$, $x \in X$. We will define $r: G \to G_0: r(g) = h = (s^{-1}(g))^{-1}g$ from the previous equality and write a formal notation $x = s^{-1}(g)$. Then there is a geometric action of G on $X \to X$ defined as follows

$$g: x \mapsto g^{-1}\cdot x = s^{-1}(g^{-1}s(x)).$$

EXAMPLE 2.1. As a subgroup G_0 we select now the center of $\mathbb{H}^n$ consisting of elements $(t, 0)$. Of course $X = G/G_0$ isomorphic to $\mathbb{C}^n$ and mapping $s : \mathbb{C}^n \to G$ simply is defined as $s(z) = (0, z)$. The Haar measure on $\mathbb{H}^n$ coincides with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ [54, § 1.1] thus the invariant measure on X also coincides with the Lebesgue measure on $\mathbb{C}^n$. Note also that composition law $s^{-1}(g \cdot s(z))$ reduces to Euclidean shifts on $\mathbb{C}^n$. We also find $s^{-1}((s(z_1))^{-1} \cdot s(z_2)) = z_2 - z_1$ and $r((s(z_1))^{-1} \cdot s(z_2)) = \frac{1}{2}\Im \bar{z}_1 z_2$.

DEFINITION 2.2. Let G, G_0, $X = G/G_0$, $s : X \to G$, $\rho : G \to \mathcal{L}(H)$ be as above. We say that $w_0 \in H$ is a *vacuum vector* if it satisfies to the following two conditions:

$$\rho(h)w_0 = \chi(h)w_0, \qquad \chi(h) \in \mathbb{C}, \quad \text{for all } h \in G_0; \tag{2.1}$$

$$\int_X |\langle w_0, \rho(s(x))w_0\rangle|^2 \, dx = \|w_0\|^2 . \tag{2.2}$$

We will say that set of vectors $w_x = \rho(x)w_0$, $x \in X$ form a family of *coherent states*.

Note that mapping $h \to \chi(h)$ from (2.1) defines a character of the subgroup G_0. The condition (2.2) could be easily achieved by a renormalisation w_0 as soon as we sure that the integral in the left hand side is finite.

CONVENTION 2.3. In that follow we will usually write $x \in X$ and $x^{-1} \in X$ instead of $s(x) \in G$ and $s(x)^{-1} \in G$ correspondingly. The right meaning of "x" could be easily found from the context (whether an element of X or G is expected there).

EXAMPLE 2.4. As a "vacuum vector" we will select the original *vacuum vector of quantum mechanics*—the Gauss function $w_0(q) = e^{-q^2/2}$ (see Figure 1), which belongs to all $L_2(\mathbb{R}^n)$. Its transformations are defined as follow:

$$\begin{aligned} w_g(q) = [\rho_{(s,z)}w_0](q) &= e^{i(2s-\sqrt{2}xq+xy)}\, e^{-(q-\sqrt{2}y)^2/2} \\ &= e^{2is-(x^2+y^2)/2} e^{((x+iy)^2-q^2)/2-\sqrt{2}i(x+iy)q} \\ &= e^{2is-z\bar{z}/2} e^{(z^2-q^2)/2-\sqrt{2}izq} . \end{aligned}$$

Particularly $[\rho_{(t,0)}w_0](q) = e^{-2it}w_0(q)$, i.e., it really is a vacuum vector in the sense of our definition with respect to G_0.

EXERCISE 2.5. Check the square integrability condition (2.2) for $w_0(q) = e^{-q^2/2}$.

The wavelet transform (similarly to the group case) could be defined as a mapping from G_0 to a space of bounded continuous functions over G via representational coefficients

$$v \mapsto \hat{v}(g) = \left\langle \rho(g^{-1})v, w_0 \right\rangle = \langle v, \rho(g)^* w_0 \rangle .$$

Due to (2.1) such functions have simple transformation properties along orbits gG_0, i.e. $\hat{v}(gh) = \bar{\chi}(h)\hat{v}(g)$, $g \in G$, $h \in G_0$. Thus they are completely defined by their values indexed by points of $X = G/G_0$. Therefore we prefer to consider so called reduced wavelet transform.

DEFINITION 2.6. The *reduced wavelet transform* $\mathcal{W}$ from a Hilbert space G_0 to a space of function $W(X)$ on a homogeneous space $X = G/G_0$ defined by a representation ρ of G on G_0, a vacuum vector w_0 is given by the formula

$$\mathcal{W} : H \to W(X) : v \mapsto \hat{v}(x) = [\mathcal{W}v](x) = \left\langle \rho(x^{-1})v, w_0 \right\rangle = \langle v, \rho^*(x)w_0 \rangle . \tag{2.3}$$

EXAMPLE 2.7. The transformation (2.3) with the kernel $[\rho_{(0,z)}w_0](q)$ is an embedding $L_2(\mathbb{R}^n) \to L_2(\mathbb{C}^n)$ and is given by the formula

$$\begin{aligned}
\hat{f}(z) &= \langle f, \rho_{s(z)} f_0 \rangle \\
&= \pi^{-n/4} \int_{\mathbb{R}^n} f(q)\, e^{-z\bar{z}/2}\, e^{-(z^2+q^2)/2+\sqrt{2}zq}\, dq \\
&= e^{-z\bar{z}/2} \pi^{-n/4} \int_{\mathbb{R}^n} f(q)\, e^{-(z^2+q^2)/2+\sqrt{2}zq}\, dq. \qquad (2.4)
\end{aligned}$$

Then $\hat{f}(g)$ belongs to $L_2(\mathbb{C}^n, dg)$ or its preferably to say that function $\breve{f}(z) = e^{z\bar{z}/2}\hat{f}(t_0, z)$ belongs to space $L_2(\mathbb{C}^n, e^{-|z|^2}dg)$ because $\breve{f}(z)$ is analytic in z. Such functions form the *Segal-Bargmann space* $F_2(\mathbb{C}^n, e^{-|z|^2}dg)$ of functions [5, 45], which are analytic by z and square-integrable with respect to the Gaussian measure $e^{-|z|^2}dz$. We use notation $\breve{\mathcal{W}}$ for the mapping $v \mapsto \breve{v}(z) = e^{z\bar{z}/2}\mathcal{W}v$. Analyticity of $\breve{f}(z)$ is equivalent to the condition $(\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}z_j I)\hat{f}(z) = 0$. The integral in (2.4) is the well-known Segal-Bargmann transform [5, 45].

EXERCISE 2.8. Check that $\breve{w}_0(z) = 1$ for the vacuum vector $w_0(q) = e^{-q^2/2}$.

There is a natural representation of G in $W(X)$. It could be obtained if we first lift functions from X to G, apply the left regular representation Λ and then pul them back to X. The result defines a representation $\lambda(g) : W(X) \to W(X)$ as follow

$$[\lambda(g)f](x) = \chi(r(g^{-1} \cdot x))f(g^{-1} \cdot x). \qquad (2.5)$$

We recall that $\chi(h)$ is a character of G_0 defined in (2.1) by the vacuum vector w_0. Of course, for the case of trivial $G_0 = \{e\}$ (2.5) becomes the left regular representation $\Lambda(g)$ of G.

PROPOSITION 2.9. *The reduced wavelet transform* $\mathcal{W}$ *intertwines* ρ *and the representation* λ (2.5) *on* $W(X)$*:*

$$\mathcal{W}\rho(g) = \lambda(g)\mathcal{W}.$$

PROOF. We have with obvious adjustments in comparison with Proposition 1.6:

$$\begin{aligned}
[\mathcal{W}(\rho(g)v)](x) &= \langle \rho(x^{-1})\rho(g)v, w_0 \rangle \\
&= \langle \rho((g^{-1}s(x))^{-1})v, w_0 \rangle \\
&= \langle \rho(r(g^{-1} \cdot x)^{-1})\rho(s(g^{-1} \cdot x)^{-1})v, w_0 \rangle \\
&= \langle \rho(s(g^{-1} \cdot x)^{-1})v, \rho^*(r(g^{-1} \cdot x)^{-1})w_0 \rangle \\
&= \chi(r(g^{-1} \cdot x)^{-1})[\mathcal{W}v](g^{-1}x) \\
&= \lambda(g)[\mathcal{W}v](x).
\end{aligned}$$

□

COROLLARY 2.10. *The function space* $W(X)$ *is invariant under the representation* λ *of* G.

EXAMPLE 2.11. Integral transformation (2.4) intertwines the Schrödinger representation (1.2) with the following realization of representation (2.5):

$$\begin{aligned}
\lambda(s,z)\hat{f}(u) &= \hat{f}_0(z^{-1} \cdot u)\bar{\chi}(s + r(z^{-1} \cdot u)) \\
&= \hat{f}_0(u - z)e^{is + i\Im(\bar{z}u)} \qquad (2.6)
\end{aligned}$$

EXERCISE 2.12. (i) Using relation $\breve{\mathcal{W}} = e^{-|z|^2/2}\mathcal{W}$ derive from above that $\breve{\mathcal{W}}$ intertwines the Schrödinger representation with the following:

$$\breve{\lambda}(s,z)\breve{f}(u) = \breve{f}_0(u - z)e^{2is - \bar{z}u - |z|^2/2}.$$

(ii) Show that infinitesimal generators of representation $\breve{\lambda}$ are:

$$d\breve{\lambda}(s,0,0) = iI, \quad d\breve{\lambda}(0,x,0) = -\partial_u - uI, \quad d\breve{\lambda}(0,0,y) = i(-\partial_z + zI)$$

We again introduce a transform adjoint to $\mathcal{W}$.

DEFINITION 2.13. The *inverse wavelet transform* $\mathcal{M}$ from $W(X)$ to H is given by the formula:

$$\begin{aligned} \mathcal{M} : W(X) \rightarrow H : \hat{v}(x) \mapsto \mathcal{M}[\hat{v}(x)] &= \int_X \hat{v}(x) w_x \, d\mu(x) \\ &= \int_X \hat{v}(x)\rho(x)\, d\mu(x) w_0. \end{aligned} \tag{2.7}$$

PROPOSITION 2.14. *The inverse wavelet transform $\mathcal{M}$ intertwines the representation λ on $W(X)$ and ρ on H:*

$$\mathcal{M}\lambda(g) = \rho(g)\mathcal{M}.$$

PROOF. We have:

$$\begin{aligned} \mathcal{M}[\lambda(g)\hat{v}(x)] &= \mathcal{M}[\chi(r(g^{-1}\cdot x))\hat{v}(g^{-1}\cdot x)] \\ &= \int_X \chi(r(g^{-1}\cdot x))\hat{v}(g^{-1}\cdot x) w_x \, d\mu(x) \\ &= \chi(r(g^{-1}\cdot x)) \int_X \hat{v}(x') w_{g\cdot x'}\, d\mu(x') \\ &= \rho_g \int_X \hat{v}(x') w_{x'}\, d\mu(x') \\ &= \rho_g \mathcal{M}[\hat{v}(x')], \end{aligned}$$

where $x' = g^{-1}\cdot x$. □

COROLLARY 2.15. *The image $\mathcal{M}(W(X)) \subset H$ of subspace $W(X)$ under the inverse wavelet transform $\mathcal{M}$ is invariant under the representation ρ.*

EXAMPLE 2.16. Inverse transformation to (2.4) is given by a realization of (2.7):

$$\begin{aligned} f(q) &= \int_{\mathbb{C}^n} \hat{f}(z) f_{s(z)}(q)\, dz \\ &= \int_{\mathbb{C}^n} \hat{f}(x,y) e^{iy(x-\sqrt{2}y)}\, e^{-(q-\sqrt{2}y)^2/2}\, dx\, dy \\ &= \int_{\mathbb{C}^n} \breve{f}(z) e^{-(\bar{z}^2+q^2)/2+\sqrt{2}\bar{z}q}\, e^{-|z|^2}\, dz. \end{aligned} \tag{2.8}$$

The transformation (2.8) intertwines the representations (2.6) and the Schrödinger representation (1.2) of the Heisenberg group.

The following proposition explain the usage of the name for $\mathcal{M}$.

THEOREM 2.17. *The operator*

$$\mathcal{P} = \mathcal{M}\mathcal{W} : H \rightarrow H \tag{2.9}$$

is a projection of H to its linear subspace for which w_0 is cyclic. Particularly if ρ is an irreducible representation then the inverse wavelet transform $\mathcal{M}$ is a left inverse *operator on H for the wavelet transform $\mathcal{W}$:*

$$\mathcal{M}\mathcal{W} = I.$$

PROOF. It follows from Propositions 2.9 and 2.14 that operator $\mathcal{M}\mathcal{W} : H \rightarrow H$ intertwines ρ with itself. Then Corollaries 2.10 and 2.15 imply that the image $\mathcal{M}\mathcal{W}$ is a ρ-invariant subspace of H containing w_0. Because of $\mathcal{M}\mathcal{W}w_0 = w_0$ we conclude that $\mathcal{M}\mathcal{W}$ is a projection.

From irreducibility of ρ by Schur's Lemma [30, § 8.2] one concludes that $\mathcal{M}\mathcal{W} = cI$ on H for a constant $c \in \mathbb{C}$. Particularly

$$\mathcal{M}\mathcal{W}w_0 = \int_X \left\langle \rho(x^{-1})w_0, w_0 \right\rangle \rho(x)w_0 \, d\mu(x) = cw_0.$$

From the condition (2.2) it follows that $\langle cw_0, w_0 \rangle = \langle \mathcal{M}\mathcal{W}w_0, w_0 \rangle = \langle w_0, w_0 \rangle$ and therefore $c = 1$. □

We have similar

THEOREM 2.18. *Operator $\mathcal{W}\mathcal{M}$ is a projection of $L_1(X)$ to $W(X)$.*

COROLLARY 2.19. *In the space $W(X)$ the strong convergence implies point-wise convergence.*

PROOF. From the definition of the wavelet transform:

$$\left|\hat{f}(x)\right| = |\langle f, \rho(x)w_0 \rangle| \leqslant \|f\| \, \|w_0\| \, .$$

Since the wavelet transform is an isometry we conclude that $\left|\hat{f}(x)\right| \leqslant c\,\|f\|$ for $c = \|w_0\|$, which implies the assertion about two types of convergence. □

EXAMPLE 2.20. The corresponding operator for the Segal-Bargmann space $\mathcal{P}$ (2.9) is an identity operator $L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ and (2.9) gives an integral presentation of the Dirac delta.

While the orthoprojection $L_2(\mathbb{C}^n, e^{-|z|^2} dg) \to F_2(\mathbb{C}^n, e^{-|z|^2} dg)$ is of a separate interest and is a principal ingredient in Berezin quantization [8,15]. We could easy find its kernel from (2.12). Indeed, $\hat{f}_0(z) = e^{-|z|^2}$, then the kernel is

$$\begin{aligned} K(z,w) &= \hat{f}_0(z^{-1} \cdot w)\bar{\chi}(r(z^{-1} \cdot w)) \\ &= \hat{f}_0(w - z)e^{i\mathfrak{I}(\bar{z}w)} \\ &= \exp\left(\frac{1}{2}(-|w-z|^2 + w\bar{z} - z\bar{w})\right) \\ &= \exp\left(\frac{1}{2}(-|z|^2 - |w|^2) + w\bar{z}\right). \end{aligned}$$

To receive the reproducing kernel for functions $\check{f}(z) = e^{|z|^2}\hat{f}(z)$ in the Segal-Bargmann space we should multiply $K(z,w)$ by $e^{(-|z|^2+|w|^2)/2}$ which gives the standard reproducing kernel $= \exp(-|z|^2 + w\bar{z})$ [5, (1.10)].

We denote by $\mathcal{W}^* : W^*(X) \to H$ and $\mathcal{M}^* : H \to W^*(X)$ the adjoint (in the standard sense) operators to $\mathcal{W}$ and $\mathcal{M}$ respectively.

COROLLARY 2.21. *We have the following identity:*

$$\langle \mathcal{W}v, \mathcal{M}^* l \rangle_{W(X)} = \langle v, l \rangle_H \, , \qquad \forall v, l \in H, \tag{2.10}$$

or equivalently

$$\int_X \left\langle \rho(x^{-1})v, w_0 \right\rangle \langle \rho(x)w_0, l \rangle \, d\mu(x) = \langle v, l \rangle \, . \tag{2.11}$$

PROOF. We show the equality in the first form (2.11) (but we will apply it often in the second one):

$$\langle \mathcal{W}v, \mathcal{M}^* l \rangle_{W(X)} = \langle \mathcal{M}\mathcal{W}v, l \rangle_H = \langle v, l \rangle_H \, .$$

□

COROLLARY 2.22. *The space $W(X)$ has the reproducing formula*

$$\hat{v}(y) = \int_X \hat{v}(x)\,\hat{b}_0(x^{-1}\cdot y)\,d\mu(x), \tag{2.12}$$

where $\hat{b}_0(y) = [\mathcal{W}w_0](y)$ is the wavelet transform of the vacuum vector w_0.

PROOF. Again we have a simple application of the previous formulas:

$$\begin{aligned}
\hat{v}(y) &= \left\langle \rho(y^{-1})v, w_0 \right\rangle \\
&= \int_X \left\langle \rho(x^{-1})\rho(y^{-1})v, w_0 \right\rangle \left\langle \rho(x)w_0, w_0 \right\rangle \,d\mu(x) \qquad (2.13)\\
&= \int_X \left\langle \rho(s(y\cdot x)^{-1})v, w_0 \right\rangle \left\langle \rho(x)w_0, w_0 \right\rangle \,d\mu(x) \\
&= \int_X \hat{v}(y\cdot x)\,\hat{b}_0(x^{-1})\,d\mu(x) \\
&= \int_X \hat{v}(x)\,\hat{b}_0(x^{-1}y)\,d\mu(x),
\end{aligned}$$

where transformation (2.13) is due to (2.11). □

3. Advanced Properties

We make the following simple but nice observation about the integral kernel of wavelet transform:

PROPOSITION 3.1. *Let e_j, $j \in \mathbb{N}$ be an orthonormal basis in H, $\hat{e}_j$ be their images under a wavelet transform $\mathcal{W}$ then the kernel $\langle \cdot, \rho(x)w_0 \rangle$ of the wavelet transform $\mathcal{W}$: $v \mapsto \langle x, \rho(x)w_0 \rangle$ has the following decomposition in the Dirac bra-ket notations:*

$$\langle \cdot, \rho(x)w_0 \rangle = \sum_{j=1}^{\infty} |\hat{e}_j\rangle \langle e_j| = \sum_{j=1}^{\infty} \langle \cdot, e_j \rangle \,\hat{e}_j.$$

Particularly if e_j are orthogonal polynomials *and $\hat{e}_j(x)$ are just powers of x then the kernel of the wavelet transform $\mathcal{W}$ is a* generating function *for e_j.*

EXERCISE 3.2. Give a proof.

EXERCISE 3.3. (i) Let $H_n(q)$ be the Hermite polynomials, show that functions $H_n(q)e^{-q^2/2}$ form an orthonormal basis in $L_2(\mathbb{R}^n)$.

(ii) Show that functions $z^n/\sqrt{n!}$ are images under the Segal-Bargmann transform $\breve{\mathcal{W}}$ of functions $H_n(q)e^{-q^2/2}$. (*Hint* use that the Hermite polynomial obtained from the Gaussian by derivatives which are infinitesimals of the Schödinger representation of the Heisenberg group).

(iii) Show that the Segal-Bargmann kernel is the generating function of Hermite polynomials.

Another example of this type is given by Bargmann in [5, § 2g]. It links representations of $SL_2(\mathbb{R})$ in the Berman space and Laguerre polynomials, we will consider it later.

PROPOSITION 3.4. *Let A be an operator $H \to H$. Then the wavelet transform $\mathcal{W}$ intertwines A with an operator $\hat{A}$ on $W(X)$ given by the integral kernel $\hat{a}(g, g')$:*

$$\hat{A}\hat{v}(x) = \int_X \hat{a}(x, x')\hat{v}(x')\,dx', \qquad \textit{where } \hat{a}(x, x') = \langle Aw_{x'}, w_x \rangle. \tag{3.1}$$

EXAMPLE 3.5. [24, (1.81)] For the Segal-Bargmann space: if an operator A on $L_2(\mathbb{R}^n)$ is given as an integral operator with a kernel $a(q, q')$ then $\hat{a}(z, z')$ is its double Segal-Bargmann transform.

EXAMPLE 3.6 (Harmonic Oscillator). Let

$$H = \frac{1}{2}\sum_{k=1}^{n}(p_k^2 + q_k^2 - 1) = \sum_{k=1}^{n} a_k^+ a_k^-,$$

be the Hamiltonian of a harmonic oscillator—the simplest non-trivial system in classic and quantum mechanics (creation a^+ and annihilation a^- operators were defined in (1.7)). The dynamics of quantum oscillator is governed by the *Schrödinger equation*:

$$\frac{d}{dt}\phi(t) = iH\phi(t),$$

and its solution $\phi(t) = e^{iHt}\phi(0)$ is given by means of the evolution operator e^{iHt}. It is easier to construct the exponent (as any other function) of H if we could diagonalise H and that is done in the Segal-Bargmann representation. Indeed, $H = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}$—the Euler operator. Its eigenvectors are z^m with eigenvalues $|m|$. Consequently the evolution of the harmonic oscillator is given by

$$e^{iHt}f(z) = f(e^{it}z),$$

which is in a nice resemblance with geometrical dynamic of classic harmonic oscillator. In the contrast, the picture in $L_2(\mathbb{R}^n)$ is not as simple. The eigenvectors of H are constructed from the Hermite polynomials (see Exercise 3.3) and dynamic is given by the complicated Mehler's formula [54, Chap. 1, (7.15)] .

CHAPTER 6

Wavelets in Banach Spaces and Functional Calculus

1. Coherent States for Banach Spaces

1.1. Abstract Nonsense. Let G be a group and G_0 be its closed normal subgroup. Let $X = G/G_0$ be the corresponding homogeneous space with an invariant measure $d\mu$ and $s : X \to G$ be a Borel section in the principal bundle $G \to G/G_0$. Let π be a continuous representation of a group G by invertible isometry operators ρ_g, $g \in G$ in a (complex) Banach space B.

The following definition simulates ones from the Hilbert space case [1, § 3.1].

DEFINITION 1.1. Let G, G_0, $X = G/G_0$, $s : X \to G$, $\pi : G \to \mathcal{L}(B)$ be as above. We say that $b_0 \in B$ is a *vacuum vector* if for all $h \in G_0$

$$\rho(h)b_0 = \chi(h)b_0, \qquad \chi(h) \in \mathbb{C}. \tag{1.1}$$

We will say that set of vectors $b_x = \rho(x)b_0$, $x \in X$ form a family of *coherent states* if there exists a continuous non-zero linear functional $l_0 \in B^*$ such that

(i) $\|b_0\| = 1$, $\|l_0\| = 1$, $\langle b_0, l_0\rangle \neq 0$;
(ii) $\rho(h)^* l_0 = \bar{\chi}(h) l_0$, where $\rho(h)^*$ is the adjoint operator to $\rho(h)$;
(iii) The following equality holds

$$\int_X \left\langle \rho(x^{-1})b_0, l_0\right\rangle \left\langle \rho(x)b_0, l_0\right\rangle \, d\mu(x) = \langle b_0, l_0\rangle \,. \tag{1.2}$$

The functional l_0 is called the *test functional*. According to the strong tradition we call the set $(G, G_0, \pi, B, b_0, l_0)$ *admissible* if it satisfies to the above conditions.

We note that mapping $h \to \chi(h)$ from (1.1) defines a character of the subgroup G_0. The following Lemma demonstrates that condition (1.2) could be relaxed.

LEMMA 1.2. *For the existence of a vacuum vector b_0 and a test functional l_0 it is sufficient that there exists a vector b'_0 and continuous linear functional l'_0 satisfying to* (1.1) *and* 1.1(ii) *correspondingly such that the constant*

$$c = \int_X \left\langle \rho(x^{-1})b'_0, l'_0\right\rangle \left\langle \rho(x)b'_0, l'_0\right\rangle \, d\mu(x) \tag{1.3}$$

is non-zero and finite.

PROOF. There exist a $x_0 \in X$ such that $\left\langle \rho(x_0^{-1})b'_0, l'_0\right\rangle \neq 0$, otherwise one has $c = 0$. Let $b_0 = \rho(x^{-1})b'_0 \left\|\rho(x^{-1})b'_0\right\|^{-1}$ and $l_0 = l'_0 \left\|l'_0\right\|^{-1}$. For such b_0 and l_0 we have 1.1(i) already fulfilled. To obtain (1.2) we change the measure $d\mu(x)$. Let $c_0 = \langle b_0, l_0\rangle \neq 0$ then $d\mu' = \left\|\rho(x^{-1})b'_0\right\| \left\|l'_0\right\| c_0 c^{-1} d\mu$ is the desired measure. □

REMARK 1.3. Conditions (1.2) and (1.3) are known for unitary representations in Hilbert spaces as *square integrability* (with respect to a subgroup G_0). Thus our definition describes an analog of square integrable representations for Banach spaces. Note that in Hilbert space case b_0 and l_0 are often the same function, thus condition 1.1(ii) is exactly (1.1). In the particular but still important case of trivial $G_0 = \{e\}$ (and thus $X = G$) all our results take simpler forms.

CONVENTION 1.4. In that follow we will usually write $x \in X$ and x^{-1} instead of $s(x) \in G$ and $s(x)^{-1}$ correspondingly. The right meaning of "x" could be easily found from the context (whether an element of X or G is expected there).

The wavelet transform (similarly to the Hilbert space case) could be defined as a mapping from B to a space of bounded continuous functions over G via representational coefficients

$$v \mapsto \widehat{v}(g) = \left\langle \rho(g^{-1})v, l_0 \right\rangle = \left\langle v, \pi(g)^* l_0 \right\rangle .$$

Due to 1.1(ii) such functions have simple transformation properties along orbits gG_0, i.e. $\widehat{v}(gh) = \bar{\chi}(h)\widehat{v}(g)$, $g \in G$, $h \in G_0$. Thus they are completely defined by their values indexed by points of $X = G/G_0$. Therefore we prefer to consider so-called reduced wavelet transform.

DEFINITION 1.5. The *reduced wavelet transform* $\mathcal{W}$ from a Banach space B to a space of function $F(X)$ on a homogeneous space $X = G/G_0$ defined by a representation π of G on B, a vacuum vector b_0 and a test functional l_0 is given by the formula

$$\mathcal{W} : B \to F(X) : v \mapsto \widehat{v}(x) = [\mathcal{W}v](x) = \left\langle \rho(x^{-1})v, l_0 \right\rangle = \left\langle v, \rho^*(x) l_0 \right\rangle . \tag{1.4}$$

There is a natural representation of G in $F(X)$. For any $g \in G$ there is a unique decomposition of the form $g = s(x)h$, $h \in G_0$, $x \in X$. We will define $r : G \to G_0 : r(g) = h = (s^{-1}(g))^{-1}g$ from the previous equality and write a formal notation $x = s^{-1}(g)$. Then there is a geometric action of G on $X \to X$ defined as follows

$$g : x \mapsto g^{-1} \cdot x = s^{-1}(g^{-1}s(x)).$$

We define a representation $\lambda(g) : F(X) \to F(X)$ as follow

$$[\lambda(g)f](x) = \chi(r(g^{-1} \cdot x)) f(g^{-1} \cdot x). \tag{1.5}$$

We recall that $\chi(h)$ is a character of G_0 defined in (1.1) by the vacuum vector b_0. For the case of trivial $G_0 = \{e\}$ (1.5) becomes the left regular representation $\rho_l(g)$ of G.

PROPOSITION 1.6. *The reduced wavelet transform $\mathcal{W}$ intertwines π and the representation λ (1.5) on $F(X)$:*

$$\mathcal{W}\rho(g) = \lambda(g)\mathcal{W}.$$

PROOF. We have:

$$\begin{aligned}
[\mathcal{W}(\rho(g)v)](x) &= \left\langle \rho(x^{-1})\rho(g)v, l_0 \right\rangle \\
&= \left\langle \rho((g^{-1}s(x))^{-1})v, l_0 \right\rangle \\
&= \left\langle \rho(r(g^{-1}\cdot x)^{-1})\rho(s(g^{-1}\cdot x)^{-1})v, l_0 \right\rangle \\
&= \left\langle \rho(s(g^{-1}\cdot x)^{-1})v, \rho^*(r(g^{-1}\cdot x)^{-1}) l_0 \right\rangle \\
&= \chi(r(g^{-1}\cdot x)^{-1})[\mathcal{W}v](g^{-1}x) \\
&= \lambda(g)[\mathcal{W}v](x).
\end{aligned}$$

□

COROLLARY 1.7. *The function space $F(X)$ is invariant under the representation λ of G.*

We will see that $F(X)$ possesses many properties of the *Hardy space*. The duality between l_0 and b_0 generates a transform dual to $\mathcal{W}$.

DEFINITION 1.8. The *inverse wavelet transform* $\mathcal{M}$ from $F(X)$ to B is given by the formula:

$$\begin{aligned}\mathcal{M}: F(X) \rightarrow B: \widehat{v}(x) \mapsto \mathcal{M}[\widehat{v}(x)] &= \int_X \widehat{v}(x) b_x \, d\mu(x) \\ &= \int_X \widehat{v}(x)\rho(x) \, d\mu(x) b_0. \end{aligned} \tag{1.6}$$

PROPOSITION 1.9. *The inverse wavelet transform $\mathcal{M}$ intertwines the representation λ on $F(X)$ and π on B:*

$$\mathcal{M}\lambda(g) = \rho(g)\mathcal{M}.$$

PROOF. We have:

$$\begin{aligned}\mathcal{M}[\lambda(g)\widehat{v}(x)] &= \mathcal{M}[\chi(r(g^{-1}\cdot x))\widehat{v}(g^{-1}\cdot x)] \\ &= \int_X \chi(r(g^{-1}\cdot x))\widehat{v}(g^{-1}\cdot x) b_x \, d\mu(x) \\ &= \chi(r(g^{-1}\cdot x)) \int_X \widehat{v}(x') b_{g\cdot x'} \, d\mu(x') \\ &= \rho_g \int_X \widehat{v}(x') b_{x'} \, d\mu(x') \\ &= \rho_g \mathcal{M}[\widehat{v}(x')],\end{aligned}$$

where $x' = g^{-1}\cdot x$. □

COROLLARY 1.10. *The image $\mathcal{M}(F(X)) \subset B$ of subspace $F(X)$ under the inverse wavelet transform $\mathcal{M}$ is invariant under the representation π.*

The following proposition explain the usage of the name for $\mathcal{M}$.

THEOREM 1.11. *The operator*

$$\mathcal{P} = \mathcal{M}\mathcal{W}: B \rightarrow B \tag{1.7}$$

is a projection of B to its linear subspace for which b_0 is cyclic. Particularly if π is an irreducible representation then the inverse wavelet transform $\mathcal{M}$ is a left inverse *operator on B for the wavelet transform $\mathcal{W}$:*

$$\mathcal{M}\mathcal{W} = I.$$

PROOF. It follows from Propositions 1.6 and 1.9 that operator $\mathcal{M}\mathcal{W}: B \rightarrow B$ intertwines π with itself. Then Corollaries 1.7 and 1.10 imply that the image $\mathcal{M}\mathcal{W}$ is a π-invariant subspace of B containing b_0. Because $\mathcal{M}\mathcal{W}b_0 = b_0$ we conclude that $\mathcal{M}\mathcal{W}$ is a projection.

From irreducibility of π by Schur's Lemma [30, § 8.2] one concludes that $\mathcal{M}\mathcal{W} = cI$ on B for a constant $c \in \mathbb{C}$. Particularly

$$\mathcal{M}\mathcal{W}b_0 = \int_X \left\langle \rho(x^{-1})b_0, l_0 \right\rangle \rho(x) b_0 \, d\mu(x) = cb_0.$$

From the condition (1.2) it follows that $\langle cb_0, l_0\rangle = \langle \mathcal{M}\mathcal{W}b_0, l_0\rangle = \langle b_0, l_0\rangle$ and therefore $c = 1$. □

We have similar

THEOREM 1.12. *Operator $\mathcal{W}\mathcal{M}$ is a projection of $L_1(X)$ to $F(X)$.*

We denote by $\mathcal{W}^*: F^*(X) \rightarrow B^*$ and $\mathcal{M}^*: B^* \rightarrow F^*(X)$ the adjoint (in the standard sense) operators to $\mathcal{W}$ and $\mathcal{M}$ respectively.

COROLLARY 1.13. *We have the following identity:*

$$\langle \mathcal{W}v, \mathcal{M}^* l\rangle_{F(X)} = \langle v, l\rangle_B, \qquad \forall v \in B, \quad l \in B^* \tag{1.8}$$

or equivalently

$$\int_X \left\langle \rho(x^{-1})v, l_0\right\rangle \left\langle \rho(x)b_0, l\right\rangle \, d\mu(x) = \langle v, l\rangle . \tag{1.9}$$

PROOF. We show the equality in the first form (1.9) (but will apply it often in the second one):

$$\langle \mathcal{W}v, \mathcal{M}^* l\rangle_{F(X)} = \langle \mathcal{M}\mathcal{W}v, l\rangle_B = \langle v, l\rangle_B .$$

□

COROLLARY 1.14. *The space* $F(X)$ *has the reproducing formula*

$$\widehat{v}(y) = \int_X \widehat{v}(x)\, \widehat{b}_0(x^{-1}\cdot y)\, d\mu(x), \tag{1.10}$$

where $\widehat{b}_0(y) = [\mathcal{W}b_0](y)$ *is the wavelet transform of the vacuum vector* b_0.

PROOF. Again we have a simple application of the previous formulas:

$$\begin{aligned}
\widehat{v}(y) &= \left\langle \rho(y^{-1})v, l_0\right\rangle \\
&= \int_X \left\langle \rho(x^{-1})\rho(y^{-1})v, l_0\right\rangle \left\langle \rho(x)b_0, l_0\right\rangle \, d\mu(x) \qquad (1.11)\\
&= \int_X \left\langle \rho(s(y\cdot x)^{-1})v, l_0\right\rangle \left\langle \rho(x)b_0, l_0\right\rangle \, d\mu(x) \\
&= \int_X \widehat{v}(y\cdot x)\, \widehat{b}_0(x^{-1})\, d\mu(x) \\
&= \int_X \widehat{v}(x)\, \widehat{b}_0(x^{-1}y)\, d\mu(x),
\end{aligned}$$

where transformation (1.11) is due to (1.9). □

REMARK 1.15. To possess a reproducing kernel—is a well-known property of spaces of analytic functions. The space $F(X)$ shares also another important property of analytic functions: it belongs to a kernel of a certain first order differential operator with Clifford coefficients (the Dirac operator) and a second order operator with scalar coefficients (the Laplace operator) [4, 33–35].

Let us now assume that there are two representations π' and π'' of the same group G in two different spaces B' and B'' such that two admissible sets $(G, G_0, \pi', B', b_0', l_0')$ and $(G, G_0, \pi'', B'', b_0'', l_0'')$ could be constructed for the same normal subgroup $G_0 \subset G$.

PROPOSITION 1.16. *In the above situation if* $F'(X) \subset F''(X)$ *then the composition* $\mathcal{T} = \mathcal{M}''\mathcal{W}'$ *of the wavelet transform* $\mathcal{W}'$ *for* π' *and the inverse wavelet transform* $\mathcal{M}''$ *for* π'' *is an intertwining operator between* π' *and* π'':

$$\mathcal{T}\pi' = \pi''\mathcal{T}.$$

$\mathcal{T}$ *is defined as follows*

$$\mathcal{T} : b \mapsto \int_X \left\langle \pi'(x^{-1})b, l_0'\right\rangle \pi''(x)b_0''\, d\mu(x). \tag{1.12}$$

This transformation defines a B''*-valued linear functional (a distribution for function spaces) on* B'.

The Proposition has an obvious proof. This simple result is a base for an alternative approach to functional calculus of operators [32, 34] and will be used in Subsection 2.2. Note also that formulas (1.4) and (1.6) are particular cases of (1.12) because $\mathcal{W}$ and $\mathcal{M}$ intertwine π and λ.

1.2. Wavelets and a Positive Cone. The above results are true for wavelets in general. In applications a Banach space B is usually equipped with additional structures and wavelets are interplay with them. We consider an example of such interaction.

We recall [36], [29, Chap. X] the notion of positivity in Banach spaces. Let $C \subset B$ be a sharp cone, i.e. $x \in C$ implies that $\lambda x \in C$ and $-\lambda x \notin C$ for $\lambda > 0$. We call such elements $x \in C$ *positive vectors*. We say also that $x \geqslant y$ iff $x - y$ is positive. There is the *dual cone* $C^* \subset B^*$ defined by the condition

$$C^* = \{f \mid f \in B^*, \ \langle b, f\rangle \geqslant 0 \ \ \forall x \in C\}.$$

An operator $A : B \to B$ is called *positive* if $Ab \geqslant 0$ for all $b \geqslant 0$. If A is positive with respect to C then A^* is positive with respect to C^*.

DEFINITION 1.17. We call a representation $\rho(g)$ *positive* if there exists a vector $b_0 \in C$ such that $\rho(x)b_0 \in C$ for all $x \in X$. A linear functional $f \in B^*$ is *positive* ($f > 0$) with respect to a vacuum vector b_0 if $\langle \rho(x)b_0, f\rangle \geqslant 0$ for all $x \in X$ and $\langle \rho(x)b_0, f\rangle$ is not identically equal to 0.

LEMMA 1.18. *For any positive representation $\rho(g)$ and vacuum vector b_0 there exists a positive test functional.*

PROOF. Obvious. □

We consider an estimation of positive linear functionals.

PROPOSITION 1.19. *Let $b \in B$ be a vector such that $b = \int_X \widehat{b}(x) b_x \, d\mu(x)$. Let*

(i) $D(b) = \{\left\langle \rho(x^{-1})b, l_0\right\rangle \mid x \in X\}$ *be the set of value of reduced wavelets transform;*
(ii) $\breve{D}(b)$ *be a convex shell of the values of* $\widehat{b}(x)$*;*
(iii) $\hat{D}(b) = \{\langle b, f\rangle \mid f \in C^*, \|f\| = 1, f \geqslant 0\}$.

Then

$$D(b) \subset \hat{D}(b) \subset \breve{D}(b).$$

PROOF. The first inclusion is obvious. The second could be easily checked:

$$\langle b, f\rangle = \left\langle \int_X \widehat{b}(x) b_x \, d\mu(x), f \right\rangle = \int_X \widehat{b}(x) \left\langle b_x, f\right\rangle \, d\mu(x).$$

□

1.3. Singular Vacuum Vectors. In many important cases the above general scheme could not be carried out because the representation π of G is not square-integrable or even not square-integrable modulo a subgroup G_0. Thereafter the vacuum vector b_0 could not be selected within the original space B which the representation π acts on. The simplest mathematical example is the Fourier transform (see Example 3.1). In physics this is the well-known problem of *absence of vacuum state* in the constructive algebraic quantum field theory [46–48]. The absence of the vacuum within the linear space of system's states is another illustration to the old thesis *Natura abhorret vacuum*[1] or even more specifically *Natura abhorret vectorem vacui*[2].

[1]Nature is horrified by (any) vacuum (Lat.).

[2]Nature is horrified by a carrier of nothingness (Lat.). This illustrates how far a humane beings deviated from Nature.

We will present a modification of our construction which works in such a situation. For a singular vacuum vector the algebraic structure of group representations could not describe the situation alone and requires an essential assistance from analytical structures.

DEFINITION 1.20. Let G, G_0, $X = G/G_0$, $s : X \to G$, $\pi : G \to \mathcal{L}(B)$ be as in Definition 1.1. We assume that there exist a topological linear space $\widehat{B} \supset B$ such that

(i) B is dense in $\widehat{B}$ (in topology of $\widehat{B}$) and representation π could be uniquely extended to the continuous representation $\widehat{\pi}$ on $\widehat{B}$.

(ii) There exists $b_0 \in \widehat{B}$ be such that for all $h \in G_0$

$$\widehat{\pi}(h)b_0 = \chi(h)b_0, \qquad \chi(h) \in \mathbb{C}. \tag{1.13}$$

(iii) There exists a continuous non-zero linear functional $l_0 \in B^*$ such that $\rho(h)^* l_0 = \bar{\chi}(h) l_0$, where $\rho(h)^*$ is the adjoint operator to $\rho(h)$;

(iv) The composition $\mathcal{M}\mathcal{W} : B \to \widehat{B}$ of the wavelet transform (1.4) and the inverse wavelet transform (1.6) maps B to B.

(v) For a vector $p_0 \in B$ the following equality holds

$$\left\langle \int_X \left\langle \rho(x^{-1})p_0, l_0 \right\rangle \, \rho(x)b_0 \, d\mu(x), l_0 \right\rangle = \langle p_0, l_0 \rangle \, , \tag{1.14}$$

where the integral converges in the weak topology of $\widehat{B}$.

As before we call the set of vectors $b_x = \rho(x)b_0$, $x \in X$ by *coherent states*; the vector b_0—a *vacuum vector*; the functional l_0 is called the *test functional* and finally p_0 is the *probe vector*.

This Definition is more complicated than Definition 1.1. The equation (1.14) is a substitution for (1.2) if the linear functional l_0 is not continuous in the topology of $\widehat{B}$. Example 3.1 shows that the Definition does not describe an empty set. The function theory in $\mathbb{R}^{1,1}$ constructed in [33] provides a more exotic example of a singular vacuum vector.

We shall show that 1.20(v) could be satisfied by an adjustment of other components.

LEMMA 1.21. *For the existence of a vacuum vector b_0, a test functional l_0, and a probe vector p_0 it is sufficient that there exists a vector b_0' and continuous linear functional l_0' satisfying to 1.20(i)–1.20(iv) and a vector $p_0' \in B$ such that the constant*

$$c = \left\langle \int_X \left\langle \rho(x^{-1})p_0, l_0 \right\rangle \, \rho(x)b_0 \, d\mu(x), l_0 \right\rangle$$

is non-zero and finite.

The proof follows the path for Lemma 1.2. The following Proposition summarizes results which could be obtained in this case.

PROPOSITION 1.22. *Let the wavelet transform $\mathcal{W}$ (1.4), its inverse $\mathcal{M}$ (1.6), the representation $\lambda(g)$ (1.5), and functional space $F(X)$ be adjusted accordingly to Definition 1.20. Then*

(i) *$\mathcal{W}$ intertwines $\rho(g)$ and $\lambda(g)$ and the image of $F(X) = \mathcal{W}(B)$ is invariant under $\lambda(g)$.*

(ii) *$\mathcal{M}$ intertwines $\lambda(g)$ and $\widehat{\pi}(g)$ and the image of $\mathcal{M}(F(B)) = \mathcal{M}\mathcal{W}(B) \subset B$ is invariant under $\rho(g)$.*

(iii) *If* $\mathcal{M}(F(X)) = B$ *(particularly if* $\rho(g)$ *is irreducible) then* $\mathcal{MW} = I$ *otherwise* $\mathcal{MW}$ *is a projection* $B \to \mathcal{M}(F(X))$*. In both cases* $\mathcal{MW}$ *is an operator defined by integral*

$$b \mapsto \int_X \left\langle \rho(x^{-1})b, l_0 \right\rangle \rho(x)b_0 \, d\mu(x), \tag{1.15}$$

(iv) *Space* $F(X)$ *has a reproducing formula*

$$\widehat{v}(y) = \left\langle \int_X \widehat{v}(x)\, \rho(x^{-1}y)b_0 \, dx, l_0 \right\rangle \tag{1.16}$$

which could be rewritten as a singular convolution

$$\widehat{v}(y) = \int_X \widehat{v}(x)\, \widehat{b}(x^{-1}y)\, dx$$

with a distribution $b(y) = \left\langle \rho(y^{-1})b_0, l_0 \right\rangle$ *defined by* (1.16).

The proof is algebraic and completely similar to Subsection 1.1.

2. Wavelets in Operator Algebras

We are going to apply the above abstract scheme to special spaces, which our main targets—wavelets on operator algebras. This gives a possibility to study operators by means of functions—symbols of operators.

2.1. Co- and Contravariant Symbols of Operators. We construct a realization of the wavelet transform as co- and contravariant symbols (also known as Wick and anti-Wick symbols) of operators. These symbols and their connections with wavelets in Hilbert spaces are known for a while [6–9]. However their realization (described below) as wavelets in Banach algebras seems to be new.

Let $\rho(g)$ be a representation of a group G in a Banach space B by isometry operators. Then we could define two new representations for groups G and $G \times G$ correspondingly in the space $\mathcal{L}(B)$ of bounded linear operators $B \to B$:

$$\widehat{\pi} \quad : \quad G \to \mathcal{L}(\mathcal{L}(B)) : A \mapsto \rho(g)^{-1} A \rho(g), \tag{2.1}$$

$$\widetilde{\pi} \quad : \quad G \times G \to \mathcal{L}(\mathcal{L}(B)) : A \mapsto \rho(g_1)^{-1} A \rho(g_2), \tag{2.2}$$

where $A \in \mathcal{L}(B)$. Note that $\widehat{\pi}(g)$ are algebra automorphisms of $\mathcal{L}(B)$ for all g. Representation $\widetilde{\pi}(g_1, g_2)$ is an algebra homomorphism from $\mathcal{L}(B)$ to the algebra $L_{(g_1,g_2)}(B)$ of linear operators on B equipped with a composition

$$A_1 \circ A_2 = A_1\, \rho(g_1)^{-1} \rho(g_2)\, A_2$$

with the usual multiplication of operators in the right-hand side. The rôle of such algebra homomorphisms in a symbolical calculus of operators was explained in [26]. It is also obvious that $\widehat{\pi}(g)$ is the restriction of $\widetilde{\pi}(g_1, g_2)$ to the diagonal of $G \times G$.

Let there are selected a vacuum vector $b_0 \in B$ and a test functional $l_0 \in B^*$ for π. Then there are the canonically associated vacuum vector $P_0 \in \mathcal{L}(B)$ and test functional $f_0 \in L^*(B)$ defined as follows:

$$P_0 \quad : \quad B \to B : b \mapsto P_0 b = \langle b, l_0 \rangle\, b_0; \tag{2.3}$$

$$f_0 \quad : \quad \mathcal{L}(B) \to \mathbb{C} : A \mapsto \langle A b_0, l_0 \rangle\,. \tag{2.4}$$

They define the following coherent states and transformations of the test functional

$$\begin{aligned} P_g &= \widehat{\pi}(g) P_0 = \langle \cdot, l_g \rangle\, b_g, & P_{(g_1,g_2)} &= \widetilde{\pi}(g_1, g_2) P_0 = \langle \cdot, l_{g_1} \rangle\, b_{g_2}, \\ f_g &= \widehat{\pi}^*(g) f_0 = \langle \cdot\, b_g, l_g \rangle\,, & f_{(g_1,g_2)} &= \widehat{\pi}^*(g_1, g_2) f_0 = \langle \cdot\, b_{g_1}, l_{g_2} \rangle \end{aligned}$$

where as usually we denote $b_g = \rho(g)b_0$, $l_g = \rho^*(g)l_0$. All these formulas take simpler forms for Hilbert spaces if $l_0 = b_0$.

DEFINITION 2.1. The *covariant (pre-)symbol* $A(x)$ $(A(x_1,x_2))$ of an operator A acting on a Banach space B defined by $b_0 \in B$ and $l_0 \in B^*$ is its wavelet transform with respect to representation $\widehat{\pi}(g)$ (2.1) ($\widetilde{\pi}(g_1,g_2)$ (2.2)) and the functional f_0 (2.4), i.e. they are defined by the formulas

$$A(x) = (\widehat{\pi}(x)A, f_0) = \left\langle \rho(x)^{-1}A\rho(x)b_0, l_0\right\rangle = \left\langle Ab_x, l_x\right\rangle, \tag{2.5}$$

$$A(x_1,x_2) = (\widetilde{\pi}(x_1,x_2)A, f_0) = \left\langle \rho(x_1)^{-1}A\rho(x_2)b_0, l_0\right\rangle = \left\langle Ab_{x_2}, l_{x_1}\right\rangle. \tag{2.6}$$

The *contravariant (pre-)symbol* of an operator A is a function $\breve{A}(x)$ (a function $\breve{A}(x_1,x_2)$ correspondingly) such that A is the inverse wavelet transform of $\breve{A}(x)$ (of $\breve{A}(x_1,x_2)$ correspondingly) with respect to $\widehat{\pi}(g)$ ($\widetilde{\pi}(g_1,g_2)$), i.e.

$$A = \int_X \breve{A}(x)\widehat{\pi}(x)P_0\, d\mu(x) = \int_X \breve{A}(x)P_x\, d\mu(x), \tag{2.7}$$

$$\begin{aligned} A &= \int_X\int_X \breve{A}(x_1,x_2)\widetilde{\pi}(x_1,x_2)P_0\, d\mu(x_1)\, d\mu(x_2) \\ &= \int_X\int_X \breve{A}(x_1,x_2)P_{(x_1,x_2)}\, d\mu(x_1)\, d\mu(x_2), \end{aligned} \tag{2.8}$$

where the integral is defined in the weak sense.

Obviously the covariant symbol $\breve{A}(x)$ is the restriction of the covariant pre-symbol $\breve{A}(x_1,x_2)$ to the diagonal of $G \times G$.

PROPOSITION 2.2. *A mapping $\sigma : A \mapsto \sigma_A(x_1,x_2)$ of operators to their covariant symbols is the algebra homomorphism from algebra of operators on B to algebra of integral operators on $F(G)$, i.e.*

$$\sigma_{A_1A_2}(x_1,x_3) = \int_X \sigma_{A_1}(x_1,x_2)\sigma_{A_2}(x_2,x_3)\, d\mu(x_2). \tag{2.9}$$

PROOF. One could easily see that:

$$\begin{aligned} &\int_X \sigma_{A_1}(x_1,x_2)\sigma_{A_2}(x_2,x_3)\, d\mu(x_2) \\ &\quad= \int_X \left\langle \rho(x_1)A_1\rho(x_2^{-1})b_0, l_0\right\rangle \left\langle \rho(x_2)A_2\rho(x_3^{-1})b_0, l_0\right\rangle\, d\mu(x_2) \\ &\quad= \int_X \left\langle \rho(x_2^{-1})b_0, A_1^*\rho^*(x_1)l_0\right\rangle \left\langle \rho(x_2)A_2\rho(x_3^{-1})b_0, l_0\right\rangle\, d\mu(x_2) \\ &\quad= \left\langle A_2\rho(x_3^{-1})b_0, A_1^*\rho^*(x_1)l_0\right\rangle \qquad (2.10) \\ &\quad= \left\langle \rho(x_1)A_1A_2\rho(x_3^{-1})b_0, l_0\right\rangle \\ &\quad= \sigma_{A_1A_2}(x_1,x_3), \end{aligned}$$

where transformation (2.10) is due to (1.9). □

The following proposition is obvious.

PROPOSITION 2.3. *An operator A could be reconstructed from its covariant presymbol $A(g_1,g_2)$ by the formula*

$$Av = \int_G\int_G A(g_1,g_2)\widehat{v}(g_2)\, d\mu(g_2) b_{g_1}\, d\mu(g_1).$$

We have a particular interest in operators closely connected with the representation ρ_g.

PROPOSITION 2.4. *Let an operator* A *on* B *is defined by the formula*

$$Av = \int_G a(g)\, \rho_g v \, d\mu(g)$$

for a function $a(g)$ *on* G*. Then* $A'\mathcal{W} = \mathcal{W}A$ *where* A' *is a two-sided convolution on* G *defined by the formula*

$$[A'\widehat{v}](h) = \int_G \int_G a(g_1)\widehat{b}_0(g_2)\widehat{v}(g_1^{-1}hg_2)\, d\mu(g_1)\, d\mu(g_2).$$

For operator algebras there are the standard notions of positivity: any operator of the form A^*A is positive; if algebra is realized as operators on a Hilbert space H then $b \in H$ defines a positive functional $f_b(A) = \langle Ab, b\rangle$. Thus the following proposition is a direct consequence of the Proposition 1.19.

PROPOSITION 2.5. [6, Thm. 1] *Let* A *be an operator, let* $D(A)$ *be the set of values of the covariant symbol* $A(x)$*, let* $\breve{D}(A)$ *be a convex shell of the values of contravariant symbol* $\breve{A}(x)$*. Let* $\hat{D}(A)$ *be the set of values of the quadratic form* $\langle Ab, b\rangle$ *for all vectors* $\|b\| = 1$*. Then*

$$D(b) \subset \hat{D}(b) \subset \breve{D}(b).$$

EXAMPLE 2.6. There are at least two very important realizations of symbolical calculus of operators. The theory of pseudodifferential operators (PDO) [17,49,53] is based on the Schrödinger representation of the Heisenberg group $\mathbb{H}^n$ (see Subsection 3.1) on the spaces of functions $L_p(\mathbb{R}^n)$ [26]. The Wick and anti-Wick symbolical calculi [6,8] arise from the Segal-Bargmann representation [5,45] (see Subsection 2.1) of the same group $\mathbb{H}^n$. Connections (intertwining operators) between these two representations were exploited in [26] to obtain fundamentals of the theory of PDO.

2.2. Functional Calculus and Group Representations. This Subsection illustrates a new approach to functional calculus of operators outlined in [32,34]. The approach uses the intertwining property for two representations instead of an algebraic homomorphism.

Let $\mathfrak{B}$ be a Banach algebra and $\mathbf{T} \subset \mathfrak{B}$ be its subset of elements. Let G be a group, G_0 be its normal subgroup and $X = G/G_0$—the corresponding homogeneous space. We assume that there is a representation τ depending from $\mathbf{T} \subset \mathfrak{B}$ defined on measurable functions from $L(X, \mathfrak{B})$ by the formula

$$\tau(g)f(x) = t(g,x)f(g^{-1}\cdot x), \qquad f(x) \in L(X, \mathfrak{B}), \tag{2.11}$$

where $t(g,x) : \mathfrak{B} \to \mathfrak{B}$ depends from $x \in X$ and $g \in G$. It is convenient to use a linear functional $l \in \mathfrak{B}'$ to make the situation more tractable by reducing it to the scalar case. Using l we could define a representation $\tau_l(x)$ on $F(X)$ by the following formula

$$\tau_l(x) : f_l(y) = \langle f(y), l\rangle \mapsto [\tau_l(x)f_l](y) = \langle \tau(x)f(y), l\rangle\,, \tag{2.12}$$

where $f(y) \in L(X, \mathfrak{B})$, $l \in \mathfrak{B}'$. We will understand convergence of all integrals involving τ in a weak sense, i.e. as convergence of all corresponding integrals with τ_l, $l \in \mathfrak{B}'$. We also say that τ is *irreducible* if all τ_l are irreducible.

REMARK 2.7. If $\mathfrak{B}$ is realized as an algebra of operators on a Banach space B then $l \in \mathfrak{B}'$ could be realized as an element of $B \otimes B'$. In this case the formula (2.12) looks like (2.5). The important difference is the following. In (2.5) the representation in the operator algebra $\mathfrak{B}$ arises from a representation in Banach space B and is the same for all elements of $\mathfrak{B}$. Representation τ_l in (2.12) is defined via the representation τ which depends in its turn from a set $\mathbf{T} \subset \mathfrak{B}$. Such representations are usually connected with some (non-linear) geometric actions of a group directly on operator algebra. Examples of these geometric actions are the representation of the Heisenberg group (3.10) leading to the Weyl functional calculus, fractional-linear transformations of operators leading [34] to Dunford-Riesz functional calculus and monogenic functional calculus [32]. Thus such representations contain important information on $\mathbf{T}$.

We also assume that there is a representation π of G in $F(X)$ with a vacuum vector b_0, a test functional l_0 and the system of wavelets (coherent states) b_x, $x \in X$, which were main actors in the previous Section. Let $\rho^*(g) = \rho(g^{-1})^*$ be the adjoint representation of $\rho(g)$ in $F'(X)$.

We need a preselected element $T_0(x) \in L(X, \mathfrak{B})$ which plays a rôle of a *vacum vector* for the representation τ, it is defined by the condition:

$$\int_X \widehat{b}_0(x')\, \tau(x')T_0(x)\, dx' = T_0(x), \tag{2.13}$$

where $\widehat{b}_0(x) = \left\langle \rho(x^{-1})b_0, l_0 \right\rangle$ is the wavelet transform of the vacuum vector $b_0 \in F(X)$ for π.

LEMMA 2.8. *A vacuum vector for τ always exists and is given by the formula*

$$T_0(x) = \int_X \widehat{b}_0(x')\, \tau(x')T(x)\, dx', \tag{2.14}$$

where $T(x) \in L(X, \mathfrak{B})$ is an arbitrary element which the integral (2.14) converges for. If τ is irreducible (i.e. all τ_l (2.12) are irreducible) in the linear span of $\tau(x')T(x)$, $x' \in X$ then $T_0(x)$ does not depend from a particular chose of $T(x)$.

PROOF. First we could easily verify condition (2.13) for the T_0 defined by (2.14):

$$\begin{aligned}
\int_X \widehat{b}_0(x)\, \tau(x)T_0(x_1)\, dx &= \int_X \widehat{b}_0(x)\, \tau(x) \int_X \widehat{b}_0(x')\, \tau(x')T(x_1)\, dx'\, dx \\
&= \int_X \int_X \widehat{b}_0(x)\, \widehat{b}_0(x')\, \tau(x)\tau(x')T(x_1)\, dx'\, dx \\
&= \int_X \left(\int_X \widehat{b}_0(x)\widehat{b}_0(x^{-1}x'')\, dx \right) \tau(x'')T(x_1)\, dx'' && (2.15)\\
&= \int_X \widehat{b}_0(x'')\, \tau(x'')T(x_1)\, dx'' && (2.16)\\
&= T_0(x_1).
\end{aligned}$$

Here we use the change of variables $x'' = x \cdot x'$ in (2.15) and reproducing property (1.10) of $\widehat{b}_0(x)$ in (2.16).

To prove that for any admissible $T(x)$ we will receive the same $T_0(x)$ is enough to pass from the representation τ to representations τ_l (2.12) defined by $l \in \mathfrak{B}'$. Then we deal with scalar valued (not operator valued) functions and knew that one could use any admissible vector $T_l(x) = \langle T(x), l\rangle$ as a vacuum vector in the reconstruction formula (1.6). □

Now we could specify the Definition 1.1 from [32] as follows.

DEFINITION 2.9. Let G, G_0, $X = G/G_0$, $\mathfrak{B}$, $\mathbf{T}$, τ, π, F, b_0, T_0 be as described above. One says that a continuous linear $\mathfrak{B}$-valued functional $\Phi_{\mathbf{T}}(\cdot, x) : F(X) \to \mathfrak{B}$, parametrized by a point $x \in X$ and depending from $\mathbf{T} \subset \mathfrak{B}$:

$$\Phi_{\mathbf{T}}(\cdot, x) : f(y) \mapsto [\Phi_{\mathbf{T}} f](x) = \int_X f(y)\Phi_{\mathbf{T}}(y, x)\, dy$$

is a *functional calculus* if

(i) $\Phi_{\mathbf{T}}$ is an intertwining operator between $\rho(g)$ and $\tau(g)$, namely

$$[\Phi_{\mathbf{T}}\rho(g)f(y)](x) = \tau(g)[\Phi_{\mathbf{T}}f(y)](x), \tag{2.17}$$

for all $g \in G$ and $f(y) \in F(X)$

(ii) $\Phi_{\mathbf{T}}$ maps the vacuum vector $b_0(y)$ for the representation π to the vacuum vector $T_0(x)$ for the representation τ:

$$[\Phi_{\mathbf{T}} b_0(y)](x) = T_0(x). \tag{2.18}$$

$\mathfrak{B}$-valued distribution $\Phi_{\mathbf{T}}(y, x_0)$, $s(x_0) = e \in G$ associated with $\mathfrak{B}$-valued linear functional on $F(X)$ is called a *spectral decomposition* of operators $\mathbf{T}$.

Representation τ in (2.17) is defined by (2.11):

$$\tau(g)[\Phi_{\mathbf{T}}f(y)](x) = t(g,x)[\Phi_{\mathbf{T}}f(y)](g^{-1}\cdot x).$$

We could state (2.17) equivalently as

$$[I_y \otimes \tau_x(g)]\Phi(y,x) = [\rho^*_y(g^{-1}) \otimes I_x]\Phi(y,x).$$

REMARK 2.10. The functional calculus $\Phi_{\mathbf{T}}(y,x)$ as defined here has the explicit covariant property with respect to variable x. Thus it could be restored by the representation τ from a single value, e.g. $\Phi_{\mathbf{T}}(y, s^{-1}(e))$, where e is the identity of G. We particularly will calculate only $[\Phi_{\mathbf{T}}f](s^{-1}(e))$ in Subsection 3.3 as the value of a functional calculus. This value is usually denoted by $f(\mathbf{T})$ and is exactly the functional calculus of operators in the traditional meaning.

In particular cases different characteristics of the spectral decomposition could give relevant information on the set of operators $\mathbf{T}$, e.g. the support $\operatorname{supp}_y \Phi_{\mathbf{T}}(y,x_0)$ of $\Phi_{\mathbf{T}}(y,x_0)$

$$f(y) = 0 \;\forall y \in \operatorname{supp}_y \Phi_{\mathbf{T}}(y,x_0) \quad \Rightarrow \quad [\Phi_{\mathbf{T}}f](x_0) = 0$$

is called *(joint) spectrum* of set $\mathbf{T} \subset \mathfrak{B}$. This definition of the spectrum is connected with the *Arveson-Connes spectral theory* [3, 16, 51] while there are several important differences mentioned in [32, Rem. 4.4].

In the paper [32] the approach was illustrated by a newly developed functional calculus for several non-commuting operators based on Möbius transformations of the unit ball in $\mathbb{R}^n$. It was shown in [34, § 7] that the classic Dunford-Riesz functional calculus is generated by a representation of $SL(2,\mathbb{R})$ within this procedure. However an abstract scheme of the approach was not presented yet. We give some its elements here.

From Proposition 1.16 we know a general form of an intertwining operator of two related representations of a group, which could be employed here. Let $l_0(y)$ be the distribution corresponding to a test functional l_0 for the representation π on $F(X)$ such that we could write

$$\langle f(y), l_0\rangle_{F(X)} = \int_X f(y) l_0(y)\, dy.$$

We denote also by $\rho^*(x)l_0(y)$, $x \in X$, $y \in X$ distributions corresponding to linear functionals $\rho^*(x)l_0$, where $\rho^*(x)$ is the adjoint representation to π on the space $F(X)$.

PROPOSITION 2.11 (Spectral syntesis). *Under assumption of Proposition 1.16 the functional calculus exists and is unique. The spectral decomposition $\Phi_{\mathbf{T}}(y,x)$ as a distribution on X is given by the formula*

$$\Phi_{\mathbf{T}}(y,x) = \int_X \rho^*(x) l_0(y)\, \tau(x) T_0(x)\, dx. \tag{2.19}$$

The functional calculus $\Phi_{\mathbf{T}}(\cdot, x)$ *as a mapping* $F(X) \to \mathfrak{B}$ *is given correspondingly*

$$\Phi_{\mathbf{T}}(\cdot, x) : f(y) \mapsto [\Phi_{\mathbf{T}} f(y)](x) = \int_X \int_X \langle f(y), \rho^*(x')l_0(y)\rangle \, \tau(x')T_0(x) \, dy \, dx'. \tag{2.20}$$

PROOF. Obviously (2.19) and (2.20) are equivalent. Thus we will prove (2.20) only. For an arbitrary $f(y) \in F(X)$ we could write

$$\begin{aligned}
[\Phi_{\mathbf{T}} f(y)](x) &= \left[\Phi_{\mathbf{T}} \int_X \widehat{f}(x')\rho(x')b_0(y)\, dx'\right](x) && (2.21)\\
&= \int_X \widehat{f}(x')[\Phi_{\mathbf{T}}\rho(x')b_0(y)](x)\, dx' && (2.22)\\
&= \int_X \widehat{f}(x')\tau(x')[\Phi_{\mathbf{T}} b_0(y)](x)\, dx' && (2.23)\\
&= \int_X \widehat{f}(x')\,\tau(x')T_0(x)\, dx' && (2.24)\\
&= \int_X \langle f(y), \rho^*(x')l_0\rangle \, \tau(x')T_0(x)\, dx' && (2.25)
\end{aligned}$$

We use in (2.21) that functions in $F(X)$ are superpositions of coherent states, transformation (2.22) is made by linearity and continuity of $\Phi_{\mathbf{T}}$, step (2.23) is due to condition (2.17) and we finally apply (2.18) to receive (2.24). Thus it is proven that the functional calculus which is continuous, linear, and satisfies to (2.17) and (2.18) (if exists) is unique and given by (2.25). Now we should check that (2.25) really gives the right answer.

We will check first that (2.25) satisfies to (2.17):

$$\begin{aligned}
\tau(g)\Phi(y,x) &= \tau(x_1) \int_X \rho^*(x')l_0(y)\, \tau(x')T_0(x)\, dx'\\
&= \int_X \rho^*(x')l_0(y)\, \tau(g \cdot x')T_0(x)\, dx'\\
&= \int_X \rho^*(g^{-1} \cdot x'')l_0(y)\, \tau(x'')T_0(x)\, dx'' && (2.26)\\
&= \rho^*(g^{-1}) \int_X \rho^*(x'')l_0(y)\, \tau(x'')T_0(x)\, dx''\\
&= \rho^*(g^{-1})\Phi(y,x),
\end{aligned}$$

where we made substitution $x'' = g \cdot x'$ in (2.26). Finally (2.18) directly follows from the condition (2.13). □

Let there exists $L_0(x) \in L'(X, \mathfrak{B})$—a test functional for a vacuum vector $T_0(x)$ and representation τ, i.e.

$$\langle T_0, L_0\rangle_{L(X,\mathfrak{B})} = \int_X \langle \tau(x^{-1}T_0), L_0\rangle_{L(X,\mathfrak{B})} \langle \tau(x)T_0, L_0\rangle_{L(X,\mathfrak{B})}\, dx,$$

where

$$\langle T_0, L_0\rangle_{L(X,\mathfrak{B})} = \int_X \langle T_0(x), L_0(x)\rangle_{\mathfrak{B}}\, dx$$

and $\langle T_0(x), L_0(x)\rangle_{\mathfrak{B}}$ is the pairing between $\mathfrak{B}$ and $\mathfrak{B}'$.

PROPOSITION 2.12 (Spectral analysis). *If a* $\mathfrak{B}$*-valued function* $F(x)$ *from* $L(X, \mathfrak{B})$ *belongs to the closer of the linear span of* $\tau(x')T_0(x)$, $x' \in X$ *then*

$$F(x) = [\Phi_{\mathbf{T}} f(y)](x),$$

where

$$f(y) = \int_X \langle \tau(x^{-1}F), L_0\rangle_{L(X,\mathfrak{B})}\, \rho(x)b_0(y)\, dx. \tag{2.27}$$

PROOF. The formula (2.27) is just another realization of intertwining operator (1.12) from Proposition 1.16. □

Let $K : F(X_1) \to F(X_2)$ be an intertwining mapping between two representations ρ_1 and ρ_2 of groups G_1 and G_2 in spaces $F(X_1)$ and $F(X_2)$ respectively. Let $K(z, y)$, $z \in X_1$, $y \in X_2$ be the Schwartz kernel of K.

THEOREM 2.13 (Mapping of spectral decompositions). *Let*

$$\begin{aligned} f_1(z) = [Kf_2](z) &= \int_{X_2} f_2(y)K(z, y)\, dy \\ \Phi_{\mathbf{T}_1}(z, x) &= \int_{X_2} K(z, y)\Phi_{\mathbf{T}_2}(y, x)\, dy, \end{aligned}$$

where a functional calculus $\Phi_{\mathbf{T}_2}$ is defined by representations ρ_2 and τ. Then $\Phi_{\mathbf{T}_2}(z, x)$ is a functional calculus for ρ_1 and τ and we have an identity:

$$[\Phi_{\mathbf{T}_1} f_1(z)](x) = [\Phi_{\mathbf{T}_2} f_2(y)](x). \tag{2.28}$$

PROOF. The intertwining property for $\Phi_{\mathbf{T}_2}(z, x)$ follows from transitivity. The identity (2.28) is a simple application of the Fubini theorem:

$$\begin{aligned} [\Phi_{\mathbf{S}} g(z)](x) &= \int_{X_1} g(z)\Phi_{\mathbf{S}}(z, x)\, dz \\ &= \int_{X_1} g(z) \int_{X_2} K(z, y)\Phi_{\mathbf{T}}(y, x)\, dy\, dz \\ &= \int_{X_2} \int_{X_1} g(z)K(z, y)\, dz\, \Phi_{\mathbf{T}}(y, x)\, dy \\ &= \int_{X_2} g(f(y))\Phi_{\mathbf{T}}(y, x)\, dy \\ &= [\Phi_{\mathbf{T}} g(f(y))](x). \end{aligned}$$

□

This Theorem could be turned in the spectral mapping theorem under suitable conditions [32, Thm. 3.19].

3. Examples

We are going to demonstrate that the above construction is not only algebraically attractive but also belongs to the heart of analysis. More examples could be found in [14, 33, 34] and will be given elsewhere.

3.1. The Heisenberg Group and Schrödinger Representation. We will consider a realization of the previous results in a particular cases of the Fourier transform and Segal-Bargmann [5, 45] type spaces $F_p(\mathbb{C}^n)$. They arise from representations of the Heisenberg group $\mathbb{H}^n$ [24, 25, 54] on $L_p(\mathbb{R}^n)$.

The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}^n$ spanned by $\{T, P_j, Q_j\}$, $n = 1, \ldots, n$ is defined by the commutation relations:

$$[P_i, Q_j] = T\delta_{ij}. \tag{3.1}$$

They are known from quantum mechanics as the canonical commutation relations of coordinates and momentum operators. An element $g \in \mathbb{H}^n$ could be represented as $g = (t, z)$ with $t \in \mathbb{R}$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and the group law is given by

$$g * g' = (t, z) * (t', z') = (t + t' + \frac{1}{2}\sum_{j=1}^{n} \Im(\bar{z}_j z'_j), z + z'), \tag{3.2}$$

where $\Im z$ denotes the imaginary part of a complex number z. The Heisenberg group is (non-commutative) nilpotent step 2 Lie group.

We take a representation of $\mathbb{H}^n$ in $L_p(\mathbb{R}^n)$, $1 < p < \infty$ by operators of shift and multiplication [54, § 1.1]:

$$g = (t, z) : f(y) \to [\sigma_{(t,z)} f](y) = e^{i(2t-\sqrt{2}vy+uv)} f(y - \sqrt{2}u), \qquad z = u + iv, \tag{3.3}$$

i.e., this is the Schrödinger type representation with parameter $\hbar = 1$. These operators are isometries in $L_p(\mathbb{R}^n)$ and the adjoint representation $\rho^*_{(t,z)} = \rho_{(-t,-z)}$ in $L_q(\mathbb{R}^n)$, $p^{-1} + q^{-1} = 1$ is given by a formula similar to (3.3).

3.2. Wavelet Transforms for the Heisenberg Group in Function Spaces.

EXAMPLE 3.1. We start from the subgroup $G_0 = \mathbb{R}^{n+1} = \{(t, z) \mid \Im(z) = 0\}$. Then $X = G/G_0 = \mathbb{R}^n$ and an invariant measure coincides with the Lebesgue measure. Mappings $s : \mathbb{R}^n \to \mathbb{H}^n$ and $r : \mathbb{H}^n \to H$ are defined by the identities $s(x) = (0, ix)$, $s^{-1}(t, z) = \Im z$, $r(t, u + iv) = (t, u)$. The composition law $s^{-1}((t, z) \cdot s(x)) = x+u$ reduces to Euclidean shifts on $\mathbb{R}^n$. We also find $s^{-1}((s(x_1))^{-1} \cdot s(x_2)) = x_2 - x_1$ and $r((s(x_1))^{-1} \cdot s(x_2)) = 0$.

We consider the representation $\sigma(g)$ of $\mathbb{H}^n$ in the space of smooth rapidly decreasing functions $B = \mathcal{S}(\mathbb{R}^n)$. As a character of $G_0 = \mathbb{R}^{n+1}$ we take the $\chi(t, u) = e^{2it}$. The corresponding test functional l_0 satisfying to 1.20(iii) is the integration $l_0(f) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y)\, dy$. Thus the wavelet transform is as follows

$$\widehat{f}(x) = \int_{\mathbb{R}^n} \sigma(s(x)^{-1}) f(y)\, dy = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\sqrt{2}xy} f(y)\, dy \tag{3.4}$$

and is nothing else but the Fourier transform[3].

Now we arrive to the absence of a vacuum vector in B, indeed there is no a $f(x) \in \mathcal{S}(\mathbb{R}^n)$ such that

$$[\sigma(t, u) f](y) = \chi(t, u) f(y) \iff e^{i2t} f(y - \sqrt{2}u) = e^{i2t} f(y).$$

There is a way out accordingly to Subsection 1.3. We take $B' = L_\infty(\mathbb{R}^n) \supset B$ and the vacuum vector $b_0(y) \equiv (2\pi)^{-n/2} \in B'$. Then coherent states are $b_x(y) = (2\pi)^{-n/2} e^{-i\sqrt{2}xy}$ and the inverse wavelet transform is defined by the inverse Fourier transform

$$f(y) = \int_{\mathbb{R}^n} \widehat{f}(y) b_x(y)\, dx = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(y) e^{-i\sqrt{2}xy}\, dx.$$

The condition 1.20(iv) $\mathcal{M}\mathcal{W} : B \to B$ follows from the composition of two facts $\mathcal{W} : B \to B$ and almost identical to it $\mathcal{M} : B \to B$, which are proved in standard analysis textbooks (see for example [31, § IV.2.3]). To check scaling (1.14) according to the tradition in analysis [25] we take a probe vector $p_0 = e^{-y^2/2} \in B$. Due to well known formula $\int_{-\infty}^{+\infty} e^{-y^2/2} dy = (2\pi)^{1/2}$ of real analysis we have

$$\begin{aligned}
\left\langle \int_X \langle \widehat{p}_0(x), l_0 \rangle\, b_x\, dx, l_0 \right\rangle &= (2\pi)^{-n} \iiint e^{i\sqrt{2}xy} e^{-y^2/2}\, dy e^{-i\sqrt{2}xw}\, dx\, dw \\
&= (2\pi)^{n/2} \int_{\mathbb{R}^n} e^{-y^2/2} dy \\
&= \langle p_0, l_0 \rangle .
\end{aligned}$$

---

[3]The *inverse* Fourier transform in fact. In our case the signs selection is opposite to the standard one, but we will neglect this difference.

Thus our scaling is correct. $\mathcal{W}$ and $\mathcal{M}$ intertwine the left regular representation — multiplication by $e^{i\sqrt{2}yv}$ with operators

$$\begin{aligned}[\lambda(g)f](x) &= \chi(r(g^{-1}\cdot x))f(g^{-1}\cdot x)\\ &= e^{i\sqrt{2}\cdot 0}f(x-\sqrt{2}u) = f(x-\sqrt{2}u),\end{aligned}$$

i.e. with Euclidean shifts. From the identity $\langle \mathcal{W}v, \mathcal{M}^*l\rangle_{F(X)} = \langle v, l\rangle_B$ (1.8) follows the Plancherel's identity:

$$\int_{\mathbb{R}^n} \widehat{v}(y)\widehat{l}(y)\,dy = \int_{\mathbb{R}^n} v(x)l(x)\,dx.$$

These are basic and important properties of the Fourier transform.

The Schrödinger representation is irreducible on $\mathcal{S}(\mathbb{R}^n)$ thus $\mathcal{M} = \mathcal{W}^{-1}$. Thereafter integral formulas (1.15) and (1.16) representing operators $\mathcal{M}\mathcal{W} = \mathcal{W}\mathcal{M} = 1$ correspondingly give an integral resolution for a convolution with the Dirac delta $\delta(x)$. We have integral resolution for the Dirac delta

$$\delta(x-y) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\xi(x-y)}\,d\xi.$$

All described results on the Fourier transform are a part of any graduate curriculum. What is a reason for a reinvention of a bicycle here? First, the same path works with minor modifications for a function theory in $\mathbb{R}^{1,1}$ described in [33]. Second we will use this interpretation of the Fourier transform in Example 3.5 for a demonstration how the Weyl functional calculus fits in the scheme outlined in [32, 34] and Subsection 2.2.

REMARK 3.2. Of course, the Heisenberg group is not the only possible source for the Fourier transform. We could consider the "$ax + b$" group [54, Chap. 7] of the affine transformations of Euclidean space $\mathbb{R}^n$. The normal subgroup $G_0 = \mathbb{R}$ of dilations generates the homogeneous space $X = \mathbb{R}^n$ on which shifts act simply transitively. The Fourier transform deduced from this setting will naturally exhibit scaling properties. We could alternatively consider a group $\mathbb{M}^n$ of Möbius transformation [12, Chap. 2] in $\mathbb{R}^{n+1}$ which map upper half plane to itself. Then there is an induced action of $\mathbb{M}^n$ on $\mathbb{R}^n$—the boundary of upper half plane. $\mathbb{M}^n$ generated by composition of the affine transformations and the Kelvin inverse [12, Chap. 2]. If we take the normal subgroup G_0 generated by dilations and the Kelvin inverse then the quotient space X will again coincide with $\mathbb{R}^n$ and we immediately arrive to the above case. On the other hand the Fourier transform derived in such a way could be easily connected with the plane wave decomposition [50] in Clifford analysis [11, 19].

EXAMPLE 3.3. As a subgroup G_0 we select now the center of $\mathbb{H}^n$ consisting of elements $(t, 0)$. Of course $X = G/G_0$ isomorphic to $\mathbb{C}^n$ and mapping $s : \mathbb{C}^n \to G$ simply is defined as $s(z) = (0, z)$. The Haar measure on $\mathbb{H}^n$ coincides with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ [54, § 1.1] thus the invariant measure on X also coincides with the Lebesgue measure on $\mathbb{C}^n$. Note also that composition law $s^{-1}(g\cdot s(z))$ reduces to Euclidean shifts on $\mathbb{C}^n$. We also find $s^{-1}((s(z_1))^{-1}\cdot s(z_2)) = z_2 - z_1$ and $r((s(z_1))^{-1}\cdot s(z_2)) = \frac{1}{2}\Im\bar{z}_1 z_2$.

As a "vacuum vector" we will select the original *vacuum vector* of quantum mechanics—the Gauss function $f_0(x) = e^{-x^2/2}$ which belongs to all $L_p(\mathbb{R}^n)$. Its transformations are defined as follow:

$$\begin{aligned}f_g(x) = [\rho_{(t,z)}f_0](x) &= e^{i(2t-\sqrt{2}vx+uv)}\,e^{-(x-\sqrt{2}u)^2/2}\\ &= e^{2it-(u^2+v^2)/2}e^{-((u-iv)^2+x^2)/2+\sqrt{2}(u-iv)x}\\ &= e^{2it-z\bar{z}/2}e^{-(\bar{z}^2+x^2)/2+\sqrt{2}\bar{z}x}.\end{aligned}$$

Particularly $[\rho_{(t,0)}f_0](x) = e^{-2it}f_0(x)$, i.e., it really is a vacuum vector in the sense of our definition with respect to G_0. For the same reasons we could take $l_0(x) = e^{-x^2/2} \in L_q(\mathbb{R}^n)$, $p^{-1}+q^{-1}=1$ as the test functional.

It could be shown that $[\rho_{(0,z)}f_0](x)$ belongs to $L_q(\mathbb{R}^n)\otimes L_p(\mathbb{C}^n)$ for all $p>1$ and $q>1$, $p^{-1}+q^{-1}=1$. Thus transformation (1.4) with the kernel $[\rho_{(0,z)}f_0](x)$ is an embedding $L_p(\mathbb{R}^n)\to L_p(\mathbb{C}^n)$ and is given by the formula

$$\begin{aligned}
\widehat{f}(z) &= \left\langle f, \rho_{s(z)} f_0 \right\rangle \\
&= \pi^{-n/4}\int_{\mathbb{R}^n} f(x)\, e^{-z\bar{z}/2}\, e^{-(z^2+x^2)/2+\sqrt{2}zx}\, dx \\
&= e^{-z\bar{z}/2}\pi^{-n/4}\int_{\mathbb{R}^n} f(x)\, e^{-(z^2+x^2)/2+\sqrt{2}zx}\, dx. \qquad (3.5)
\end{aligned}$$

Then $\widehat{f}(g)$ belongs to $L_p(\mathbb{C}^n, dg)$ or its preferably to say that function $\breve{f}(z) = e^{z\bar{z}/2}\widehat{f}(t_0,z)$ belongs to space $L_p(\mathbb{C}^n, e^{-|z|^2}dg)$ because $\breve{f}(z)$ is analytic in z. Such functions for $p=2$ form the *Segal-Bargmann space* $F_2(\mathbb{C}^n, e^{-|z|^2}dg)$ of functions [5,45], which are analytic by z and square-integrable with respect the Gaussian measure $e^{-|z|^2}dz$. For this reason we call the image of the transformation (3.5) by *Segal-Bargmann type space* $F_p(\mathbb{C}^n, e^{-|z|^2}dg)$. Analyticity of $\breve{f}(z)$ is equivalent to condition $(\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}z_j I)\widehat{f}(z) = 0$.

The integral in (3.5) is the well-known Segal-Bargmann transform [5,45]. Inverse to it is given by a realization of (1.6):

$$\begin{aligned}
f(x) &= \int_{\mathbb{C}^n} \widehat{f}(z) f_{s(z)}(x)\, dz \\
&= \int_{\mathbb{C}^n} \widehat{f}(u,v) e^{iv(u-\sqrt{2}x)}\, e^{-(x-\sqrt{2}u)^2/2}\, du\, dv \qquad (3.6)\\
&= \int_{\mathbb{C}^n} \breve{f}(z) e^{-(\bar{z}^2+x^2)/2+\sqrt{2}\bar{z}x}\, e^{-|z|^2}\, dz.
\end{aligned}$$

The corresponding operator $\mathcal{P}$ (1.7) is an identity operator $L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)$ and (1.7) gives an integral presentation of the Dirac delta.

Integral transformations (3.5) and (3.6) intertwines the Schrödinger representation (3.3) with the following realization of representation (1.5):

$$\begin{aligned}
\lambda(t,z)f(w) &= \widehat{f}_0(z^{-1}\cdot w)\bar{\chi}(t + r(z^{-1}\cdot w)) \qquad (3.7)\\
&= \widehat{f}_0(w-z)e^{it+i\Im(\bar{z}w)} \qquad (3.8)
\end{aligned}$$

Meanwhile the orthoprojection $L_2(\mathbb{C}^n, e^{-|z|^2}dg) \to F_2(\mathbb{C}^n, e^{-|z|^2}dg)$ is of a separate interest and is a principal ingredient in Berezin quantization [8,15]. We could easy find its kernel from (1.10). Indeed, $\widehat{f}_0(z) = e^{-|z|^2}$, then the kernel is

$$\begin{aligned}
K(z,w) &= \widehat{f}_0(z^{-1}\cdot w)\bar{\chi}(r(z^{-1}\cdot w)) \\
&= \widehat{f}_0(w-z)e^{i\Im(\bar{z}w)} \\
&= \exp\left(\frac{1}{2}(-|w-z|^2 + w\bar{z} - z\bar{w})\right) \\
&= \exp\left(\frac{1}{2}(-|z|^2 - |w|^2) + w\bar{z}\right).
\end{aligned}$$

To receive the reproducing kernel for functions $\breve{f}(z) = e^{|z|^2}\widehat{f}(z)$ in the Segal-Bargmann space we should multiply $K(z,w)$ by $e^{(-|z|^2+|w|^2)/2}$ which gives the standard reproducing kernel $= \exp(-|z|^2 + w\bar{z})$ [5, (1.10)].

3.3. Operator Valued Representations of the Heisenberg Group. We proceed now with our main targets: wavelets in operator algebras. We shell show that well-known and new functional calculi are realizations of the scheme from Subsection 2.2.

CONVENTION 3.4. [2] Let B be a Banach space. We will say that an operator $A : B \to B$ is *unitary* if A is invertible and $\|Ab\| = \|b\|$ for all $b \in B$. An operator $A : B \to B$ is called *self-adjoint* if the operator $\exp(iA)$ is unitary. In the Hilbert space case this convention coincides with the standard definition.

Let $T_1, \ldots, T_n$ be an n-tuples of selfadjoint linear operators on a Banach space B. We put for our convenience $T_0 = I$—the identical operator. It follows from the Trotter-Daletskii[4] formula [44, Thm. VIII.31] that any linear combination $\sum_{j=0}^{n} a_j T_j$ is again a selfadjoint operator. We will consider a set of unitary operators

$$T(a_0, a_1, \ldots, a_n) = \exp\left(i \sum_{j=0}^{n} a_j T_j\right) \tag{3.9}$$

parametrized by vectors $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$. Particularly $T(0, 0, \ldots, 0) = I$. A family of their transformations $\omega(t, z)$, $t \in \mathbb{R}$, $z \in \mathbb{C}^n$ is defined by the rule

$$\begin{aligned}\omega(t,z)T(a_0, a_1, \ldots, a_n) = T\Bigg(a_0 + t + \sum_{j=1}^{n}(u_j v_j - \sqrt{2} a_j u_j),\\ a_1 - \sqrt{2} v_1, \ldots, a_n - \sqrt{2} v_n\Bigg),\end{aligned} \tag{3.10}$$

where $z_j = u_j + i v_j$. A direct calculation shows that $\omega(t', z')\omega(t'', z'') = \omega(t' + t'' + \frac{1}{2}\Im(\bar{z}' z''), z' + z'')$—this is a non-linear geometric representation of the Heisenberg group $\mathbb{H}^n$. We could observe that

$$T(a_0, a_1, \ldots, a_n) = \omega(a_0, a)T(0, 0, \ldots, 0) = \omega(a_0, a)T_0 = \omega(a_0, a)I,$$

where $a = (ia_1, \ldots, ia_n)$. Obviously all transformations $\omega(t, z)$ are isometries if the norm of elements $T(a_0, a_1, \ldots, a_n)$ is defined as their operator norm.

The representation ω (3.10) is not linear and we would like to use the procedure outlined in Remark 1.2. We construct the linear space of operator valued functions $L(\mathbb{R}^n, \mathfrak{B})$ for a $\mathbb{H}^n$-homogeneous space X as follows

$$[\mathcal{T}f](t) = \int_X f(x)\, \omega(s(x))\, dx\, T(t), \qquad t \in \mathbb{R}^n,\ s(x) \in \mathbb{H}^n. \tag{3.11}$$

We also extend the representation ω to $L(X, \mathfrak{B})$ as follows:

$$\omega : [\mathcal{T}f](t) \mapsto \omega(g)[\mathcal{T}f](t) = \int_X f(g \cdot x)\, \omega(s(x))\, dx\, T(t), \tag{3.12}$$

where $[\mathcal{T}f](t) \in L_1(\mathbb{H}^n)$, $g \in \mathbb{H}^n$.

We will go on with coherent states defined by such a representation. In the notations of Subsection 2.2 operators $T_1, \ldots, T_n$ form a set $\mathbf{T}$ defining the representation $\tau = \omega$ in (3.10) with $T_0(x) = I$ being a vacuum vector.

EXAMPLE 3.5. We are ready to demonstrate that the Weyl functional calculus is an application of Definition 2.9 and Example 3.1 as was announced in [32, Remark 4.3]. Consider again the subgroup $G_0 = \mathbb{R}^{n+1} = \{(t, z) \mid \Im(z) = 0\}$ and a realization of scheme from Subsection 1.3 for this subgroup. Then the first paragraph of Example 3.1 is applicable here.

[4]The formula is usually attributed to Trotter alone. It is widely unknown that the result appeared in [18] also.

We could easily see that $\omega(t,u_1,\ldots,u_n)I = e^{it+i\sum_1^n u_j}I$ and we select the identity operator I times $(2\pi)^{-n/2}$ as the vacuum vector $T_0(x)$ of the representation ω. Thereafter the transformation $\mathcal{T} : S(\mathbb{R}^n) \to L(\mathbb{R}^n, \mathfrak{B})$ (3.11) is exactly the inverse wavelet transform for the representation ω. This transformation is defined at least for all $f \in S(\mathbb{R}^n)$. The space $S(\mathbb{R}^n)$ is the image of the wavelet (Fourier) transform (3.4). Thus as outlined in Proposition 2.11 we could construct an intertwining operator $\mathcal{F}$ between σ (3.3) and ω (3.10) from the formula (2.20) as follow (see Remark 2.10):

$$
\begin{aligned}
[\Phi_{\mathbf{T}}f](0) &= \mathcal{M}_\omega \mathcal{W}_\sigma f = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{f}(x)\,\omega(0,x_1,\ldots,x_n)\,I\,dx \\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{f}(x)\, e^{i\sum_1^n x_jT_j}\,dx. \qquad (3.13)
\end{aligned}
$$

This formula is exactly the integral formula for the Weyl functional calculus [2, 43, 52]. As an example one could define a function

$$
e^{-\sum_1^n T_j^2/2} = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-\sum_1^n x_j^2/2} e^{i\sum_1^n x_jT_j}\,dx. \qquad (3.14)
$$

As we have seen (2.19) one could formally write the integral kernel from (3.13) as convolution of the integral kernels for $\mathcal{W}_\sigma$ and $\mathcal{M}_\omega$:

$$
\Phi(y,0) = \int_{\mathbb{R}^n} e^{-i\sum_1^n y_jx_j} e^{i\sum_1^n x_jT_j}\,dx. \qquad (3.15)
$$

This expression looks very formal, but it is possible to give it a precise mathematical meaning as an operator valued distribution. Such an approach was explored by ANDERSON in [2]. The support of this distribution was defined as the *Weyl joint spectrum* for n-tuple of non-commuting operators $T_1,\ldots,T_n$ and studied in [2].

REMARK 3.6. As was mentioned in Remark 3.2 one could construct the Fourier transform from representations of $ax+b$ group or the group $\mathbb{M}^n$ of Möbius transformations of the upper half plane. Analogously one could deduce the Weyl functional calculus as an intertwining operator between two representation of this group. The Cauchy kernel $G(x)$ [11, § 9] in Clifford analysis is the kernel of the Cauchy integral transform

$$
f(y) = \int_{\partial X} G_y(x)\vec{n}(x)f(x)\,d\sigma(x), \qquad y \in X,
$$

where $\vec{n}(x)$ is the outer unit vector orthogonal to ∂X and $d\sigma(x)$ is the surface element at a point x. The Cauchy integral formula (as any wavelet transform) intertwines two representations acting on X and ∂X of $\mathbb{M}^n$ similarly to the case of complex analysis [34, § 6]. Thus we could apply here Theorem 2.13 on a mapping of spectral distributions. The formula (2.28) take the form

$$
\Phi_W f = \int_{\partial X} \Phi_W(G)(x)\vec{n}(x)f(x)\,d\sigma(x),
$$

where Φ_W stands for the Weyl functional calculus. This gives another interpretation for the main result of the paper [28, Thm. 5.4].

There is no a reason to restrict ourselves only to the case of subgroup $G_0 = \mathbb{R}^{n+1} = \{(t,z) \mid \Im(z) = 0\}$. Thus we proceed with the next example.

EXAMPLE 3.7. In an analogy with Example 3.3 let us consider now the wavelet theory associated to the subgroup $G_0 = \mathbb{R}^1 = \{(t,0)\}$ and the representation ω. The first paragraph of Example 3.3 depends only on $G = \mathbb{H}^n$ and $G_0 = \mathbb{R}^1$ and thus is applicable in our case.

It is easy to see from formula (3.10) that any operator valued function $[\mathcal{T}f](x)$ (3.11) is an eigen vector for $\omega(h)$, $h \in G_0$. To be concise with function models we select as a vacuum vector the operator $\exp(-\sum_1^n T_j^2)$ (3.14). Then the condition (2.13) immediately follows from (3.14). Thus we could define a functional calculus Φ : $F_p(\mathbb{C}^n) \to L(\mathbb{C}^n, \mathfrak{B})$ by the formula (see Remark 2.10):

$$\begin{aligned}
[\Phi_{\mathbf{T}} f](0) &= \int_{\mathbb{C}^n} f(z)\omega(0,z)\exp(-\sum_1^n T_j^2/2)\,dz \\
&= \int_{\mathbb{C}^n} f(z)\omega(0,z)(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\sum_1^n x_j^2/2} e^{i\sum_1^n x_j T_j}\,dx\,dz \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{C}^n} f(z) \int_{\mathbb{R}^n} e^{-\sum_1^n x_j^2/2}\omega(0,z) e^{i\sum_1^n x_j T_j}\,dx\,dz \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{3n}} \exp\sum_{j=1}^n \left(-\frac{x_j^2}{2} + i(v_j - \sqrt{2}x_j)u_j + i(x_j - \sqrt{2}v_j)T_j\right) \\
&\qquad \times f(z)\,dx\,du\,dv. \qquad (3.16)
\end{aligned}$$

The last formula could be rewritten for mutually commuting operators T_j as follows:

$$\begin{aligned}
[\Phi_{\mathbf{T}} f](0) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} \exp\sum_{j=1}^n \left(-\frac{x_j^2}{2} + i x_j(T_j - \sqrt{2}u_j)\right)\,dx \\
&\qquad \times \exp\sum_{j=1}^n i(v_j u_j - \sqrt{2}v_j T_j)\, f(z)\,du\,dv \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{C}^n} \exp\sum_{j=1}^n \left(i v_j(u_j - \sqrt{2}T_j) - \frac{(T_j - \sqrt{2}u_j)^2}{2}\right) f(z)\,dz
\end{aligned}$$

where the exponent of operator is defined in the standard sense, e.g. via the Weyl functional calculus (3.13) or the Taylor expansion. The last formula is similar to (3.6). This is very natural for commuting operators as well as that for non-commuting operators fromula (3.16) is more complicated.

The spectral distribution

$$\Phi_{\mathbf{T}}(z,0) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\sum_{j=1}^n \left(-\frac{x_j^2}{2} + i(v_j - \sqrt{2}x_j)u_j + i(x_j - \sqrt{2}v_j)T_j\right)\,dx$$

derived from (3.16) contains at least as much information on operators $T_1, \ldots, T_n$ as the Weyl distribution (3.15) and deserves a careful separate investigation. We will just mention in conclusion that the Segal-Bargmann space is an example of the Fock space—space of second quantization for bosonic fields. Thus the functional calculus based on the Segal-Bargmann model sketched here seems to be an appropriate model for quantized bosonic fields.

Acknowledgements

I am grateful to Mohamed Bugatma for many suggestions, which help to improve these notes.

Bibliography

[1] S. Twareque Ali, J.-P. Antoine, J.-P. Gazeau, and U.A. Mueller. Coherent states and their generalizations: A mathematical overview. *Rev. Math. Phys.*, 7(7):1013–1104, 1995. Zbl # 837.43014. 47

[2] Robert F. V. Anderson. The Weyl functional calculus. *J. Functional Analysis*, 4:240–267, 1969. 63, 64

[3] William Arveson. *The Harmonic Analysis of Automorphisms Groups*. American Mathematical Society, Amer. Math. Soc. Summer Institute, 1980. 57

[4] Michael Atiyah and Wilfried Schmid. A geometric construction of the discrete series for semisimple Lie group. In J.A. Wolf, M. Cahen, and M. De Wilde, editors, *Harmonic Analysis and Representations of Semisimple Lie Group*, volume 5 of *Mathematical Physics and Applied Mathematics*, pages 317–383. D. Reidel Publishing Company, Dordrecht, Holland, 1980. 38, 50

[5] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform. Part I. *Comm. Pure Appl. Math.*, 3:215–228, 1961. 39, 40, 42, 44, 45, 55, 59, 62

[6] F. A. Berezin. Covariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR Ser. Mat.*, 36:1134–1167, 1972. Reprinted in [8, pp. 228–261]. 53, 55

[7] F. A. Berezin. Quantization. *Izv. Akad. Nauk SSSR Ser. Mat.*, 38:1116–1175, 1974. 53

[8] F. A. Berezin. *Metod vtorichnogo kvantovaniya*. "Nauka", Moscow, second edition, 1986. Edited and with a preface by M. K. Polivanov. 44, 53, 55, 62, 67

[9] Felix A. Berezin. Quantization in complex symmetric spaces. *Math. USSR-Izv.*, 9:341–379, 1975. 53

[10] David Bernier and Keith F. Taylor. Wavelets from square-integrable representations. *SIAM J. Math. Anal.*, 27(2):594–608, 1996. MR97h:22004. 33

[11] F. Brackx, R. Delanghe, and F. Sommen. *Clifford Analysis*, volume 76 of *Research Notes in Mathematics*. Pitman Advanced Publishing Program, Boston, 1982. 61, 64

[12] Jan Cnops. *Hurwitz Pairs and Applications of Möbius Transformations*. Habilitation dissertation, Universiteit Gent, Faculteit van de Wetenschappen, 1994. See also [13]. 61

[13] Jan Cnops. *An introduction to Dirac operators on manifolds*, volume 24 of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2002. 67

[14] Jan Cnops and Vladimir V. Kisil. Monogenic functions and representations of nilpotent Lie groups in quantum mechanics. *Mathematical Methods in the Applied Sciences*, 22(4):353–373, 1998. `http://arXiv.org/abs/math/9806150`arXiv:math/9806150. MR2000b:81044. Zbl1005.22003. 59

[15] Lewis A. Coburn. Berezin-Toeplitz quantization. In *Algebraic Mettods in Operator Theory*, pages 101–108. Birkhäuser Verlag, New York, 1994. 44, 62

[16] Alain Connes. Une classification des facteurs de type III. *Ann. Sci. École Norm. Sup. (4)*, 6:133–252, 1973. 57

[17] H.O. Cordes. *Pseudodifferential Operators — An Abstract Theory*, volume 756 of *Lect. Notes Math.* Springer-Verlag, Berlin, 1979. 55

[18] Yu.L. Daletski. On representation of solutions of operator equations in a form of functional integrals. *Dokl. Akad. Nauk SSSR*, 134(5):1013–1016, 1960. 63

[19] Richard Delanghe, Frank Sommen, and Vladimir Souček. *Clifford Algebra and Spinor-Valued Functions*. Kluwer Academic Publishers, Dordrecht, 1992. 61

[20] Jacques Dixmier. *Les C^*-algebres et Leurs Representations*. Gauthier-Villars, Paris, 1964. 25

[21] Jacques Dixmier. *C^*-algebras*. North-Holland Publishing Co., Amsterdam, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15. 36

[22] M. Duflo and Calvin C. Moore. On the regular representation of a nonunimodular locally compact group. *J. Functional Analysis*, 21(2):209–243, 1976. 36

[23] V. A. Fock. Konfigurationsraum und zweite quantelung. *Z. Phys. A*, 75:622–647, 1932. 40

[24] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989. 12, 45, 59

[25] Roger Howe. On the role of the Heisenberg group in harmonic analysis. *Bull. Amer. Math. Soc. (N.S.)*, 3(2):821–843, 1980. 19, 59, 60

[26] Roger Howe. Quantum mechanics and partial differential equations. *J. Funct. Anal.*, 38(2):188–254, 1980. 53, 55

[27] Roger Howe and Eng Chye Tan. *Non-Abelian Harmonic Analysis: Applications of* $SL(2,\mathbb{R})$. Universitext. Springer-Verlag, New York, 1992. 19

[28] Brian Jefferies and Alan McIntosh. The Weyl calculus and Clifford analysis. *Bull. Austral. Math. Soc.*, 57(2):329–341, 1998. 64

[29] L. V. Kantorovich and G. P. Akilov. *Functional analysis*. Pergamon Press, Oxford, second edition, 1982. Translated from the Russian by Howard L. Silcock. 51

[30] A. A. Kirillov. *Elements of the theory of representations*. Springer-Verlag, Berlin, 1976. Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220. 17, 18, 21, 29, 30, 31, 32, 35, 36, 40, 44, 49

[31] Alexander A. Kirillov and Alexei D. Gvishiani. *Theorems and Problems in Functional Analysis*. Problem Books in Mathematics. Springer-Verlag, New York, 1982. 9, 10, 17, 28, 29, 31, 39, 60

[32] Vladimir V. Kisil. Möbius transformations and monogenic functional calculus. *Electron. Res. Announc. Amer. Math. Soc.*, 2(1):26–33, 1996. (electronic) MR98a:47018. 51, 55, 56, 57, 59, 61, 63

[33] Vladimir V. Kisil. Analysis in $\mathbb{R}^{1,1}$ or the principal function theory. *Complex Variables Theory Appl.*, 40(2):93–118, 1999. http://arXiv.org/abs/funct-an/9712003arXiv:funct-an/9712003. MR2000k:30078. 31, 38, 50, 52, 59, 61

[34] Vladimir V. Kisil. Two approaches to non-commutative geometry. In H. Begehr, O. Celebi, and W. Tutschke, editors, *Complex Methods for Partial Differential Equations*, chapter 14, pages 219–248. Kluwer Academic Publishers, Netherlands, 1999. http://arXiv.org/abs/funct-an/9703001arXiv:funct-an/9703001, MR2001a:01002. 38, 39, 50, 51, 55, 56, 57, 59, 61, 64

[35] A.W. Knapp and N.R. Wallach. Szegö kernels associated with discrete series. *Invent. Math.*, 34(2):163–200, 1976. 38, 50

[36] M. G. Kreĭn and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. *Uspehi Matem. Nauk (N. S.)*, 3(1(23)):3–95, 1948. MR10:256c. 51

[37] Serge Lang. *Algebra*. Addison-Wesley, New York, 1969. 19, 20

[38] Serge Lang. $SL_2(\mathbf{R})$, volume 105 of *Graduate Text in Mathematics*. Springer-Verlag, New York, 1985. 19

[39] George W. Mackey. *Mathematical Foundations of Quantum Mechanics*. W.A. Benjamin, Inc., New York, Amsterdam, 1963. 39

[40] Stéphane Mallat. *A wavelet tour of signal processing*. Academic Press Inc., San Diego, CA, 1998. Illustration from the book are on-line: http://www.cs.nyu.edu/cs/faculty/mallat/book.html. 14

[41] Willard Miller, Jr. *Lie Theory and Special Functions*. Academic Press, New York, 1968. Mathematics in Science and Engineering, Vol. 43. 17

[42] Vladimir Nazaikinskii and Boris Sternin. Wave packet transform in symplectic geometry and asymptotic quantization. In Komrakov B.P., Krasil'shchik I.S., Litvinov G.L., and Sossinsky A.B., editors, *Lie Groups and Lie Algebras. Their Representations, Generalizations and Applications*, number 433 in Mathematics and Its Applications, pages 47–70, Dordrecht-Boston-London, 1998. Kluwer Academic Publishers. 40

[43] E. Nelson. A functional calculus for non-commuting operators. In F.E. Browder, editor, *Functional analysis and related fields, Proceedings of a conference in honour of Professor Marshal Stone, Univ. of Chicago, May 1968*, pages 172–187. Springer-Verlag, Berlin, Heidelberg, New York, 1970. 64

[44] Michael Reed and Barry Simon. *Functional Analysis*, volume 1 of *Methods of Modern Mathematical Physics*. Academic Press, Orlando, second edition, 1980. 63

[45] Irving E. Segal. *Mathematical Problems of Relativistic Physics*, volume II of *Proceedings of the Summer Seminar (Boulder, Colorado, 1960)*. American Mathematical Society, Providence, R.I., 1963. 39, 40, 42, 55, 59, 62

[46] Irving E. Segal. Algebraic quantization and stability. In William B. Arveson and Ronald G. Douglas, editors, *Operator Theory: Operator Algebras and Applications*, number 51 in Proceedings of Symposia in Mathematics, pages 503–518. American Mathematical Society, Providence, R.I., 1990. 51

[47] Irving E. Segal. C^*-algebras and quantization. In Robert Doran, editor, *C^*-Algebras: 1943–1993*, number 167 in Contemporary Mathematics, pages 55–65. American Mathematical Society, Providence, R.I., 1994. 51

[48] Irving E. Segal. Rigorous covariant form of the correspondence principle. In William Arveson et al., editors, *Quantization, Nonlinear Partial Differential Equations, and Operator Algebra (Cambridge, MA, 1994)*, volume 59 of *Proc. Sympos. Pure Math.*, pages 175–202. American Mathematical Society, Providence, R.I., 1996. 51

[49] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson. 55

[50] Frank Sommen. Plane wave decompositions of monogenic functions. *Annales Pol. Math.*, 49:101–114, 1988. 61

[51] Masamichi Takesaki. *Structure of Factors and Automorphism Groups*, volume 51 of *Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I., 1983. 57

[52] Michael E. Taylor. Functions of several self-adjoint operators. *J. Funct. Anal.*, 19:91–98, 1968. 64

[53] Michael E. Taylor. *Pseudodifferential operators*, volume 34 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1981. 55

[54] Michael E. Taylor. *Noncommutative harmonic analysis*, volume 22 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1986. 17, 18, 24, 31, 40, 41, 46, 59, 60, 61

[55] N. Ja. Vilenkin. *Special Functions and the Theory of Group Representations*. American Mathematical Society, Providence, R. I., 1968. Translated from the Russian by V. N. Singh. Translations of Mathematical Monographs, Vol. 22. 17, 28, 29, 31

Index

$SB_2(\mathbb{C})$, 8
$\Lambda(\mathfrak{g})$, 22
$SL_2(\mathbb{R})$, 14
$\mathbb{R}^n_\pm$, 15
δ_{jk}, 20
tr, 21
$\langle\cdot,\cdot\rangle$, 20
$ax+b$ group, 14

abstract group, 13
action
  transitive, 15
adjoint representation, 20
admissible vector, 27
annihilation operator, 38
annihilation operator, 32
associativity, 13

Bargmann projection, *see* Segal-Bargmann projection
Bergman projection, 8
Bergman space, 7
bra and ket, 6

calculus
  functional, *see* functional calculus
Cauchy integral formula, 7, 25
character
  of group, 21
  of representation, 21
character of a group, 21
character of representation, 21
classes of equivalent representations, 20
coherent states, 3, 11, 25
commutation relations, 31
commutative, 14
commuting operator, 22
configuration space, 31
conjugation, 15
continuous group, 14
convolution, 16
convolution operator, 16
creation operator, 32, 38
cyclic vector, 22

dicrete series, 27
Dirac
  bra and ket, 6
dual object, 22
dual space, 22

equation Schrödinger, 38
equivalent representations, 20
exact representation, 20

faithful representation, 20
fiducial vector, 25
finite dimensional representation, 20
functional calculus
  Riesz-Dunford, 9

generating function, 37
ground state, 25
group, 13
  $SL_2(\mathbb{R})$, 14
  $ax+b$, 14
  abstract, 13
  commutative, 14
  continuous, 14
  Heisenberg, 14
  Lie, 14
  locally compact, 14
  noncommutative, 14
  of characters, 21
  representation, 19
  subgroup
    isotropy, 15
  transformation, 13
group multiplication, 13

Haar measure, 15
Hardy space, 7
Heisenberg commutation relations, 31
Heisenberg group, 14
  Schrödinger representation, 19
Hilbert space, 5
  orthonormal base, 5
homogeneous space, 15

identity, 13
  Parseval(-Pythagoras), 5
  Plancherel, 6
  Pythagoras, 5
infinite dimensional representation, 20
intertwining operator, 22
invariant measure, 15
invariant subspace, 21
inverse, 13
inverse wavelet transform, 26
irreducible representation, 22
isotropy (sub)group, 15

kernel
  reproducing, 28
Kroneker delta, 20

Lebesgue measure, 15
left shift, 15
lemma
  Schur's, 23
Lie groups, 14
linear-fractional transformations, 15
linearization procedure, 19
locally compact, 14
loop, 15
lower (upper) half plane, 15

main problem of representation theory, 22
matrix elements, 20
measure, 15
  Haar, 15
  Lebesgue, 15
  left invariant, 15
measurement, 31
momentum space, 31
mother wavelet, 11, 25

noncommutative, 14
nontrivial invariant subspaces, 22

observables, 31
operator
  annihilation, 32, 38
  commuting, 22
  convolution, 16
  creation, 32, 38
  intertwining, 22
  Töplitz, 8
  unitary, 20
orbits, 14
orthogonal polynomials, 37
orthonormal base, 5
orthonormal basis, 20

Parseval(-Pythagoras) identity, 5
Plancherel identity, 6
projection
  Bergman, 8
  Segal-Bargmann, 9

reducible representation, 22
regular representation, 22
relation of equivalence, 20
relations
  commutation, 31
representaion
  linearization, 19
representation, 19
  adjoint, 20
  continuous, 19
  exact, 20
  faithful, 20
  finite dimensional, 20
  infinite dimensional, 20
  irreducible, 22
  linear, 19
  reducible, 22
  regular, 22
  Schrödinger, 19
  series
    discrete, 27
  square integrable, 25, 27
    modulo subgroup, 31
  trivial, 20
  unitary, 21
representation of a group, 19
representation space, 20
representations, 19
  equivalent, 20
    unitary, 21
reproducing kernel, 6, 28
restriction of representation, 22
Riesz-Dunford functional calculus, 9
right shift, 15

scalar product, 20
Schrödinger equation, 38
Schrödinger representation, 19, 32
Schur's lemma, 23
Segal-Bargmann projection, 9
Segal-Bargmann space, 8
set
  total, 25
shift
  left, 15
  right, 15
signal processing, 9
singular integral operator, 7
singular integral operator
  Szegö, 7
space
  Bergman, 7
  Hardy, 7
  Hilbert, 5
  Segal-Bargmann, 8
square integrable modulo subgroup, 31
square integrable representation, 25, 27
state
  ground, 25
states, 31
Stone-von Neumann theorem, 32
subrepresentation, 22
subspace
  invariant, 21
Szegö singular integral operator, 7

Töplitz operator, 8, 9
theorem
  Stone-von Neumann, 32
total set, 25
trace, 21
transform
  wavelet, 25
transformation
  linear-fractional, 15
transformation group, 13
transitive, 15
trivial representation, 20

unit circle, 7
unit disk, 7

unitary equivalent representations, 21
unitary operator, 20
unitary representation, 21
upper (lower) half plane, 15

vacuum vector, 11, 25
vacuum vector of quantum mechanics, 33
vector
  admissible, 27
  cyclic, 22
  fiducial, 25
  vacuum, 11, 25

wave functions, 31
wavelet
  mother, 11, 25
wavelet transform, 25
  inverse, 26
wavelets, 3, 5, 11, 25
WFT, *see* Windowed Fourier Transform
Windowed Fourier Transform, 9